Generalized Geometry and Gravity

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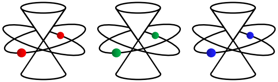
Dualities and Generalized Geometries
Corfu, September 2018

JACOBS UNIVERSITY

cost Action MP 1405
Quantum Structure of Spacetime

Outline

- Generalized geometry (briefly)
- Interaction via deformation
- Geometric ladder to higher/generalized geometry
- Generalized geometry and gravity



Generalized Geometry

Idea: consider extensions of the tangent bundle (typically doubled); unify symplectic, complex and Riemannian geometry; string symmetries

Courant algebroid

vector bundle $E \xrightarrow{\pi} M$, anchor $h \in \mathrm{Hom}(E, TM)$,
$\mathbb{R}$-bilinear bracket $[-,-]$, and fiber-wise metric $\langle -,- \rangle$,
s.t. for $e, e', e'' \in \Gamma E$:

$$[e,[e',e'']] = [[e,e'],e''] + [e',[e,e'']] \tag{1}$$

$$2\langle [e,e'],e' \rangle \overset{(2a)}{=} h(e)\langle e',e' \rangle \overset{(2b)}{=} 2\langle [e',e'],e \rangle \tag{2}$$

Consequences:

$$[e,fe'] = h(e).f\, e' + f[e,e'] \tag{3}$$

$$h([e,e']) = [h(e),h(e')]_{\mathrm{Lie}} \tag{4}$$

Remarks: (2a+b) can be polarized
(1) and (3) are the axioms of a Leibniz algebroid

Generalized Geometry

Example: Standard Courant algebroid

Treat vector fields and forms on equal footing:

$$0 \to T^*M \xrightarrow{j} E \xrightarrow{h} TM \to 0$$

with j the natural embedding and h the natural projection.

$$E = TM \oplus T^*M \text{ "generalized tangent bundle"}$$
$$V = X + \xi = X^i(x)\partial_i + \xi_i(x)dx^i \in \Gamma E$$

With the Dorfman bracket

$$[X+\xi, Y+\eta]_D = [X,Y] + \mathcal{L}_X\eta - \iota_Y d\xi \quad (+\text{twisting/flux terms}),$$

the natural pairing $\langle -, - \rangle$ of TM and T^*M and the projection $h : E \to TM$ (anchor) we obtain a Courant algebroid.

Symmetries: diffeomorphisms, B-transform, θ-transform

Generalized Metric

The pairing $\langle -, - \rangle$ has signature (n, n). An idempotent self-adjoint homomorphism τ can turn it into a positive definite generalized metric

$$\mathbb{G}(V, W) := \langle \tau(V), W \rangle \qquad (\mathbb{G}_{\alpha\beta}) = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}$$

The homomorphism τ can be defined in terms of its eigenbundles

$$E_{\pm} = \{(X, (\pm g + B)(X)) \,|\, X \in \Gamma TM\} \qquad \tau(E_{\pm}) = \pm E_{\pm}$$

via θ-transform: closed-open string relations, NC gauge theory etc.

$$\frac{1}{g + B} = \frac{1}{G + \Phi} + \theta$$

Generalized Geometry and (super)gravity

established approach: choose Courant algebroid and follow the scheme

Generalized metric $\to$ Bismut connection $\to$ set torsion zero (add further conditions as needed) $\to$ curvature $\to$ equations of motion $\leftrightarrow$ action

- the good: very advanced, available for all flavors of SUGRA
- the bad: problems with covariance, ambiguities
- the ugly: assumptions, fine-tuning, reverse-engineering
- the players: Coimbra, Minasian, Strickland-Constable, Triendl, Waldram, Blumenhagen, Deser, Plauschinn, Rennecke, Garcia-Fernandez, Grana, Jurco, Vysoky + many more

DFT approach: Hull, Hohm, Zwiebach + many more

new alternative approach: graded geometry, deformation (this talk)

