Four and five-loop renormalization of QCD

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DESY

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Outline

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Introduction

- Renormalization constants are on there own fundamental quantities of the given theory
- Quark masses can be measured with high precision using different renormalization schemes
  $\Rightarrow$ have to be able to translate between them
- on-shell renomalization constants required for the calculation of processes with massive particles
- pole masses suffer from (infrared) renormalon problems
  $\Rightarrow$ try to quantify the effect
- all $\overline{\mathrm{MS}}$ renormalization constants, in particular the β function and mass anomaous dimension, are now available at five loops.

Scheme definitions

One considers the renormalized quark propagator

$$S_F(q) = \frac{-i\,Z_2}{\not{q} - Z_m m + \Sigma(q,m)}$$

with $\Sigma(q,m)$ the quark two-point function.
For the $\overline{\mathrm{MS}}$ scheme we require

$$S_F(q) \quad \text{finite}$$

and for the on-shell scheme we require a pole at the position of the mass

$$S_F(q) \overset{q^2 \to M^2}{\longrightarrow} \frac{-i}{\not{q} - M}$$

Outline

Setup of the calculation

expand the quark two-point function

$$\begin{aligned}\Sigma(q,M) &\approx M\,\Sigma_1(M^2,M)\\ &\quad +(\not{q}-M)\left(2M^2\frac{\partial}{\partial q^2}\Sigma_1(q^2,M)\Big|_{q^2=M^2}+\Sigma_2(M^2,M)\right)+\ldots\end{aligned}$$

renormalization constants are then given by

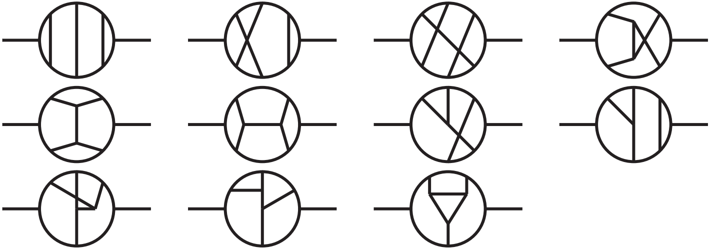
$$\begin{aligned}Z_m^{\mathrm{OS}} &= 1+\Sigma_1(M^2,M)\,,\\ \left(Z_2^{\mathrm{OS}}\right)^{-1} &= 1+2M^2\frac{\partial}{\partial q^2}\Sigma_1(q^2,M)\Big|_{q^2=M^2}+\Sigma_2(M^2,M)\,.\end{aligned}$$

best to apply the projector

$$\begin{aligned}\mathrm{Tr}\left\{\frac{\not{Q}+M}{4M^2}\Sigma(q,M)\right\} &= \Sigma_1(q^2,M)+t\Sigma_2(q^2,M)\\ &= \Sigma_1(M^2,M)+\left(2M^2\frac{\partial}{\partial q^2}\Sigma_1(q^2,M)\Big|_{q^2=M^2}+\Sigma_2(M^2,M)\right)t\end{aligned}$$

Setup of the calculation cont'd

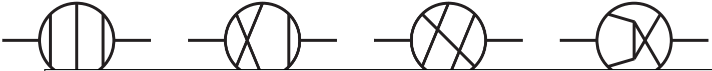
Need to calculate 4-loop on-shell diagrams of the form

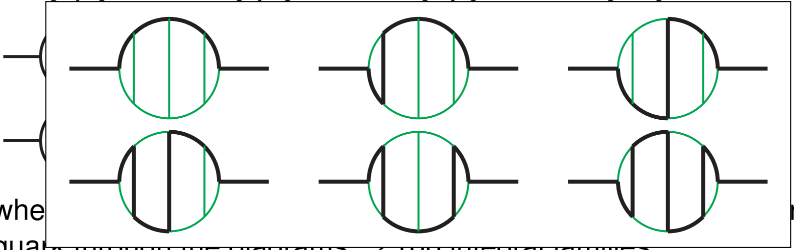


where we have to consider all possible ways to route the external quark through the diagrams $\Rightarrow$ 100 integral families.

