Symplectic realization and quantization of electric charge in field of monopole distributions

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The problem: we are interested in the hamiltonian description and quantization of the electrically charged particle in the field of the magnetic monopole distributions,

$$\{x^i, x^j\} = 0 \ , \quad \{x^i, \bar{\pi}_j\} = \delta^i{}_j \quad \text{and} \quad \{\bar{\pi}_i, \bar{\pi}_j\} = e\, \varepsilon_{ijk}\, B^k(\vec{x}) \ ,$$
$$\{\bar{\pi}_i, \bar{\pi}_j, \bar{\pi}_k\} := \tfrac{1}{3}\{\bar{\pi}_i, \{\bar{\pi}_j, \bar{\pi}_k\}\} + \text{cyclic} = e\, \varepsilon_{ijk}\, \vec{\nabla}\cdot\vec{B} \ . \tag{0.1}$$

What is wrong with the Dirac monopole? We consider the situation, where the source of the non-associativity cannot be removed by imposing the appropriate boundary condition.

Why now? This work was manly motivated by the locally non-geometric backgrounds in closed string theory, like the constant R-flux:

$$\{x^i, x^j\} = \frac{\ell_s^3}{\hbar^2}\, R^{ijk}\, p_k \ , \qquad \{x^i, p_j\} = \delta^i_j \qquad \{p_i, p_j\} = 0 \ .$$

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- Lorentz force law from symplectic realizations
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Symplectic realizations

Consider extended phase space, $x^I = (x^i, \tilde{x}^i)$ and $p_I = (p_i, \tilde{p}_i)$, with

$$\{x^I, x^J\} = \{p_I, p_J\} = 0 \qquad \text{and} \qquad \{x^I, p_J\} = \delta^I{}_J \ .$$

Covariant momenta are $\pi_I = (\pi_i, \tilde{\pi}_i) = p_I - e\, A_I(x^I)$. Choose

$$\vec{A}(x^I) = -\tfrac{1}{2}\, \vec{\tilde{x}} \times \vec{B}(\vec{x}\,) \qquad \text{and} \qquad \vec{\tilde{A}}(x^I) = \vec{0} \ ,$$

with the property $\vec{\tilde{\nabla}} \times \vec{A} = \vec{B}$.

Define $\vec{\bar{\pi}} = \vec{\pi} + \vec{\tilde{\pi}}$. The algebra of PB becomes:

$$\begin{aligned}
\{x^i, \bar{\pi}_j\} &= \{\tilde{x}^i, \bar{\pi}_j\} = \{\tilde{x}^i, \tilde{\pi}_j\} = \delta^i{}_j \ , \\
\{\bar{\pi}_i, \bar{\pi}_j\} &= e\, \varepsilon_{ijk}\, B^k(\vec{x}\,) + \tfrac{e}{2}\left(\varepsilon_{ijk}\, \partial_l B^k(\vec{x}\,) - \varepsilon_{ijl}\, \partial_k B^k(\vec{x}\,)\right) \tilde{x}^l \ , \\
\{\bar{\pi}_i, \tilde{\pi}_j\} &= \{\tilde{\pi}_i, \bar{\pi}_j\} = \tfrac{e}{2}\, \varepsilon_{ijk}\, B^k(\vec{x}\,) \ .
\end{aligned} \tag{0.2}$$

The corresponding symplectic two-form is given by

$$\begin{aligned}\Omega \;=\; & \tfrac{e}{2}\left[\,\varepsilon_{ijk}\,B^k(\vec{x}\,)+\tfrac{1}{2}\left(\varepsilon_{ijk}\,\partial_l B^k(\vec{x}\,)-\varepsilon_{ijl}\,\partial_k B^k(\vec{x}\,)\right)\tilde{x}^l\,\right]\mathrm{d}x^i\wedge\mathrm{d}\tilde{x}^j \\ & +\tfrac{e}{2}\,\varepsilon_{ijk}\,B^k(\vec{x}\,)\,\mathrm{d}x^i\wedge\mathrm{d}x^j+\,\mathrm{d}\bar{\pi}_i\wedge\mathrm{d}x^i+\mathrm{d}\bar{\pi}_i\wedge\mathrm{d}\tilde{x}^i+\mathrm{d}\tilde{\pi}_i\wedge\mathrm{d}\tilde{x}^i\ .\end{aligned}$$

The projection to the constraint surface $\vec{\tilde{x}}=\vec{\tilde{\pi}}=\vec{0}$ coincides exactly with the almost symplectic structure

$$\omega=\tfrac{e}{2}\,\varepsilon_{ijk}\,B^k(\vec{x}\,)\,\mathrm{d}x^i\wedge\mathrm{d}x^j+\mathrm{d}\bar{\pi}_i\wedge\mathrm{d}x^i\ ,$$

corresponding to the monopole algebra (0.1).