Interaction via deformation

Electrodynamics as deformed quantum mechanics

$\vec{B} = \nabla \times \vec{A}$ implies $\nabla \cdot B = 0$, hence we cannot work with canonical momenta and covariant derivatives in the presence of magnetic sources.

alternative: deformed canonical commutation relations

$$[x^i, x^j]' = 0 \ , \ [p_i, x^j]' = \tfrac{\hbar}{i}\delta^i_j \ , \ [p_i, p_j]' = i\hbar e F_{ij} \quad \text{(where } F_{ij} = \epsilon_{ijk} B_k\text{)}$$

Let $\mathbf{p} = \sigma^i p_i$ and $H = \dfrac{\mathbf{p}^2}{2m} \quad \Rightarrow \quad$ Pauli Hamiltonian:

$$H = \frac{1}{2m}\left(\frac{1}{4}[\sigma^i, \sigma^j]_+ [p_i, p_j]'_+ + \frac{1}{4}[\sigma^i, \sigma^j][p_i, p_j]'\right) = \frac{\vec{p}^{\,2}}{2m} - \frac{\hbar e}{2m}\vec{\sigma} \cdot \vec{B}$$

Lorentz-Heisenberg equations of motion:

$$\frac{d\vec{r}}{dt} = \frac{i}{\hbar}[H, \vec{r}\,]' = \frac{\vec{p}}{m} \ , \quad \frac{d\vec{p}}{dt} = \frac{i}{\hbar}[H, \vec{p}\,]' = \frac{e}{2m}\left(\vec{p} \times \vec{B} - \vec{B} \times \vec{p}\right)$$

this formalism allows $\nabla \cdot B \neq 0$: magnetic sources, non-associativity

Interaction via deformation

. . . relativistically (with appropriate mass shell constraint implementation)

$$[p_\mu, x^\nu] = \tfrac{\hbar}{i}\delta_\mu^\nu \qquad [p_\mu, p_\nu] = i\hbar e F_{\mu\nu}(x) \qquad [\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}\,.$$

Gravitational interaction via deformation of the γ-algebra

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}(x)$$

gives an algebraic approach to the geodesic equation, connections, curvature, etc. Properties like metricity follow from associativity. Local inertial coordinates are reinterpreted as Darboux charts.

classical $\leftrightarrow$ quantum correspondence:

$$\begin{array}{rcl} \theta^\mu & \leftrightarrow & \gamma^\mu \\ 2\theta^\mu\theta^\nu & \leftrightarrow & [\gamma^\mu, \gamma^\nu]_- \\ \{\theta^\mu, \theta^\nu\} & \leftrightarrow & [\gamma^\mu, \gamma^\nu]_+ \end{array}$$

Interaction via deformation

Graded Poisson algebra

$$\{\underset{a}{\theta^\mu}, \underset{a}{\theta^\nu}\} = 2\underset{0}{g^{\mu\nu}}(x) \qquad \{\underset{c}{p_\mu}, \underset{0}{x^\nu}\} = \underset{0}{\delta^\nu_\mu} \qquad \{p_\mu, f(x)\} = \partial_\mu f(x)$$

Since $g^{\mu\nu}(x)$ has degree 0, the Poisson bracket must have degree $b = -2a$ for θ^μ of degree a, i.e. it is an even bracket.

Since $g^{\mu\nu}(x)$ is symmetric, we must have $-(-1)^{b+a^2} \overset{!}{=} +1$, i.e. a is odd.

wlog: $\{,\}$ is of degree $b = -2$, θ^μ are Grassmann variables of degree 1, $\theta^\mu\theta^\nu = -\theta^\nu\theta^\mu$, and the momenta p_μ have degree $c = -b = 2$

$\Leftrightarrow$ a metric structur on TM and natural symplectic structure on T^*M, shifted in degree and combined into a graded Poisson structure on

$$\underset{p_\mu}{T^*[2]} \oplus \underset{\theta^\mu}{T[1]}\,\underset{x^\mu}{M}$$