Setup of the calculation

Need to calculate 4-loop on-shell diagrams of the form





whe … nal

quark through the diagrams ⇒ 100 integral families.

Setup of the calculation

Follow the *standard* procedure for multi-loop calculations

- exploit that the appearing integrals are not linear independent but related through Integration-By-Parts identities [Chetyrkin, Tkachov '81]
- reduce all appearing integrals to a small set of basis integrals using `FIRE` [Smirnov] or `Crusher` [PM,Seidel]
- evaluate the remaining basis integrals ($\mathcal{O}(350)$) using analytic or numerical techniques (Mellin-Barnes or Sector Decomposition)[`MB.m` [Czakon] `FIESTA` [Smirnov]]

Setup of the calculation

- Obtained renormalization constant Z_m^{OS} by calculating four-loop on-shell integrals
- Together with the renormalization constant in the $\overline{\mathrm{MS}}$-scheme $Z_m^{\overline{\mathrm{MS}}}$

  [Chetyrkin '97; Larin, van Rittbergen, Vermaseren '97; Baikov,Chetyrkin,Kühn '14]

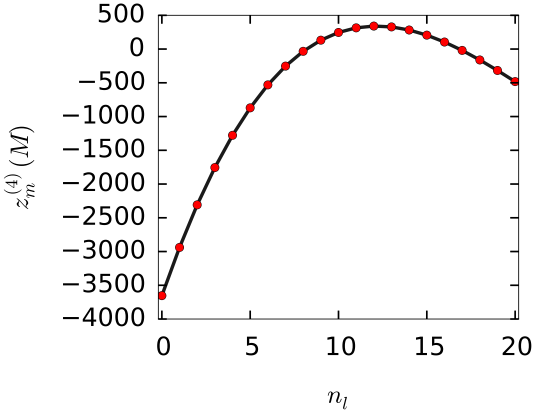
  we get for the relation between the two schemes

$$\left.\begin{aligned} m_{\mathrm{bare}} &= Z_m^{\mathrm{OS}} M \\ m_{\mathrm{bare}} &= Z_m^{\overline{\mathrm{MS}}} m \end{aligned}\right\} \Rightarrow m = M \frac{Z_m^{\mathrm{OS}}}{Z_m^{\overline{\mathrm{MS}}}}$$

  - 1-loop [Tarrach'81]
  - 2-loop [Gray,Broadhurst,Grafe,Schilcher'90]
  - 3-loop [Chetyrkin,Steinhauser'99; Melnikov, v. Ritbergen'00; Marquard,Mihaila,Piclum,Steinhauser'07]

$\overline{\text{MS}}$–on-shell relation at four-loop order

$$z_m^{(4)} = -3654.15 \pm 1.64 + (756.942 \pm 0.040) n_l - 43.4824 n_l^2 + 0.678141 n_l^3 \,.$$



$\overline{\text{MS}}$–on-shell relation at four-loop order

$\overline{\text{MS}} \rightarrow$ on-shell

$$
\begin{aligned}
m_t(m_t) &= M_t \big(1 - 0.4244\,\alpha_s - 0.9246\,\alpha_s^2 - 2.593\,\alpha_s^3 \\
&\quad -(8.949 \pm 0.018)\,\alpha_s^4\big) \\
&= 173.34 - 7.924 - 1.859 - 0.562 \\
&\quad - (0.209 \pm 0.0004)\ \text{GeV}
\end{aligned}
$$

[PM, Steinhauser, Smirnov, Smirnov, Wellmann ’16]

$\overline{\text{MS}}$–on-shell relation at four-loop order

$\overline{\text{MS}} \to$ on-shell

$$
\begin{aligned}
m_t(m_t) &= M_t \left(1 - 0.4244\,\alpha_s - 0.9246\,\alpha_s^2 - 2.593\,\alpha_s^3 \right. \\
&\quad \left. -(8.949 \pm 0.018)\,\alpha_s^4\right) \\
&= 173.34 - 7.924 - 1.859 - 0.562 \\
&\quad - (0.209 \pm 0.0004)\ \text{GeV}
\end{aligned}
$$