For the exemple of $\vec{B}_{\rm spher}(\vec{x}\,)=\frac{\rho}{3}\,\vec{x}$, one has

$$\begin{aligned}\{x^i,\bar{\pi}_j\} &= \{\tilde{x}^i,\bar{\pi}_j\} = \{\tilde{x}^i,\tilde{\pi}_j\} = \delta^i{}_j\ ,\\ \{\bar{\pi}_i,\bar{\pi}_j\} &= \tfrac{e\,\rho}{3}\,\varepsilon_{ijk}\left(x^k-\tilde{x}^k\right)\ ,\\ \{\bar{\pi}_i,\tilde{\pi}_j\} &= \{\tilde{\pi}_i,\bar{\pi}_j\} = \tfrac{e\,\rho}{6}\,\varepsilon_{ijk}\,x^k\ . \qquad (0.3)\end{aligned}$$

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Our aim now is to obtain the Lorentz force equation

$$m\,\ddot{\vec{x}} = e\,\dot{\vec{x}} \times \vec{B} + e\,\vec{E}\ ,$$

from (0.2) by choosing an appropriate Hamiltonian. Let,

$$H(x^I,\pi_I) = \frac{1}{2}\left(\vec{\pi}\quad \vec{\tilde{\pi}}\ \right)\cdot\begin{pmatrix} a & b \\ b & c\end{pmatrix}\begin{pmatrix}\vec{\pi} \\ \vec{\tilde{\pi}}\end{pmatrix} + V(x^I)\ .$$

Hamiltonian first order eqs lead to the following second order ODEs

$$\begin{aligned}\ddot{x}^i &= \{a\,\pi^i + b\,\tilde{\pi}^i, \pi_j\}\,\dot{x}^j + \{a\,\pi^i + b\,\tilde{\pi}^i, \tilde{\pi}_j\}\,\dot{\tilde{x}}^j - a\,\partial^i V - b\,\tilde{\partial}^i V\ ,\\ \ddot{\tilde{x}}^i &= \{b\,\pi^i + c\,\tilde{\pi}^i, \pi_j\}\,\dot{x}^j + \{b\,\pi^i + c\,\tilde{\pi}^i, \tilde{\pi}_j\}\,\dot{\tilde{x}}^j - b\,\partial^i V + c\,\tilde{\partial}^i V\ .\end{aligned}$$

If $a = 0$, $b = 2/m$, $\{\tilde{\pi}_i, \tilde{\pi}_j\} = 0$ and $\{\tilde{\pi}_i, \pi_j\} = (e/2)\,\varepsilon_{ijk}\,B^k(\vec{x})$, one obtains the contribution $(e/m)\,\dot{\vec{x}} \times \vec{B}$. Then, from $-b\,\vec{\tilde{\nabla}}V = \frac{e}{m}\,\vec{E}$, one has

$$V(x^I) = -\tfrac{e}{2}\left(\vec{\tilde{x}}\cdot\vec{E}(\vec{x}) + \nu(\vec{x})\right)\ .$$

Lorentz force law from symplectic realizations.

If we redefine $\vec{A}$ by adding an arbitrary smooth vector field $\vec{\alpha}(\vec{x}\,)$,

$$\vec{A}(x^I) = \vec{\alpha}(\vec{x}\,) - \tfrac{1}{2}\,\vec{\tilde{x}} \times \vec{B}(\vec{x}\,)\ ,$$

the Poisson bracket $\{\tilde{\pi}_i, \pi_j\}$ is unchanged, so is the corresponding eom. The resulting Hamiltonian is $O(3,3) \times O(3,3)$ symmetric,

$$H(x^I, \pi_I) = \tfrac{2}{m}\,\vec{\pi} \cdot \vec{\tilde{\pi}} - \tfrac{e}{2}\left(\vec{\tilde{x}} \cdot \vec{E}(\vec{x}\,) + \nu(\vec{x}\,)\right)\ .$$

The equations of motion are

$$\begin{aligned}
\ddot{x}_i &= \tfrac{e}{m}\,\varepsilon_{ijk}\,\dot{x}^j\,B^k(\vec{x}\,) + \tfrac{e}{m}\,E_i(\vec{x}\,)\ ,\\
\ddot{\tilde{x}}_i &= \tfrac{e}{m}\left[\partial_i\alpha_j(\vec{x}\,) - \partial_j\alpha_i(\vec{x}\,) + \left(\varepsilon_{ijk}\,\partial_l B^k(\vec{x}\,) - \varepsilon_{ijl}\,\partial_k B^k(\vec{x}\,)\right)\tilde{x}^l\right]\dot{x}^j\\
&\quad + \tfrac{e}{m}\,\varepsilon_{ijk}\,\dot{\tilde{x}}^j\,B^k(\vec{x}\,) + \tfrac{e}{m}\left(\tilde{x}^j\,\partial_i E_j(\vec{x}\,) + \partial_i\nu(\vec{x}\,)\right)\ .
\end{aligned}$$

The price to pay for the inclusion of generic magnetic $\vec{B}(\vec{x}\,)$ and electric $\vec{E}(\vec{x}\,)$ fields here is the presence of the auxiliary variables $\tilde{x}^i$.