Interaction via deformation

Graded Poisson algebra on $T^*[2] \oplus T[1]M$

$$\{\underset{1}{\theta}^{\mu}, \underset{1}{\theta}^{\nu}\} = 2\underset{0}{g}^{\mu\nu}(x) \qquad \{\underset{2}{p}_{\mu}, \underset{0}{x}^{\nu}\} = \underset{0}{\delta}{}^{\nu}_{\mu} \qquad \{p_\mu, f(x)\} = \partial_\mu f(x)$$

associativity/Jacobi identity $\Leftrightarrow$ metric connection

$$\{p_\mu, \{\theta^\alpha, \theta^\beta\}\} = 2\partial_\mu g^{\alpha\beta} = \{\{p_\mu, \theta^\alpha\}, \theta^\beta\} + \{\theta^\alpha, \{p_\mu, \theta^\beta\}\}$$

$$\{\underset{2}{p}_{\mu}, \underset{1}{\theta}^{\alpha}\} = \nabla_\mu \theta^\alpha = \Gamma^{\alpha}_{\mu\beta} \underset{1}{\theta}^{\beta}$$

and curvature

$$\{\{p_\mu, p_\nu\}, \theta^\alpha\} = [\nabla_\mu, \nabla_\nu]\theta^\alpha = \theta^\beta R_\beta{}^\alpha{}_{\mu\nu}$$

$$\Rightarrow \quad \{\underset{2}{p}_{\mu}, \underset{2}{p}_{\nu}\} = \tfrac{1}{4} \underset{1}{\theta}^{\beta} \underset{1}{\theta}^{\alpha} R_{\beta\alpha\mu\nu}$$

Interaction via deformation

Equations of motion

$$\frac{dx^\mu}{d\tau} = \{\tfrac{1}{2} g^{\alpha\beta} p_\alpha p_\beta, x^\mu\} = g^{\mu\nu} p_\nu$$

$$\frac{dp_\nu}{d\tau} = \{\tfrac{1}{2} g^{\alpha\beta} p_\alpha p_\beta, p_\nu\} = \tfrac{1}{2}(\partial_\mu g^{\alpha\beta}) p_\alpha p_\beta = {}^{g}\Gamma_\mu{}^{\alpha\beta} p_\alpha p_\beta$$

with *any* metric-compatible connection ${}^{g}\Gamma$.

Geodesic equation:

$$\frac{d^2 x^\mu}{d\tau^2} = \{\tfrac{1}{2} g^{\alpha\beta} p_\alpha p_\beta, g^{\mu\nu} p_\nu\} = -\frac{dx^\alpha}{d\tau} {}^{LC}\Gamma_{\alpha\beta}{}^\mu \frac{dx^\beta}{d\tau}$$

...climbing up the geometric ladder

Step 0: Poisson manifold T^*M

- degree 0: x^i "coordinates", p_i "momenta"

symplectic 2-form: $\omega = dp_i \wedge dx^i$

even (degree 0) Poisson bracket

$$\{x^i, x^j\} = 0, \quad \{p_i, x^j\} = \delta_i^j, \quad \{p_i, p_j\} = 0$$

generalization, deformation, quantization

$$\{f, g\}_\theta = \theta^{ij}(x)\partial_i f\, \partial_j g \quad \rightsquigarrow \quad f \star g = fg + \frac{\hbar}{2}\{f, g\} + \dots$$

Step 1: Super Poisson manifold $T^*[1]M$

(M itself is assumed to be Poisson here)

- degree 0: x^i "coordinates"
- degree 1: ξ_i "momenta", $\xi_i\xi_j = -\xi_j\xi_i$

symplectic 2-form: $\omega = d\xi_i \wedge dx^i$

odd (degree -1) Poisson bracket defined on functions $f(x,\xi)$

$$\{x^i, x^j\} = 0, \quad \{\xi_i, x^j\} = \delta_i^j, \quad \{\xi_i, \xi_j\} = 0$$