[PM, Steinhauser, Smirnov, Smirnov, Wellmann ’16]

$$
\begin{aligned}
M_b &= m_b(m_b) \left(1 + 0.4244\,\alpha_s + 0.9401\,\alpha_s^2 + 3.045\,\alpha_s^3 \right. \\
&\quad \left. +(12.685 \pm 0.025)\,\alpha_s^4\right) \\
&= 4.163 + 0.398 + 0.199 + 0.145 + (0.136 \pm 0.0003)\ \text{GeV}
\end{aligned}
$$

Light quark mass dependence

- starts at 2 loops [Gray,Broadhurst,Grafe,Schilcher '90]
- available up to 3 loops [Bekavac,Grozin,Seidel,Steinhauser '07]
- for top quark
    - 11 MeV @ two loop
    - 16 MeV @ three loop
    - getting more important at higher orders due to renormalon enhancement [Hoang, Lepenik, Preisser '17]

Threshold masses, e.g. PS mass

$$m^{\rm PS} = M - \delta m(\mu_f)$$

$$\delta m(\mu_f) = -\frac{1}{2}\int_{|\vec{q}|<\mu_f} \frac{{\rm d}^3 q}{(2\pi)^3} V(\vec{q}) = \mu_f \frac{C_F \alpha_s}{\pi}(1 + \alpha_s \ldots)$$

need static potential @ three loops

[Smirnov,Smirnov,Steinhauser '09; Anzai ,Kiyo,Sumino '09]

$$\begin{aligned} m_t^{\rm PS}(\mu_f = 80~\text{GeV}) &= 163.508 + (7.531 - 3.685) \\ &\quad +(1.607 - 0.989) + (0.495 - 0.403) \\ &\quad +(0.195 - 0.211 \pm 0.0004)~\text{GeV} \\ &= 163.508 + 3.847 + 0.618 + 0.092 \\ &\quad -(0.016 \pm 0.0004)~\text{GeV} \end{aligned}$$

- large cancellations between contributions from OS-MS and PS-OS
- good convergence

Wave function renormalization: Overview

- final missing building block to complete the on-shell renormalization procedure of QCD at four loops.

  [PM, Smirnov, Smirnov, Steinhauser ’18]

- gauge dependence starts only at 3 loops

  [Broadhurst,Gray,Schilcher]

  [Melnikov,van Ritbergen;Marquard,Mihaila,Piclum,Steinhauser]

- computationally more involved due to additional derivative and gauge dependence

4-loop results, $N_c = 3$

$\xi = 0$	$1/\epsilon^4$	$1/\epsilon^3$	$1/\epsilon^2$	$1/\epsilon$	ϵ^0
n_l^0	-1.77242 ± 0.00040	-27.6674 ± 0.0041	-317.093 ± 0.029	-3142.15 ± 0.33	-28709.9 ± 3.2
n_l^1	0.460936 ± 0.000016	6.69143 ± 0.00023	74.6540 ± 0.0013	696.6612 ± 0.0076	6174.290 ± 0.084
n_l^2	-0.039931	-0.51572	-5.5055	-48.777	-418.93
n_l^3	0.00115741	0.0125386	0.126757	1.07105	8.9160

ξ^1	$1/\epsilon^4$	$1/\epsilon^3$	$1/\epsilon^2$	$1/\epsilon$	ϵ^0
n_l^0	-0.018555 ± 0.000011	0.034239 ± 0.000089	-0.05678 ± 0.00052	5.2230 ± 0.0028	36.820 ± 0.017
n_l^1	0.00173611	-0.0052083	0.0224269	-0.34863	-1.61105

ξ^2	$1/\epsilon^4$	$1/\epsilon^3$	$1/\epsilon^2$	$1/\epsilon$	ϵ^0
n_l^0	0.0000002 ± 0.0000038	0.001952 ± 0.000026	-0.03022 ± 0.00012	-0.18686 ± 0.00061	-2.9266 ± 0.0028

In the QED limit with $n_l = 0$ in good agreement with analytic calculation

[Laporta '18]

Results for individual color factors also available.

