Doubling, T-Duality and Generalized Geometry: a Simple Model

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Dualities and Generalized Geometries
Corfu Summer Institute - September, 12th 2018

Plan of the talk

1. Introduction and Motivation

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2. The 3-D Isotropic Rigid Rotator

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3. $SL(2,\mathbb{C})$ as a Drinfeld Double

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- Underlying DFT is a manifestly T-dual invariant formulation of the string world-sheet [Siegel, Tseytlin, Hull, Park], the *doubled world-sheet*, in which actually the *doubling* refers to the fields living on it and not to its coordinates.
- Interplay between GG and doubled world-sheet/DFT emerges out from the identification of the appropriate carrier space of the dynamics. It could be enlarged also to non-Abelian T-duality and Poisson-Lie T-duality.

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- A dual model is introduced having the Lie dual of $SU(2)$ as configuration space, the group $SB(2, \mathbb{C})$ of Borel 2x2 complex matrices. They are dual partners in the description of the group $SL(2, \mathbb{C})$ as a Drinfeld double.

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- A doubled parent action with configuration space $SL(2,C)$ is then defined: it reduces to the original action of the rotator or to its dual through a suitable gauging procedure.
- Geometric structures can be understood in terms of GG.

- Action:

$$S_0 = -\frac{1}{4}\int_{\mathbb{R}} \mathrm{Tr}(g^{-1}dg \wedge^{*} g^{-1}dg) = -\frac{1}{4}\int_{\mathbb{R}} \mathrm{Tr}\ (g^{-1}\dot{g})^2 dt$$

$g : t \in \mathbb{R} \to SU(2)$, with $g^{-1}dg = i\alpha^k\sigma_k$ the Maurer-Cartan left-invariant (Lie algebra-valued) one-form, α^k are the basic left-invariant one-forms, $*$ the Hodge star operator on the source space $\mathbb{R}$, $*dt = 1$, Tr the trace over the Lie algebra $\to$ group-valued field theory, reduction of Principal Chiral Model to 0+1 dimensions.

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- Parametrization with $\mathbb{R}^4$ coordinates $g = y^0\sigma_0 + iy^i\sigma_i$ $\equiv 2(y^0 e_0 + iy^i e_i)$ with $(y^0)^2 + \sum_i (y^i)^2 = 1$ and σ_0 and σ_i respectively the identity matrix $\mathbb{I}$ and the Pauli matrices: $y^i = -\frac{i}{2}\mathrm{Tr}(g\sigma_i)$, $y^0 = \frac{1}{2}\mathrm{Tr}(g\sigma_0)$, $i = 1, .., 3$

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- In terms of the left generalized velocities $\dot{Q}^i$

$$\dot{Q}^i := (y^0 \dot{y}^i - y^i \dot{y}^0 + \epsilon^i{}_{jk} y^j \dot{y}^k)$$

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- Equations of motion:

$$\ddot{Q}^i = 0 \qquad or \qquad \frac{d}{dt}\left(g^{-1}\frac{dg}{dt}\right) = 0$$

- Cotangent bundle $T^*SU(2)$ coordinates: (Q^i, I_i), being the I_i's the left momenta:

$$I_i = \frac{\partial \mathcal{L}_0}{\partial \dot{Q}^i} = \delta_{ij}\dot{Q}^j \quad ; \quad I = iI_i e^{i*}$$

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- The dynamics is obtained from $\mathcal{H}_0$ through the canonical Poisson brackets on the cotangent bundle:

$$\{y^i, y^j\} = 0 \quad \{I_i, I_j\} = \epsilon_{ij}{}^k I_k \quad \{y^i, I_j\} = \delta^i_j y^0 + \epsilon^i{}_{jk} y^k$$

derived from the first-order formulation of the action

$$S_1 = \int < I|g^{-1}\dot{g} > dt - \int \mathcal{H}_0 dt \equiv \int \theta - \int \mathcal{H}_0 dt$$

being θ the canonical one-form defining the symplectic form $\omega = d\theta = dI_i \wedge \delta^i_j \alpha^j - \frac{1}{2} I_i \delta^i_j \epsilon^j{}_{kl} \alpha^k \wedge \alpha^l$ with $d\alpha^k = \frac{i}{2}\epsilon^k_{ij}\alpha^i \wedge \alpha^j$. Inverting ω leads to the Poisson algebra.