Relation to dissipative systems

The motion of a charged particle in the field $\vec{B}_{\rm spher}(\vec{x}\,) = \frac{\rho}{3}\,\vec{x}$, is analogous to the motion in in the Dirac monopole field with an additional time-dependent friction in eom [Bakas, Lüst' 13].

Damped harmonic oscillator,

$$m\,\ddot{x} + \lambda\,\dot{x} + \omega^2\,x = 0\ .$$

The Lagrangian is: $L_{\rm dho} = \tilde{x}\left(m\,\ddot{x} + \lambda\,\dot{x} + \omega^2\,x\right)$, with $\tilde{x}$ being the Lagrange multiplier, physically representing the reservoir,

$$m\,\ddot{\tilde{x}} - \lambda\,\dot{\tilde{x}} + \omega^2\,\tilde{x} = 0.$$

The Hamiltonian is

$$H_{\rm dho} = \frac{1}{m}\,p\,\tilde{p} - \lambda(xp - \tilde{x}\tilde{p}) + \omega^2\,x\,\tilde{x}\ .$$

The auxiliary degrees of freedom are needed to conserve the total energy.
Quantization [Feshbach' 77].

Hamiltonian reduction

Here we investigate whether it is possible to eliminate the auxiliary degrees of freedom preserving the Lorentz force equation.

It is easy to see that $\vec{\phi} = \vec{\tilde{x}} \approx \vec{0}$, eliminates all propagating degrees of freedom. Indeed, $\dot{\phi}^i = \{\phi^i, H\} = \frac{2}{m}\pi^i \approx 0$, thus results in $\vec{\psi} = \vec{\pi} \approx \vec{0}$. All $\Phi^I = (\phi^i, \psi_i)$ are of first class, $\{\Phi^I, \Phi^J\} \approx 0$.

Starting from the symplectic realisation of the magnetic monopole algebra, can we find $\vec{\phi}(\vec{x}, \vec{\tilde{x}}\,) \approx \vec{0}$ and H, s.t. the reduced Hamiltonian dynamics reproduces the Lorentz force equation. Let us start with a generic form of $A_I(x^I) = (\vec{A}, \vec{\tilde{A}})$.

$$\vec{\phi} = \vec{\tilde{x}} - \zeta\,\vec{x} \approx \vec{0}\ .$$

Its conservation results in

$$\frac{\mathrm{d}\phi^i}{\mathrm{d}t} = \{\phi^i, H\} = (\zeta\, a - b)\,\pi^i + (\zeta\, b - c)\,\tilde{\pi}^i \approx 0\ .$$

If $\zeta = \frac{b}{a} = \frac{c}{b}$, one obtains the free motion, $\ddot{x}^i = \ddot{\tilde{x}}^i = 0$.

Assuming that $\zeta \neq \frac{c}{b}$, we obtain the secondary constraint

$$\vec{\psi} = \vec{\tilde{\pi}} - \gamma\,\vec{\pi} \approx \vec{0}\ , \qquad \gamma = -\frac{\zeta\, a - b}{\zeta\, b - c}\ .$$

Writing $H_{\rm tot} = H + \vec{u}\cdot\vec{\phi} + \vec{v}\cdot\vec{\psi}$, we may find u_i and v^i. Since $\{\psi_i, \phi^j\} = -(1+\zeta\,\gamma)\,\delta_i{}^j$, from $\{\phi^i, H_{\rm tot}\} \approx 0$ one finds $\vec{v} = \vec{0}$. Then from $\{\psi_i, H_{\rm tot}\} \approx 0$, one has

$$u_i = \frac{1}{1+\zeta\,\gamma}\left((a+\gamma\, b)\,\{\tilde{\pi}_i - \gamma\,\pi_i, \pi_j + \zeta\,\tilde{\pi}_j\}_\zeta\,\pi^j + (\tfrac{1}{\zeta} - \gamma)\,\partial_i V_\zeta\right)\ .$$

The constraint eom are

$$\dot{x}^i \approx \{x^i, H_{\rm tot}\} = (a+\gamma\, b)\,\pi^i\ ,$$

$$\dot{\pi}_i \approx \{\pi_i, H_{\rm tot}\} = \frac{1}{1+\zeta\,\gamma}\left((a+\gamma\, b)\,\{\pi_i + \zeta\,\tilde{\pi}_i, \pi_j + \zeta\,\tilde{\pi}_j\}_\zeta\,\pi^j - (2-\zeta\,\gamma\right.$$

implying,

$$\ddot{x}^i = \frac{a+\gamma\, b}{1+\zeta\,\gamma}\left(\{\pi^i + \zeta\,\tilde{\pi}^i, \pi_j + \zeta\,\tilde{\pi}_j\}_\zeta\,\dot{x}^j - (2-\zeta\,\gamma)\,\partial^i V_\zeta\right)\ .$$