Outline

$\overline{\text{MS}}$–on-shell relation beyond 4 loops

$$m_P = m(\mu_m)\left(1 + \sum_{n=1}^{\infty} c_n(\mu, \mu_m, m(\mu))\, \alpha_s^n(\mu)\right)$$

for large n

$$c_n(\mu, \mu_m, m(\mu_m)) \underset{n\to\infty}{\longrightarrow} N c_n^{(\mathrm{as})}(\mu, m(\mu_m)) \equiv N \frac{\mu}{m(\mu_m)}\, \tilde{c}_n^{(\mathrm{as})}\,,$$

[Beneke,Braun '94;Beneke '94 '99]

where

$$\tilde{c}_{n+1}^{(\mathrm{as})} = (2b_0)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{s_1}{n+b} + \frac{s_2}{(n+b)(n+b-1)} + \cdots\right).$$

b_0, b, s_1, s_2: Combinations of coefficients of the β-function.

Beyond 4 loops

Fit N to 4-loop term and take higher orders from asymptotic formula

j	$\tilde{c}_j^{(\mathrm{as})}$	$\tilde{c}_j^{(\mathrm{as})} \alpha_s^j$
5	0.985499×10^2	0.001484
6	0.641788×10^3	0.001049
7	0.495994×10^4	0.000880
8	0.443735×10^5	0.000854
9	0.451072×10^6	0.000942
10	0.513535×10^7	0.001164

Asymptotic series $\Rightarrow$ converges only up to $\approx$ 8 loop

5+ loops and remaining ambiguity

An asymptotic series f can (sometimes) be summed by using the Borel transform $B[f]$

$$f(\alpha_s) = \sum_{n=0}^{\infty} c_n \alpha_s^n \qquad \Rightarrow \qquad B[f](t) = \sum_{n=0}^{\infty} c_{n+1} \frac{t^n}{n!}$$

the Borel integral

$$\int_0^{\infty} \mathrm{d}t \, e^{-t/\alpha_s} B[f](t)$$

has the same series expansion as $f(\alpha_s)$ and the same value.

5+ loops and remaining ambiguity cont'd

In our case we have

$$c_{n+1} = (2b_0)^n n! \qquad \Rightarrow \qquad \int_0^\infty \mathrm{d}t\, e^{-t/\alpha_s} \frac{1}{1-2b_0 t}$$

Not integrable due to pole at $1 - 2b_0 t = 0$
Possible prescription for the integral:
Take principle value and assign ambiguity Im/π

$$\delta^{(5+)} m_P = 0.250^{+0.015}_{-0.038}\,(N) \pm 0.001\,(c_4) \pm 0.010\,(\alpha_s) \pm 0.071 \text{ (ambiguity) GeV}$$

5+ loops and remaining ambiguity: light mass effects

- take 2-loop and 3-loop mass effects into account
- 4 loop: massless value using five-flavour theory
- 5 loop: massless value using four-flavour theory
- 6+ loops: massless value using three-flavour theory
- final estimate depends on $\Lambda^{(3)}_{\mathrm{QCD}}$

$$\begin{aligned}\delta^{(5+)} m_P &= 0.304^{+0.012}_{-0.063}\,(N) \pm 0.030\,(m_{b,c}) \\ &\pm 0.009\,(\alpha_s) \pm 0.108\text{ (ambiguity) GeV}\end{aligned}$$

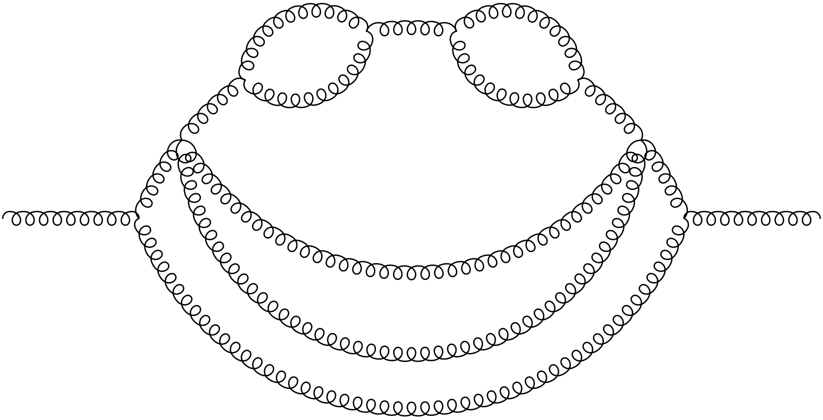
[Beneke, PM, Nason, Steinhauser]