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- As a group $T^*SU(2) \simeq SU(2) \ltimes \mathbb{R}^3$ (semi-direct product) with the corresponding Lie algebra given by:

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- The Poisson brackets governing the dynamics are the brackets induced by the coadjoint action.

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- Maximally isotropic: the subspace cannot be enlarged while preserving the property of isotropy.
- $\mathfrak{D} = \mathcal{G} \bowtie \tilde{\mathcal{G}}$ is a Lie bialgebra in which the role of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ can be symmetrically interchanged.The triple $(\mathfrak{D}, \mathcal{G}, \tilde{\mathcal{G}})$ is referred to as a Manin triple.

$SL(2,\mathbb{C})$, $SU(2)$ and $SB(2,\mathbb{C})$

- The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is spanned by $e_i = \sigma_i/2, b_i = ie_i$

$$[e_i, e_j] = i\epsilon_{ij}^k e_k, \qquad [e_i, b_j] = i\epsilon_{ij}^k b_k, \qquad [b_i, b_j] = -i\epsilon_{ij}^k e_k$$

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- $\{\tilde{e}^i\}$ span the Lie algebra $\mathfrak{sb}(2,\mathbb{C})$ *dual* of $\mathfrak{su}(2)$ $< \tilde{e}^i, e_j >:= \delta^i_j$

The $O(d,d)$ invariant metric

- Introduce the *doubled* notation

$$e_I = \begin{pmatrix} e_i \\ \tilde{e}^i \end{pmatrix}, \qquad e_i \in \mathfrak{su}(2), \quad \tilde{e}^i \in \mathfrak{sb}(2,\mathbb{C}),$$

The scalar product $< u, v >= 2\mathrm{Im}(\mathrm{Tr}(uv))$ yields

$$< e_I, e_J >= \eta_{IJ} = \begin{pmatrix} 0 & \delta_i^j \\ \delta_j^i & 0 \end{pmatrix}$$

This is the $O(3,3)$ invariant metric.

The Generalized Metric

- Through the scalar product $(u, v) = 2\mathrm{Re}[\mathrm{Tr}(uv)]$

$$(e_i, e_j) = -(b_i, b_j) = \delta_{ij}, \qquad (e_i, b_j) = 0$$

it is possible to define a non-degenerate scalar product a $((\,,\,))$, giving rise to a Riemannian metric:

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$$((e_I, e_J)) = \mathcal{H}_{IJ} = \begin{pmatrix} \delta_{ij} & {\epsilon_{3i}}^{j} \\ -{\epsilon^{i}}_{j3} & \delta^{ij} + {\epsilon^{i}}_{l3}\delta^{lk}{\epsilon^{j}}_{k3} \end{pmatrix}$$

satisfying the relation: $\mathcal{H}^T \eta \mathcal{H} = \eta$.

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- $\mathcal{H}$ is an $O(3,3)$ matrix having the same structure as the $O(d,d)$ *generalized metric* of DFT with δ_{ij} playing the role of G^{ij} and ϵ_{ij3} playing the role of B_{ij}!

Geometry of the Dual Model

- A suitable action for the dual system is the following:

$$\tilde{S}_0 = -\frac{1}{4}\int_{\mathbb{R}} \mathcal{T}r[\tilde{g}^{-1}d\tilde{g}\wedge * \tilde{g}^{-1}d\tilde{g}] = -\frac{1}{4}\int_{\mathbb{R}} \mathcal{T}r[(\tilde{g}^{-1}\dot{\tilde{g}})(\tilde{g}^{-1}\dot{\tilde{g}})]dt$$

with $\tilde{g} : t \in \mathbb{R} \to SB(2,\mathbb{C})$, the group-valued target space coordinates, so that

$$\tilde{g}^{-1}d\tilde{g} = i\beta_k \tilde{e}^k$$

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- The trace $\mathcal{T}r$ is defined by the scalar product $((,))$ that makes the dual Lagrangian only left-$SB(2,\mathbb{C})$ invariant, differently from the Lagrangian of the rigid rotator which is invariant under both left and right actions of both groups.