$=$ Schouten bracket $[\,,\,]_S$ of vector fields $v = v^i(x)\xi_i$: $\{v, w\} = [v, w]_{\text{Lie}}$

degree 2 "Hamiltonian" (Poisson bi-vector): $\theta = \frac{1}{2}\theta^{ij}(x)\xi_i\xi_j \quad \{\theta, \theta\} = 0$

derived bracket: $\{\{f, \theta\}, g\} = \theta^{ij}(x)\partial_i f\, \partial_j g$

i.e. we recover the previous bracket (with a few more bells and whistles)

<u>$n = 1$ (open string)</u>

Poisson sigma model

2-dimensional topological field theory, $E = T^*M$

$$S^{(1)}_{\mathrm{AKSZ}} = \int_{\Sigma_2} \Big(\xi_i \wedge \mathrm{d}X^i + \frac{1}{2}\,\theta^{ij}(X)\,\xi_i \wedge \xi_j \Big) \ ,$$

with $\theta = \frac{1}{2}\,\theta^{ij}(x)\,\partial_i \wedge \partial_j$, $\xi = (\xi_i) \in \Omega^1(\Sigma_2, X^*T^*M)$

perturbative expansion $\Rightarrow$ star product $\star$, Kontsevich formality

(valid on-shell ($[\theta,\theta]_S = 0$) as well as off-shell, e.g. twisted Poisson)

Kontsevich (1997), Cattaneo, Felder (2000)

AKSZ construction: action functionals in BV formalism of sigma model QFT's in $n+1$ dimensions for symplectic Lie n-algebroids E

Alexandrov, Kontsevich, Schwarz, Zaboronsky (1995/97)

Geometric ladder to generalized geometry

Step 2: Graded Poisson manifold $T^*[2]T[1]M$

- degree 0: x^i "coordinates"
- degree 1: $\xi^\alpha = (\theta^i, \chi_i)$
- degree 2: p_i "momenta"

symplectic 2-form

$$\omega = dp_i \wedge dx^i + \frac{1}{2} G_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta = dp_i \wedge dx^i + d\chi_i \wedge d\theta^i$$

even (degree -2) Poisson bracket on functions $f(x, \xi, p)$

$$\{x^i, x^j\} = 0, \quad \{p_i, x^j\} = \delta_i^j, \quad \{\xi^\alpha, \xi^\beta\} = G^{\alpha\beta}$$

metric $G^{\alpha\beta}$: natural pairing of TM, T^*M:

$$\{\chi_i, \theta^j\} = \delta_i^j \ , \quad \{\chi_i, \chi_j\} = 0 \ , \quad \{\theta^i, \theta^j\} = 0$$

Geometric ladder to generalized geometry

Generalized geometry as a derived structure

degree 3 "Hamiltonian":

$$\Theta = \xi^\alpha h^i_\alpha(x) p_i \quad (+\text{twisting/flux terms})$$

For $e = e_\alpha(x)\xi^\alpha$ (degree 1, odd):

- pairing: $\langle e, e' \rangle = \{e, e'\}$
- anchor: $h(e)f = \{\{e, \Theta\}, f\}$
- bracket: $[e, e']_D = \{\{e, \Theta\}, e'\}$

$$\{\Theta, \Theta\} = 0 \qquad \Leftrightarrow \qquad \text{Courant algebroid axioms}$$

Geometric ladder to generalized geometry

$n = 2$ (open membrane)

Courant sigma model

standard Courant algebroid $C = TM \oplus T^*M$
TFT with 3-dimensional membrane world volume Σ_3

$$S^{(2)}_{\mathrm{AKSZ}} = \int_{\Sigma_3} \Big(\phi_i \wedge \mathrm{d}X^i + \tfrac{1}{2}\, G_{IJ}\, \alpha^I \wedge \mathrm{d}\alpha^J - h_I{}^i(X)\, \phi_i \wedge \alpha^I \\ + \tfrac{1}{6}\, C_{IJK}(X)\, \alpha^I \wedge \alpha^J \wedge \alpha^K\Big)$$

embedding maps $X : \Sigma_3 \to M$, 1-form α, aux. 2-form ϕ, fiber metric G, anchor h, 3-form C (e.g. H-flux, f-flux, Q-flux, R-flux).