Hamiltonian reduction

This is exactly the Lorentz force equation corresponding to the effective magnetic field

$$\begin{aligned} B^i_{\rm eff} &= \frac{m}{e}\,\frac{a+\gamma\,b}{1+\zeta\,\gamma}\,\varepsilon^{ijk}\left(\{\pi_j+\zeta\,\tilde\pi_j,\pi_k+\zeta\,\tilde\pi_k\}_\zeta\right) \\ &= \frac{m\,(\zeta+1)\,(a+\gamma\,b)}{1+\zeta\,\gamma}\,\vec\nabla\times\left(\vec A_\zeta+\zeta^2\,\vec{\tilde A}_\zeta\right)\ , \end{aligned}$$

where $\vec A_\zeta(\vec x\,):=\vec A(\vec x,\zeta\,\vec x\,)$, and the effective electric field

$$\vec E_{\rm eff}=-\frac{m}{e}\,\frac{(a+\gamma\,b)\,(2-\zeta\,\gamma)}{1+\zeta\,\gamma}\,\vec\nabla V_\zeta\ .$$

Since, $\vec\nabla\cdot\vec B_{\rm eff}=0$, it cannot be sourced by monopoles; $\vec\nabla\times\vec E_{\rm eff}=\vec 0$ and hence cannot be sourced by magnetic currents. Writing $\vec B_{\rm mag}=\vec B-\vec B_{\rm eff}$, we can decompose the original $\vec B$ as

$$\vec B=\vec B_{\rm mag}+\frac{m\,(\zeta+1)\,(a+\gamma\,b)}{1+\zeta\,\gamma}\,\vec\nabla\times\left(\vec A_\zeta+\zeta^2\,\vec{\tilde A}_\zeta\right)\ ,$$

where $\vec B_{\rm mag}$ with $\vec\nabla\cdot\vec B_{\rm mag}=\vec\nabla\cdot\vec B$, accounts for magnetic charge distributions, while $\vec B_{\rm eff}$ is created by electric currents and time-varying electric fields.

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$$\begin{aligned} B^i_{\text{eff}} &= \tfrac{m}{e}\,\tfrac{a+\gamma\,b}{1+\zeta\,\gamma}\,\varepsilon^{ijk}\left(\{\pi_j+\zeta\,\tilde{\pi}_j,\pi_k+\zeta\,\tilde{\pi}_k\}_\zeta\right) \\ &= \tfrac{m\,(\zeta+1)\,(a+\gamma\,b)}{1+\zeta\,\gamma}\,\vec{\nabla}\times\left(\vec{A}_\zeta+\zeta^2\,\vec{\tilde{A}}_\zeta\right)\ , \end{aligned}$$

where $\vec{A}_\zeta(\vec{x}\,):=\vec{A}(\vec{x},\zeta\,\vec{x}\,)$, and the effective electric field

$$\vec{E}_{\text{eff}}=-\tfrac{m}{e}\,\tfrac{(a+\gamma\,b)\,(2-\zeta\,\gamma)}{1+\zeta\,\gamma}\,\vec{\nabla}V_\zeta\ .$$

Since, $\vec{\nabla}\cdot\vec{B}_{\text{eff}}=0$, it cannot be sourced by monopoles; $\vec{\nabla}\times\vec{E}_{\text{eff}}=\vec{0}$ and hence cannot be sourced by magnetic currents. Writing $\vec{B}_{\text{mag}}=\vec{B}-\vec{B}_{\text{eff}}$, we can decompose the original $\vec{B}$ as

$$\vec{B}=\vec{B}_{\text{mag}}+\tfrac{m\,(\zeta+1)\,(a+\gamma\,b)}{1+\zeta\,\gamma}\,\vec{\nabla}\times\left(\vec{A}_\zeta+\zeta^2\,\vec{\tilde{A}}_\zeta\right)\ ,$$

where $\vec{B}_{\text{mag}}$ with $\vec{\nabla}\cdot\vec{B}_{\text{mag}}=\vec{\nabla}\cdot\vec{B}$, accounts for magnetic charge distributions, while $\vec{B}_{\text{eff}}$ is created by electric currents and time-varying electric fields.

Hamiltonian reduction

For specific choice of $\vec{A} = \vec{\alpha}(\vec{x}\,) - \frac{1}{2}\,\vec{\tilde{x}} \times \vec{B}(\vec{x}\,)$ and H, we find $(\zeta = 1)$,

$$\begin{aligned} \vec{B}_{\rm eff}(\vec{x}\,) &= \vec{\nabla} \times \left(\vec{\alpha}(\vec{x}\,) - \tfrac{1}{2}\,\vec{x} \times \vec{B}(\vec{x}\,)\right) , \\ \vec{E}_{\rm eff}(\vec{x}\,) &= \tfrac{1}{2}\,\vec{\nabla}\left(\,\vec{x} \cdot \vec{E}(\vec{x}\,) + \nu(\vec{x}\,)\right) . \end{aligned}$$

We can now ask for which original $\vec{B}$ and $\vec{E}$ do the constrained eom coincide with given Lorentz force eq., i.e., $\vec{B}_{\rm eff} = \vec{B}$ and $\vec{E}_{\rm eff} = \vec{E}$?