- Parametrization: the group manifold $SB(2,\mathbb{C})$ can be parametrized with $\mathbb{R}^4$ coordinates: $\tilde{g} = 2(u_0\tilde{e}^0 + iu_i\tilde{e}^i)$, being $u_0^2 - u_3^2 = 1$ and $\tilde{e}^0 = \mathbb{I}/2$ with

$$u_i = \frac{1}{4}((i\tilde{g},\tilde{e}^i)),\ i = 1,2, \quad u_3 = \frac{1}{2}((i\tilde{g},\tilde{e}^3)), \quad u_0 = \frac{1}{2}((\tilde{g},\tilde{e}^0))$$

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$$\tilde{L}_0 = \dot{\tilde{Q}}_i\dot{\tilde{Q}}_r h^{ir}$$

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- Tangent bundle $TSB(2,\mathbb{C})$ coordinates: $(\tilde{Q}_i, \dot{\tilde{Q}}_i)$ with the $\tilde{Q}_i$'s implicitly defined.
- Equations of motion:

$$(\delta^{ij} + \epsilon^i{}_{k3}\epsilon^j{}_{l3}\delta^{kl})\ddot{\tilde{Q}}_j = 0$$

- Cotangent bundle $T^*SB(2,\mathbb{C})$ coordinates: $(\tilde{Q}_i, \tilde{I}^i)$, being the $\tilde{I}^i$'s the conjugate left momenta.

The cotangent bundle $T^*SB(2,\mathbb{C})$

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- The linear combination over the dual basis is introduced:

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- Poisson brackets:

$$\begin{aligned} \{u_i, u_j\} &= 0 \\ \{\tilde{I}^i, \tilde{I}^j\} &= f^{ij}{}_k \tilde{I}^k \\ \{u_i, \tilde{I}^j\} &= \delta^j_i u_0 - f_i{}^{jk} u_k \end{aligned}$$

which are derived from the first order formulation of the action.

- As a group, $T^*SB(2,\mathbb{C}) \simeq SB(2,\mathbb{C}) \ltimes \mathbb{R}^3$, with Lie algebra;

$$\begin{aligned} [B_i, B_j] &= i f_{ij}{}^k B_k \\ [S_i, S_j] &= 0 \\ [B_i, S_j] &= i f_{ij}{}^k S_k. \end{aligned}$$

- The two models can be obtained from the same *parent action* defined on the *whole* $SL(2,\mathbb{C}) \to$ they are dual.
- Goal: to define a dynamical model symmetric under $SU(2) \leftrightarrow SB(2,\mathbb{C})$ on the Drinfeld double $SL(2,\mathbb{C})$.

The doubled action

- Introduce an action functional on $TSL(2,\mathbb{C})$ (doubled coordinates) as a $(0+1)$-dimensional field theory which is group-valued with $\gamma : t \in \mathbb{R} \to \gamma(t) \in SL(2,\mathbb{C})$.

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- Left invariant one-form on the group manifold:

$$\gamma^{-1}\mathrm{d}\gamma = \gamma^{-1}\dot{\gamma}\; dt \equiv \dot{\mathbf{Q}}^I e_I \; dt \tag{1}$$

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- Defining the decomposition $\dot{\mathbf{Q}}^I \equiv (A^i, B_i)$ implies:

$$\gamma^{-1}\dot{\gamma}\, dt = (A^i e_i + B_i \tilde{e}^i) dt.$$

where both components are tangent bundle coordinates for $SL(2,\mathbb{C})$.

- Proposed action:

$$S = \frac{1}{2}\int_{\mathbb{R}} (k_1 < \gamma^{-1}\mathrm{d}\gamma \stackrel{\wedge}{,} *\gamma^{-1}\mathrm{d}\gamma > + k_2((\gamma^{-1}\mathrm{d}\gamma \stackrel{\wedge}{,} *\gamma^{-1}\mathrm{d}\gamma))),$$

k_1, k_2 are real parameters and

$$\begin{aligned} < \gamma^{-1}\mathrm{d}\gamma \stackrel{\wedge}{,} *\gamma^{-1}\mathrm{d}\gamma > &= \dot{\mathbf{Q}}^I\dot{\mathbf{Q}}^J < e_I, e_J >= \dot{\mathbf{Q}}^I\dot{\mathbf{Q}}^J\eta_{IJ} \\ ((\gamma^{-1}\mathrm{d}\gamma \stackrel{\wedge}{,} *\gamma^{-1}\mathrm{d}\gamma)) &= \dot{\mathbf{Q}}^I\dot{\mathbf{Q}}^J((e_I, e_J)) = \dot{\mathbf{Q}}^I\dot{\mathbf{Q}}^J\mathcal{H}_{IJ}. \end{aligned}$$