Cmp. analysis by [Hoang,Lepenik, Preisser '17] using different prescription

$$\frac{1}{2} \sum_{c_n < f c_n^{min}} c_n \to 250\mathrm{MeV} \qquad \text{with} \quad f = 5/4$$

Outline

Sample Diagram



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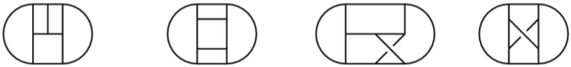
results based on [Luthe,Maier,PM,Schröder]

Method

- Need to calculate the relevant propagators and vertices at five-loop order.
- Renormalization constants (or anomalous dimensions) can be read off from the single pole.
- Only interested in the poles and since we work in the $\overline{\text{MS}}$ scheme we can use a mass as infrared regulator and calculate fully massive five-loop tadpoles instead.
- Prize to pay: No multiplicative renormalization and a gluon mass counterterm.

Some details and statistics

- qq - 83,637, cc - 83,637, gg - 509,777, gcc - 1,444,756 five-loop diagrams generated with `QGRAF` [Nogueira '91]
- Mapped onto 4 12-line topologies



- Reduction to master integrals using IBPs in a Laporta-like implementation in `Crusher` [Marquard, Seidel] and `TIDE` [Luthe] with `Fermat` [Lewis] as backend
- Calculation done using `FORM` [Vermaseren]
- Color algebra done with `color` [van Ritbergen,Schellekens,Vermaseren '99]

Master integrals from Factorial Series

- The idea of the method goes back to Laporta who suggested to calculate Feynman integrals in form of a factorial series. [Laporta '01]
- Take an integral and raise the power of one propagator to the power x e.g. $I(1,1,1) \to I(x) = I(x,1,1)$
- Using IBP relations one can obtain a difference equation for the integral

$$\sum_{k+0}^{R} p_k(x) I(x+k) = \sum_i \sum_{k=0}^{R_i} p_{ik}(x) J_i(x+k)$$

  where J_i are integrals of simpler sectors
- Make an ansatz for $I(x)$ in terms of a factorial series (N.B. not the most general one)

$$I(x) = \sum_{s=0}^{\infty} \frac{\Gamma(x+1)}{\Gamma(x+d/2+s+1)} a_s$$

Master integrals cont'd

- Inserting the ansatz into the difference equation results in a recurrence relation for a_s

$$\sum_{k=0}^{R'} g_k(s) a_{s+k} = \sum_i \sum_{k=0}^{R'_i} g_{ik}(s) a_{i,s+k}$$

- given the initial values $a_0, a_1, \ldots$ are known, an arbitrary number of values for a_n can be calculated.
- using the obtained values for a_n $I(x)$ can be calculated

$$I(x) = \sum_{s=0}^{\infty} \frac{\Gamma(x+1)}{\Gamma(x+d/2+s+1)} a_s$$
$$= \frac{\Gamma(x+1)}{\Gamma(x+d/2+1)} \Big(a_0 + \frac{a_1}{(x+d/2+1)} + \frac{a_2}{(x+d/2+1)(x+d/2+2)} + \cdots \Big)$$

Master integrals cont'd

- In this way we can obtain a numerical result for the master integrals with a very high precision
- For individual master integrals it is not always possible to reconstruct the analytic form since we do not know the alphabet
- For the final results for the anomalous dimensions the expressions are precise enough to reconstruct the analytic form in terms of ζ values using PSLQ