To answer we need to fix the ambiguity $\vec{\alpha}(\vec{x}\,)$ and $\nu(\vec{x}\,)$.

$$\vec{A}(x^I) = \vec{a}(\vec{x}\,) - \tfrac{1}{2}\,\vec{x} \times \vec{B}_{\rm mag}(\vec{x}\,) - \tfrac{1}{2}\left(\vec{\tilde{x}} - \vec{x}\,\right) \times \vec{B}(\vec{x}\,) ,$$

where here $\vec{\nabla} \times \vec{a} = \vec{B}_{\rm eff}$, and

$$V(x^I) = e\,\phi(\vec{x}\,) - \tfrac{e}{2}\,\vec{x} \cdot \vec{E}_{\rm mag}(\vec{x}\,) - \tfrac{e}{2}\left(\vec{\tilde{x}} - \vec{x}\,\right) \cdot \vec{E}(\vec{x}\,) ,$$

where $\vec{\nabla}\phi = \vec{E}_{\rm eff}$, and $\vec{E}_{\rm mag} = \vec{E} - \vec{E}_{\rm eff}$.

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where $\vec{\nabla}\phi = \vec{E}_{\rm eff}$, and $\vec{E}_{\rm mag} = \vec{E} - \vec{E}_{\rm eff}$.

Hamiltonian reduction

- Independently of the choice of $A_I(x^I)$ and $H(x^I, \pi_I)$, Hamiltonian constraints $\vec{\phi} = \vec{\tilde{x}} - \zeta\,\vec{x} \approx \vec{0}$, lead to the Lorentz force with the source-free magnetic $\vec{B}_{\rm eff}$ and electric $\vec{E}_{\rm eff}$ fields.
- If original $\vec{B}$ and $\vec{E}$ are source-free, the appropriate choice of $A_I(x^I)$ and $V(x^I)$ ensures that the constrained eom coincide with the original Lorentz force law.
- Only in this case the auxiliary variables can be eliminated in the consistent way.

Examples: For $\vec{B}_{\rm spher} = \frac{\rho_m}{3}\,\vec{x}$, $\vec{B}_{\rm eff} = 0$, and $\vec{B}_{\rm mag} = \vec{B}$, the vector potencial $\vec{A}(x^I) = -\frac{\rho_m}{6}\,\vec{\tilde{x}} \times \vec{x}$, $\vec{\tilde{A}} = 0$, and

$$H(x^I, p_I) = \frac{2}{m}\,\vec{\pi}\cdot\vec{\tilde{\pi}} = \frac{2}{m}\,\vec{p}\cdot\vec{\tilde{p}} + \frac{e\rho_m}{3m}\,\left(\vec{\tilde{x}}\times\vec{x}\right)\cdot\vec{\tilde{p}}\,.$$

For Dirac monopole

$$H(x^I, p_I) = \frac{2}{m}\,\vec{p}\cdot\vec{\tilde{p}} + \frac{eg}{m\,|\vec{x}\,|^3}\,\left(\vec{\tilde{x}}\times\vec{x}\right)\cdot\vec{\tilde{p}}\,.$$

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Integrability

For $\vec{B}_{\rm spher} = \frac{\rho_m}{3}\,\vec{x}$, and $\vec{E} = \vec{0}$ the Hamiltonian flow eqs. are:

$\dot{\vec{x}} = \frac{2}{m}\,\vec{\tilde{\pi}}$, $\dot{\vec{\tilde{x}}} = \frac{2}{m}\,\vec{\pi}$, $\dot{\vec{\pi}} = \frac{e\,\rho}{3m}\left(\vec{\pi}\times\vec{x} - 2\,\vec{\tilde{\pi}}\times\vec{\tilde{x}}\right)$ and $\dot{\vec{\tilde{\pi}}} = \frac{e\,\rho}{3m}\,\vec{\tilde{\pi}}\times\vec{x}$.

The known integrals of motion are:

$$I_1 = H = \frac{2}{m}\,\vec{\pi}\cdot\vec{\tilde{\pi}}, \qquad I_2 = \frac{2}{m}\,\vec{\tilde{\pi}}^{\,2}, \qquad I_3 = -\vec{L}^{\,2} = 4\,(\vec{x}\cdot\vec{\tilde{\pi}}\,)^2 - 4\,\vec{x}^{\,2}\,\vec{\tilde{\pi}}^{\,2}.$$

For the Dirac monopole one has I_1, I_2 and also the Poincre vector,

$$\vec{K} = \frac{2}{m}\,\vec{x}\times\vec{p} - \frac{eg}{m}\,\frac{\vec{x}}{|\vec{x}|}\ ,$$

in particular, by quantisation of angular momentum one finds the Dirac charge quantisation condition $e\,g = \frac{\hbar}{2}\,n$ with $n\in\mathbb{Z}$. The conservation of $\vec{K}$ ensures that the charged particle never reaches the location of the monopole.