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- Explicitly

$$\hat{L} = \frac{1}{2}(k\,\eta_{IJ} + \mathcal{H}_{IJ})\dot{\mathbf{Q}}^I\dot{\mathbf{Q}}^J$$

$k_1/k_2 \equiv k$.

Recovering both the actions

- Fix a local decomposition for the elements of the double group $SL(2,\mathbb{C})$: $\gamma = \tilde{g}g$, $g \in SU(2)$ and $\tilde{g} \in SB(2,\mathbb{C})$.

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- Promote the $SB(2,\mathbb{C})_L$ invariance to a gauge symmetry for getting the usual description of the rotator.
- Promote the global invariance of $\hat{L}$ under right action of the group $SU(2)$ for getting the dual rotator.

- In the doubled description the left generalized momenta are given by:

$$\mathbf{P}_I = \frac{\partial L}{\partial \dot{\mathbf{Q}}^\mathbf{I}} = (\eta_{IJ} + k\mathcal{H}_{IJ})\dot{\mathbf{Q}}^\mathbf{J}$$

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- The Hamiltonian equations on the Drinfeld double are obtained through the determination of Poisson brackets from the first order action:

$$\widehat{\mathcal{S}} = \int \langle \mathbf{P} | \gamma^{-1} d\gamma \rangle - \int \widehat{H} dt \equiv \int \boldsymbol{\theta} - \int \widehat{H} dt$$

with

$$\begin{aligned} \mathbf{P} &= i\, \mathbf{P}_I e^{I^*} = i\,(I_i e^{i^*} + \tilde{I}^i \tilde{e}_i^*) \\ \gamma^{-1} d\gamma &= i\, \boldsymbol{\alpha}^J e_J = (\alpha^k e_k + \beta_k \tilde{e}^k)\ . \end{aligned}$$

- $\mathbf{P}_{\mathbf{I}}$, $\boldsymbol{\alpha}^J$ are respectively generalized momenta and basis one-forms on the doubled configuration space $SL(2,\mathbb{C})$.

Poisson Brackets on $T^*SL(2,\mathbb{C})$

- The symplectic form $\boldsymbol{\omega} = d\boldsymbol{\theta}$ on $T^*SL(2,\mathbb{C}) \simeq SL(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})^*$ yields for the generalized momenta the Poisson brackets to:

$$\begin{aligned}
\{I_i, I_j\} &= \epsilon_{ij}{}^k I_k \\
\{\tilde{I}^i, \tilde{I}^j\} &= f^{ij}{}_k \tilde{I}^k \\
\{I_i, \tilde{I}^j\} &= \epsilon^j{}_{il}\tilde{I}^l - I_l f^{lj}{}_i \quad \{\tilde{I}^i, I_j\} = -\epsilon^i{}_{jl}\tilde{I}^l + I_l f^{li}{}_j
\end{aligned}$$

  while the Poisson brackets between momenta and configuration space variables $g, \tilde{g}$ are unchanged with respect to $T^*SU(2)$, $T^*SB(2,\mathbb{C})$.

- equation of motion for $\mathbf{P_I}$:

$$\frac{d}{dt}\mathbf{P}_I = \{\mathbf{P}_I, \widehat{H}\} = [(\eta + k\,\mathcal{H})^{-1}]^{JK} C_{IJ}^L \mathbf{P}_L \mathbf{P}_K$$

  with

$$\{\mathbf{P}_I, \mathbf{P}_J\} = C_{IJ}^K \mathbf{P}_K$$

- The double group $SL(2,\mathbb{C})$ can be endowed with PB's (Heisenberg double) which generalize both those of $T^*SU(2)$ and of $T^*SB(2\mathbb{C})$ [[Semenov-Tyan-Shanskii '91, Alekseev-Malkin '94]]

$$\{\gamma_1,\gamma_2\} = -\gamma_1\gamma_2 r^* - r\gamma_1\gamma_2 \qquad \gamma_1 = \gamma\otimes 1, \gamma_2 = 1\otimes\gamma_2$$