Geometric ladder

hierarchie of actions, brackets, extended objects and algebras

AKSZ-model:	Poisson-sigma (open string) $T^*[1]M$	Courant-sigma (open membrane) $T^*[2]T[1]M$	...
derived bracket:	Poisson T^*M	Dorfman $TM \oplus T^*M$	...
			
object:	point particle	closed string	...
algebraic structure:	non-commutative	non-associative	...

Generalized Geometry and Gravity

setup: $T^*[2]T[1]M$ ("step 2") even bracket with odd variables
deformation by a non-symmetric metric $\mathcal{G} = j \circ (g + B) \circ h$

$$\{\chi_i, \chi_j\} = 0 \quad \to \quad \{\chi_i, \chi_j\}' = 2g_{ij}(x)$$

$\Rightarrow$ for $X = X^i(x)\chi_i$ and $v = v^i(x)p_i$, the Poisson structure implies

$$\{v, X\}' = \nabla^{\mathcal{G}}_v X \ , \quad \{v, v'\}' = [v, v']_{\mathrm{Lie}} + R(v, v')$$

$\{\Theta, \Theta\} = 0 \Leftrightarrow R(v, v') = 0$ (no curvature!) Weitzenböck connection

$$\nabla^{\mathcal{G}}_i \chi_j = -(\partial_i \mathcal{G}_{jl})\, \theta^l$$

the derived bracket involves the Levi-Civita connection ∇^{LC} (no torsion!)

$$[X, Y]' = [X, Y]_D + 2g(\nabla^{\mathrm{LC}} X, Y) + H(-, X, Y)$$

plus skew symmetric torsion $H = dB$.

Khoo, Boffo; PS

Generalized Geometry and Gravity

generalized Koszul formula for nonsymmetric $\mathcal{G} = g + B$

$$\begin{aligned} 2g(\nabla_Z X, Y) &= \langle Z, [X, Y]' \rangle' \\ &= X\mathcal{G}(Y, Z) - Y\mathcal{G}(X, Z) + Z\mathcal{G}(X, Y) \\ &\quad -\mathcal{G}(Y, [X, Z]_{\mathrm{Lie}}) - \mathcal{G}([X, Y]_{\mathrm{Lie}}, Z) + \mathcal{G}(X, [Y, Z]_{\mathrm{Lie}}) \\ &= 2g(\nabla_X^{LC} Y, Z) + H(X, Y, Z) \end{aligned}$$

$\Rightarrow$ non-symmetric Ricci tensor

$$R_{jl} = R_{jl}^{LC} - \frac{1}{2}\nabla_i^{LC} H_{jl}{}^{i} - \frac{1}{4} H_{lm}{}^{i} H_{ij}{}^{m} \qquad R = \mathcal{G}_{ij} g^{ik} g^{jl} R_{kl}$$

$\Rightarrow$ gravity action (closed string effective action) after partial integration:

$$S_{\mathcal{G}} = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g} \left(R^{LC} - \frac{1}{12} H_{ijk} H^{ijk} \right)$$

Khoo, Vysoky, Jurco, Boffo; PS

Generalized Geometry and Gravity

This formulation consistently combines all approaches of Einstein: Non-symmetric metric, Weitzenböck and Levi-Civita connections, without any of the usual drawbacks.

The dilaton $\phi(x)$ rescales the generalized tangent bundle. The deformation can be formulated in terms of vielbeins

$$E = e^{-\frac{\phi}{3}} \begin{pmatrix} 1 & 0 \\ g + B & 1 \end{pmatrix} \qquad E^{-1}\partial_i E = \begin{pmatrix} -\frac{1}{3}\partial_i\phi & 0 \\ \partial_i(g+B) & -\frac{1}{3}\partial_i\phi \end{pmatrix}$$