Integrability

For $\vec{B}_{\rm spher} = \frac{\rho_m}{3}\,\vec{x}$, and $\vec{E} = \vec{0}$ the Hamiltonian flow eqs. are:

$\dot{\vec{x}} = \frac{2}{m}\,\vec{\tilde{\pi}}$, $\dot{\vec{\tilde{x}}} = \frac{2}{m}\,\vec{\pi}$, $\dot{\vec{\pi}} = \frac{e\,\rho}{3m}\left(\vec{\pi}\times\vec{x} - 2\,\vec{\tilde{\pi}}\times\vec{\tilde{x}}\right)$ and $\dot{\vec{\tilde{\pi}}} = \frac{e\,\rho}{3m}\,\vec{\tilde{\pi}}\times\vec{x}$.

The known integrals of motion are:

$$I_1 = H = \frac{2}{m}\,\vec{\pi}\cdot\vec{\tilde{\pi}}, \qquad I_2 = \frac{2}{m}\,\vec{\tilde{\pi}}^{\,2}, \qquad I_3 = -\vec{L}^{\,2} = 4\,(\vec{x}\cdot\vec{\tilde{\pi}}\,)^2 - 4\,\vec{x}^{\,2}\,\vec{\tilde{\pi}}^{\,2}.$$

For the Dirac monopole one has I_1, I_2 and also the Poincre vector,

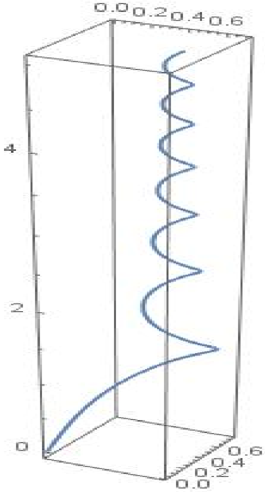
$$\vec{K} = \frac{2}{m}\,\vec{x}\times\vec{p} - \frac{eg}{m}\,\frac{\vec{x}}{|\vec{x}|}\ ,$$

in particular, by quantisation of angular momentum one finds the Dirac charge quantisation condition $e\,g = \frac{\hbar}{2}\,n$ with $n\in\mathbb{Z}$. The conservation of $\vec{K}$ ensures that the charged particle never reaches the location of the monopole.

Motion of the charged particle in the axial field, $\vec{B} = (0, 0, \rho z)$,

$$\dot{\tilde{\pi}}_x = \omega\, z\, \tilde{\pi}_y \ , \qquad \dot{\tilde{\pi}}_y = -\omega\, z\, \tilde{\pi}_x \qquad \text{and} \qquad \dot{\tilde{\pi}}_z = 0 \ ,$$

where $\omega = e\,\rho/m$ and we assume here that $e\,\rho > 0$.



We represent $\widehat{x}^I$ as multiplication, $\big(\widehat{x}^I\Psi\big)(x) = x^I\,\Psi(x)$ with $x^I = (x^i, \tilde{x}^i)$, and $\widehat{p}_I$ as differentiation, $\big(\widehat{p}_I\Psi\big)(x) = -\,\mathrm{i}\,\hbar\,\partial_I\Psi(x)$ with $\partial_I = (\partial_i, \tilde{\partial}_i)$. Then,

$$\widehat{H} = \frac{1}{m}\left(\widehat{\pi}_i\,\widehat{\tilde{\pi}}^i + \widehat{\tilde{\pi}}_i\,\widehat{\pi}^i\right) = \frac{2}{m}\left(-\,\mathrm{i}\,\hbar\,\vec{\nabla} - e\,\vec{A}\right)\cdot\left(-\,\mathrm{i}\,\hbar\,\vec{\tilde{\nabla}} - e\,\vec{\tilde{A}}\right)\,.$$

We may impose the constraints,

$$\widehat{\phi}^i\Psi_{\rm phys} = \big(\widehat{\tilde{x}}^i - \widehat{x}^i\big)\Psi_{\rm phys} = 0; \qquad \widehat{\psi}_i\Psi_{\rm phys} = \big(\widehat{\tilde{\pi}}_i - \widehat{\pi}_i\big)\Psi_{\rm phys} = 0\,.$$

If $\vec{B} = \vec{\nabla}\times\vec{a}$ everywhere, then $A_I(x^I)$ can be defined as

$$\vec{A}(x^I) = \tfrac{1}{2}\left(\vec{a}(\vec{x}) - (\vec{x}\cdot\vec{\nabla})\vec{a}(\vec{x}) - \vec{\tilde{x}}\times\vec{B}(\vec{x})\right) \qquad \text{and} \qquad \vec{\tilde{A}}(x^I) = \vec{0}\,.$$

Solving the constraints we come to the effective theory,

$$\widehat{H}_{\rm eff} = \frac{1}{2m}\,\widehat{\pi}_i\,\widehat{\pi}^i\,, \qquad \left[\widehat{\pi}_i, \widehat{\pi}_j\right] = \mathrm{i}\,\hbar\,e\,\varepsilon_{ijk}\,B^k_{\rm eff} = \mathrm{i}\,\hbar\,(\partial_i a_j - \partial_j a_i)\,.$$