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- By writing γ as $\gamma = \tilde{g}g$ it can be shown that these brackets are compatible with

$$\begin{aligned}
&\{\tilde{g}_1,\tilde{g}_2\} = -[r,\tilde{g}_1\tilde{g}_2],\\
&\{\tilde{g}_1,g_2\} = -\tilde{g}_1 r g_2, \qquad \{g_1,\tilde{g}_2\} = -\tilde{g}_2 r^* g_1\\
&\{g_1,g_2\} = [r^*, g_1 g_2].
\end{aligned}$$

- In the limit $\lambda \to 0$, with $r = \lambda \tilde{e}^i \otimes e_i$, $\tilde{g}(\lambda) = 1 + i\lambda I_i e^i + O(\lambda^2)$ $g = y^0\sigma_0 + iy^i\sigma_i$ they reproduce correctly the canonical Poisson brackets on the cotangent bundle of $SU(2)$.

Recovering the PB's for $T^*SU(2)$ and $T^*SB(2,\mathbb{C})$

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- Consider now $r^* = -\mu e_k \otimes e^k$ as a solution of the Yang Baxter equation and expand $g \in SU(2)$ as a function of the parameter μ: $g = \mathbf{1} + i\mu \tilde{I} e_i + O(\mu^2)$.

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- By repeating the same analysis as above one gets back the canonical Poisson structure on $T^*SB(2,C)$, with position coordinates and momenta now interchanged. In particular:

$$\{\tilde{I}^i, \tilde{I}^j\} = f^{ij}{}_k \tilde{I}^k$$

Connection with Generalized Geometry

- It is possible to consider a different Poisson structure on the double [Semenov]: $\{\gamma_1, \gamma_2\} = \frac{\lambda}{2}\left[\gamma_1(r^* - r)\gamma_2 - \gamma_2(r^* - r)\gamma_1\right]$

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- Expand $\gamma \in D$ as $\gamma = \mathbf{1} + i\lambda I_i \tilde{e}^i + i\lambda \tilde{I}^i e_i$ and rescale r, r^* by the same parameter $\lambda \Longrightarrow$

$$
\begin{aligned}
\{I_i, I_j\} &= \epsilon_{ij}{}^k I_k; \qquad \{\tilde{I}^i, \tilde{I}^j\} = f^{ij}{}_k \tilde{I}^k \\
\{I_i, \tilde{I}^j\} &= -f_i^{jk} I_k - \tilde{I}^k \epsilon_{ki}{}^j
\end{aligned}
$$

which are the Poisson brackets induced by the Lie bi-algebra structure of the double.

C-Brackets

- By using the compact notation $I = iI_i e^{i^*}, \tilde{I} = i\tilde{I}_i \tilde{e}_i^*$, one can rewrite the Poisson algebra as follows:

$$\{I+\tilde{I}, J+\tilde{J}\} = \{I,J\} - \{J,\tilde{I}\} + \{I,\tilde{J}\} + \{\tilde{I},\tilde{J}\}.$$

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- Very interesting structure representing a Poisson realization of the C-bracket for the generalized bundle $T \oplus T^*$ over $SU(2)$, once one considers the isomorphisms

$$TSL(2,\mathbb{C}) \simeq SL(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$$

with the fiber:

$$\mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{su}(2) \oplus \mathfrak{sb}(2,\mathbb{C}) \simeq TSU(2) \oplus T^*SU(2).$$

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- $I = iI_i e^{i^*}, J = iJ_i e^{i^*}$ are considered as one-forms, with e^{i^*} being a basis over T^* and $\tilde{I} = \tilde{I}^i \tilde{e}_i^*, \tilde{J} = \tilde{J}^i \tilde{e}_i^*$ as vector-fields, with $\tilde{e}_i^*$ a basis over T. Namely, the couple $(I_i, \tilde{I}^i)$ identifies the fiber coordinate of the generalized bundle $T \oplus T^*_-$ of $SU(2)$.