Going through the same steps as before we find in $d = 10$

$$S = \frac{1}{2\kappa}\int d^{10}x\, e^{-2\phi}\sqrt{-g}\Big(R^{\mathrm{LC}} - \tfrac{1}{12}H^2 + 4(\nabla\phi)^2\Big)$$

Boffo, PS

Quantization and the dilaton

$x^i, p_i, \theta^i, \chi_i \quad \rightsquigarrow$ differential ops on $\psi(x,\theta) \in \Lambda^\bullet T^*M$ (spinors):

$$p \rightsquigarrow \partial_x \qquad \chi \rightsquigarrow \partial_\theta = \mathbf{i}_\chi \qquad x \rightsquigarrow x\,\cdot \qquad \theta \rightsquigarrow \theta\wedge$$

θ, χ: finite dimensional representation by γ-matrices:

$$V \rightsquigarrow \gamma_V = V^\alpha(x)\gamma_\alpha\ , \qquad [\gamma_V, \gamma_W]_+ = G(V,W) \text{ etc.}$$

Symmetry Lie algebra generators: $M^\alpha{}_\beta \xi_\alpha \tilde{\xi}^\beta$
$M^i{}_j$ picks up tr$M$ "anomaly" after quantization

$$\Lambda^\bullet T^*M \rightsquigarrow \Lambda^\bullet T^*M \otimes \det^{\frac{1}{2}} TM$$

requiring the introduction of the dilaton field ϕ for covariance.

Generalized Geometry and Gravity

Fully deformed Poisson structure on $T^*[2]T[1]M$

$$\{v, f\} = v.f$$
$$\{V, W\} = G(V, W) \equiv \langle V, W \rangle$$
$$\{v, V\} = \nabla_v V \qquad \leftarrow \text{connection metric wrt. } G$$
$$\{v, w\} = [v.w]_{\text{Lie}} + R(v, w) \qquad \leftarrow \text{curvature of } \nabla$$

with

- degree 0: $f(x)$
- degree 1: $V = V^\alpha(x)\xi_\alpha$ "generalized vectors" $\in \Gamma(TM \oplus T^*M)$
- degree 2: $v = v^i(x)p_i$ "vector fields" $\in \Gamma(TM)$

general Hamiltonian

$$\Theta = \tilde{\xi}^\alpha h(\xi_\alpha) + \frac{1}{6} C_{\alpha\beta\gamma}\tilde{\xi}^\alpha\tilde{\xi}^\beta\tilde{\xi}^\gamma \qquad \leftarrow \text{general flux (H,f,Q,R)}$$

derived bracket

$$\{\{\{V,\Theta\},W\},X\} = \langle \nabla_V W, X\rangle - \langle \nabla_W V, X\rangle + \langle \nabla_X V, W\rangle + C(V,W,X)$$

$$\{\{\{\xi_\alpha,\Theta\},\xi_\beta\},\xi_\gamma\} = \underbrace{\Gamma_{\alpha\beta\gamma} - \Gamma_{\beta\alpha\gamma}}_{\text{torsion}} + \Gamma_{\gamma\alpha\beta} + C_{\alpha\beta\gamma} =: \Gamma^{\text{new}}_{\gamma\alpha\beta}$$

"mother of all brackets"

$$\begin{aligned}[V,W] &= \nabla_V W - \nabla_W V + \langle \nabla V, W\rangle + C(V,W,-)\\ &= [V,W]_{\text{Lie}} + T(V,W) + \langle \nabla V, W\rangle + C(V,W,-)\end{aligned}$$

In order to obtain a regular Courant algebroid, impose

$$\{\Theta,\Theta\} = 0 \quad \Leftrightarrow \quad \nabla C + \frac{1}{2}\{C,C\} = 0\,, \quad G^{-1}|_h = 0\,, \ldots$$

Summary + Discussion

- generalized geometry provides a perfect setting for the formulation of theories of gravity
- our approach is based on the deformation of a Poisson structure ...
- ... other more traditional approaches focus on the generalized metric and Bismut connection ($\rightarrow$ covariance and uniqueness problems)
- string effective action without string theory; target space approach.