We represent $\widehat{x}^I$ as multiplication, $\big(\widehat{x}^I\Psi\big)(x) = x^I\,\Psi(x)$ with $x^I = (x^i, \tilde{x}^i)$, and $\widehat{p}_I$ as differentiation, $\big(\widehat{p}_I\Psi\big)(x) = -\,\mathrm{i}\,\hbar\,\partial_I\Psi(x)$ with $\partial_I = (\partial_i, \tilde{\partial}_i)$. Then,

$$\widehat{H} = \frac{1}{m}\left(\widehat{\pi}_i\,\widehat{\tilde{\pi}}^i + \widehat{\tilde{\pi}}_i\,\widehat{\pi}^i\right) = \frac{2}{m}\left(-\,\mathrm{i}\,\hbar\,\vec{\nabla} - e\,\vec{A}\right)\cdot\left(-\,\mathrm{i}\,\hbar\,\vec{\tilde{\nabla}} - e\,\vec{\tilde{A}}\right)\,.$$

We may impose the constraints,

$$\widehat{\phi}^{\,i}\Psi_{\rm phys} = \left(\widehat{\tilde{x}}^{\,i} - \widehat{x}^{\,i}\right)\Psi_{\rm phys} = 0; \qquad \widehat{\psi}_i\Psi_{\rm phys} = \left(\widehat{\tilde{\pi}}_i - \widehat{\pi}_i\right)\Psi_{\rm phys} = 0\,.$$

If $\vec{B} = \vec{\nabla}\times\vec{a}$ everywhere, then $A_I(x^I)$ can be defined as

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Solving the constraints we come to the effective theory,

$$\widehat{H}_{\rm eff} = \frac{1}{2m}\,\widehat{\pi}_i\,\widehat{\pi}^{\,i}\,, \qquad \left[\widehat{\pi}_i, \widehat{\pi}_j\right] = \mathrm{i}\,\hbar\,e\,\varepsilon_{ijk}\,B^k_{\rm eff} = \mathrm{i}\,\hbar\,(\partial_i a_j - \partial_j a_i)\,.$$

It is convenient to represent

$$\widehat{H} = \widehat{H}_+ - \widehat{H}_- := \frac{1}{m}\,\widehat{\pi}_{i\,+}\,\widehat{\pi}^{\,i}_{+} - \frac{1}{m}\,\widehat{\pi}_{i\,-}\,\widehat{\pi}^{\,i}_{-} \ ,$$

where, $\widehat{\pi}_{i\,\pm} = \frac{1}{\sqrt{2}}\left(\widehat{\pi}_i \pm \widehat{\tilde{\pi}}_i\right)$.

For $\vec{B} = (0,0,B)$, we have, $\vec{A}(x^I) = \frac{B}{2}\,(-\tilde{y},\tilde{x},0)$, and $\vec{\tilde{A}}(x^I) = \vec{0}$. Consequently, $\left[\widehat{\pi}_{x\,\pm},\widehat{\pi}_{y\,\pm}\right] = \pm\,\mathrm{i}\,\hbar\,e\,B$. We introduce,

$$\widehat{a}_{\pm} = \frac{1}{\sqrt{2\,\hbar\,e\,B}}\left(\widehat{\pi}_{x\,\pm} + \mathrm{i}\,\widehat{\pi}_{y\,\pm}\right) \qquad \text{and} \qquad \widehat{a}^{\dagger}_{\pm} = \frac{1}{\sqrt{2\,\hbar\,e\,B}}\left(\widehat{\pi}_{x\,\pm} - \mathrm{i}\,\widehat{\pi}_{y\,\pm}\right) \ ,$$

satisfying, $\left[\widehat{a}_{\pm},\widehat{a}^{\dagger}_{\pm}\right] = 1$. Then the Hamiltonian becomes

$$\widehat{H} = \widehat{H}_+ - \widehat{H}_- := \hbar\,\omega_{\rm cyc}\left(\widehat{a}^{\dagger}_{+}\,\widehat{a}_{+} - \widehat{a}^{\dagger}_{-}\,\widehat{a}_{-} - 1\right) \ ,$$

where $\omega_{\rm cyc} = \frac{e\,B}{m}$. The eigenvalues are $\mathcal{E}_{n_+,n_-} = \hbar\,\omega_{\rm cyc}\,(n_+ - n_- - 1)$ with corresponding eigenstates $|n_+,n_-\rangle$. To recover the standard results, the $\widehat{a}_-$-oscillator must be kept in its ground state for which $\widehat{a}_-|n_+,0\rangle = 0$.

It is convenient to represent

$$\widehat{H}=\widehat{H}_{+}-\widehat{H}_{-}:=\frac{1}{m}\,\widehat{\pi}_{i\,+}\,\widehat{\pi}^{\,i}_{+}-\frac{1}{m}\,\widehat{\pi}_{i\,-}\,\widehat{\pi}^{\,i}_{-}\ ,$$

where, $\widehat{\pi}_{i\,\pm}=\frac{1}{\sqrt{2}}\left(\widehat{\pi}_{i}\pm\widehat{\tilde{\pi}}_{i}\right)$.