Poisson-Lie Symmetries

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- Example. Right transformations of $SU(2)$ on $SL(2,\mathbb{C})$:

$$\gamma \to \gamma h, \quad h \in SU(2) \ ,, \ \ \gamma \in SL(2,\mathbb{C})$$

and left action of $SB(2,\mathbb{C})$ on $SL(2,\mathbb{C})$:

$$\gamma \to \tilde{h}\gamma \quad \tilde{h} \in SB(2,\mathbb{C}) \ , \ \ \gamma \in SL(2,\mathbb{C})$$

.

- In terms of the coordinates $(\tilde{g}, g)$ such that $\gamma = \tilde{g}g$ the right transformation:

$$g \to gh \ , \quad \tilde{g} \to \tilde{g} \quad \text{and} \quad g \to g \quad , \quad \tilde{g} \to \tilde{h}\tilde{g}$$

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- They preserve the Poisson brackets *only if* the paramaters of the transformation $h = h_1 h_2$ satisfy the Poisson brackets:

$$\{h_1, h_2\} = [r^*, h_1 h_2]$$

and zero Poisson brackets with g and $\tilde{g}$. The $SU(2)$ right multiplication becomes a Poisson-Lie group transformation.

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- The left $SL(2,\mathbb{C})$ multiplication is a Poisson-Lie transformation if

$$\{\tilde{h}_1, \tilde{h}_2\} = \left[r^*, \tilde{h}_1 \tilde{h}_2\right]$$

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- In terms of the coordinates $(\tilde{g}, g)$ such that $\gamma = \tilde{g}g$ the right transformation:

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- The left $SL(2,\mathbb{C})$ multiplication is a Poisson-Lie transformation if

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- Both of them become canonical in the limit $\lambda \to 0$.

- The Poisson brackets

$$\{\tilde{g}_1, \tilde{g}_2\} = -[r, \tilde{g}_1\tilde{g}_2],$$
$$\{\tilde{g}_1, g_2\} = -\tilde{g}_1 r g_2, \qquad \{g_1, \tilde{g}_2\} = -\tilde{g}_2 r^* g_1$$
$$\{g_1, g_2\} = [r^*, g_1 g_2].$$

are invariant under the simultaneous action of both $SU(2)$ and $SL(2,\mathbb{C})$ if $\{\tilde{h}_1, h_2\} = 0$.

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- The dynamics on the group manifold of $SL(2,\mathbb{C})$ has Poisson-Lie group symmetries only when endowed with those brackets.

- The Poisson brackets

$$\begin{aligned}
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  are invariant under the simultaneous action of both $SU(2)$ and $SL(2,\mathbb{C})$ if $\{\tilde{h}_1, h_2\} = 0$.
- The dynamics on the group manifold of $SL(2,\mathbb{C})$ has Poisson-Lie group symmetries only when endowed with those brackets.
- The symplectic structure $\{I_i, I_j\} = \epsilon_{ij}{}^k$ is obtained from $\{\tilde{g}_1, \tilde{g}_2\} = -[\tilde{g}_1\tilde{g}_2]$, while $\left\{\tilde{I}^I, \tilde{I}^j\right\} = f^{ij}{}_k I^k$ from $\{g_1, g_2\} = [r^*, g_1 g_2]$.

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- The dynamics on the group manifold of $SL(2,\mathbb{C})$ has Poisson-Lie group symmetries only when endowed with those brackets.
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- The momentum variables of each model inherit their Poisson brackets from the Poisson-Lie structure of the dual group, which in turn exhibits Poisson-Lie symmetry in the sense elucidated above.

Conclusions

- The double formulation of a mechanical system in terms of dual configuration spaces has been discussed.
- The geometrical structures of DFT have been reproduced ($O(d,d)$-invariant metric and Generalized Metric).
- Poisson brackets for the generalized momenta (C-brackets) have been derived establishing a connection with Generalized Geometry.
- Poisson-Lie symmetries of the dual models have been revealed.
- The model is simple, but it is readily generalizable, for instance, to the Principal Chiral Model (work in progress); in fact, by adding one space dimension to the source space one has a 2-d field theory, modeled on the rigid rotator, which is duality invariant and has all the richness of the Double and Generalized Geometries.

The End

Thank you for your attention.