For $\vec{B}=(0,0,B)$, we have, $\vec{A}(x^{I})=\frac{B}{2}\,(-\tilde{y},\tilde{x},0)$, and $\vec{\tilde{A}}(x^{I})=\vec{0}$.
Consequently, $\left[\widehat{\pi}_{x\,\pm},\widehat{\pi}_{y\,\pm}\right]=\pm\,\mathrm{i}\,\hbar\,e\,B$. We introduce,

$$\widehat{a}_{\pm}=\frac{1}{\sqrt{2\,\hbar\,e\,B}}\left(\widehat{\pi}_{x\,\pm}+\mathrm{i}\,\widehat{\pi}_{y\,\pm}\right)\qquad\text{and}\qquad\widehat{a}^{\dagger}_{\pm}=\frac{1}{\sqrt{2\,\hbar\,e\,B}}\left(\widehat{\pi}_{x\,\pm}-\mathrm{i}\,\widehat{\pi}_{y\,\pm}\right)\ ,$$

satisfying, $\left[\widehat{a}_{\pm},\widehat{a}^{\dagger}_{\pm}\right]=1$. Then the Hamiltonian becomes

$$\widehat{H}=\widehat{H}_{+}-\widehat{H}_{-}:=\hbar\,\omega_{\mathrm{cyc}}\left(\widehat{a}^{\dagger}_{+}\,\widehat{a}_{+}-\widehat{a}^{\dagger}_{-}\,\widehat{a}_{-}-1\right)\ ,$$

where $\omega_{\mathrm{cyc}}=\frac{e\,B}{m}$. The eigenvalues are $\mathcal{E}_{n_{+},n_{-}}=\hbar\,\omega_{\mathrm{cyc}}\,(n_{+}-n_{-}-1)$ with corresponding eigenstates $|n_{+},n_{-}\rangle$. To recover the standard results, the $\widehat{a}_{-}$-oscillator must be kept in its ground state for which $\widehat{a}_{-}|n_{+},0\rangle=0$.

Axial magnetic fields

For the magnetic field, $\vec{B} = (0, 0, \rho z)$, the corresponding vector potential reads,

$$\vec{A}_{\rm axial}(x^I) = \tfrac{\rho}{4}\, z\, (y - 2\tilde{y}, 2\tilde{x} - x, 0) \qquad \text{and} \qquad \vec{\tilde{A}}_{\rm axial}(x^I) = \vec{0}\ ,$$

and the Hamiltonian is

$$\widehat{\mathcal{H}} = \tfrac{1}{m}\left(\widehat{\pi}^2_{x\,+} + \widehat{\pi}^2_{y\,+}\right) - \tfrac{1}{m}\left(\widehat{\pi}^2_{x\,-} + \widehat{\pi}^2_{y\,-}\right) + \tfrac{2}{m}\,\widehat{p}_z\,\widehat{\tilde{p}}_z\ ,$$

with

$$\begin{aligned} \left[\widehat{\pi}_{x\,\pm}, \widehat{\pi}_{y\,\pm}\right] &= \pm\,\tfrac{\mathrm{i}\,\hbar}{4}\, e\,\rho\,\widehat{z}\ , \\ \left[\widehat{\pi}_{x\,\pm}, \widehat{p}_z\right] &= \pm\,\tfrac{\mathrm{i}\,\hbar}{4\sqrt{2}}\, e\,\rho\left(\widehat{y} - 2\,\widehat{\tilde{y}}\right)\ , \\ \left[\widehat{\pi}_{y\,\pm}, \widehat{p}_z\right] &= \pm\,\tfrac{\mathrm{i}\,\hbar}{4\sqrt{2}}\, e\,\rho\left(2\,\widehat{\tilde{x}} - \widehat{x}\right)\ . \end{aligned}$$

The energy spectrum is continuous.

Discussion

- By introducing the auxiliary degrees of freedom we construct a symplectic realization of the monopole algebra, $[\pi_i, \pi_j] = \mathrm{i}\, e\varepsilon_{ijk}B^k(x)$.
- The letter is used to construct the Hamiltonian description of a charged particle interacting with distribution of magnetic monopoles, reproducing the Lorentz force law for x.
- If $\nabla \cdot B = 0$, one can eliminate the auxiliary variables $\tilde{x}$ by imposing the Hamiltonian constraint $\phi = \tilde{x} - x = 0$, and thus obtain the standard Hamiltonian formulation.
- In the presence of the magnetic monopoles, i.e., $\nabla \cdot B \neq 0$, the auxiliary degrees of freedom are necessary for the consistent Hamiltonian description.
- This is analogous to the locally "non-geometric" backgrounds in string theory, wherein there are no local expressions for the geometry and the background fields require the extended space of double field theory for their proper definition.