Quark anomalous dimensions

The anomalous dimensions are defined as usual

$$\gamma_2 = -\partial_{\ln\mu^2} \ln Z_2 =$$
$$- c_f\, a \left\{ (1-\xi) + \gamma_{21}\, a + \gamma_{22}\, a^2 + \gamma_{23}\, a^3 + \gamma_{24}\, a^4 + \dots \right\},$$
$$\gamma_m = \partial_{\ln\mu^2} \ln m_{\mathrm{q}}(\mu) \; =$$
$$- c_f\, a \left\{ 3 + \gamma_{m1}\, a + \gamma_{m2}\, a^2 + \gamma_{m3}\, a^3 + \gamma_{m4}\, a^4 + \dots \right\},$$
$$\text{with } a \equiv \frac{C_{\mathrm{A}}\, g^2(\mu)}{16\pi^2}$$

Known up to four loops from

[Tarrach ’81; Tarasov; Larin ’93; Larin, van Ritbergen, Vermaseren ’97; Chetyrkin ’97]

Colour factors

Besides the usual color factors

$$c_f = \frac{C_{\mathrm{F}}}{C_{\mathrm{A}}} \quad , \quad n_f = \frac{N_{\mathrm{f}}\, T_{\mathrm{F}}}{C_{\mathrm{A}}}$$

we also need

$$\begin{aligned} d_1 &= \frac{[\mathrm{sTr}(T^a T^b T^c T^d)]^2}{N_{\mathrm{A}}\, T_{\mathrm{F}}^2 C_{\mathrm{A}}^2} \,, \\ d_2 &= \frac{\mathrm{sTr}(T^a T^b T^c T^d)\, \mathrm{sTr}(F^a F^b F^c F^d)}{N_{\mathrm{A}}\, T_{\mathrm{F}} C_{\mathrm{A}}^3} \,, \\ d_3 &= \frac{[\mathrm{sTr}(F^a F^b F^c F^d)]^2}{N_{\mathrm{A}}\, C_{\mathrm{A}}^4} \end{aligned}$$

Color structures with a trace over 3 generators do not contribute.

γ_m, 2 – 4 loop

The quark mass anomalous dimension is gauge independent

$$3^1\,\gamma_{m1} = n_f\Big[-10\Big] + \Big[(9c_f + 97)/2\Big]\,,$$

$$3^3\,\gamma_{m2} = n_f^2\Big[-140\Big] + n_f\Big[54(24\zeta_3-23)c_f - 4(139+324\zeta_3)\Big]$$
$$+\Big[(6966c_f^2-3483c_f+11413)/4\Big],$$

$$3^4\,\gamma_{m3} = n_f^3\Big[-8(83-144\zeta_3)\Big] + n_f^2\Big[48(19-270\zeta_3+162\zeta_4)c_f$$
$$+2(671+6480\zeta_3-3888\zeta_4)\Big] + n_f\Big[-216(35-207\zeta_3+180\zeta_5)c_f^2$$
$$-3(8819-9936\zeta_3+7128\zeta_4-2160\zeta_5)c_f$$
$$-(65459/2+72468\zeta_3-21384\zeta_4-32400\zeta_5)+2592(2-15\zeta_3)d_1\Big]$$
$$+\tfrac{9}{8}\Big[-9(1261+2688\zeta_3)c_f^3+6(15349+3792\zeta_3)c_f^2$$
$$-2(34045+5472\zeta_3-15840\zeta_5)c_f+(70055+11344\zeta_3-31680\zeta_5)$$
$$-1152(2-15\zeta_3)d_2\Big]$$

γ_m, 5 loop

At five loops we get

$$6^5\,\gamma_{m4} = \gamma_{m44}\left[4n_f\right]^4 + \gamma_{m43}\left[4n_f\right]^3 + \gamma_{m42}\left[4n_f\right]^2 + \gamma_{m41}\left[4n_f\right] + \gamma_{m40}\ ,$$

with the coefficients

$$\gamma_{m44} = -6(65 + 80\zeta_3 - 144\zeta_4)\ ,$$

$$\gamma_{m43} = 3(4483+4752\zeta_3-12960\zeta_4+6912\zeta_5)c_f$$
$$+\,(18667/2+32208\zeta_3+29376\zeta_4-55296\zeta_5)\ .$$

$$\gamma_{m42} = \left\{c_f^2, c_f, d_1, 1\right\}.\left\{9(45253 - 230496\zeta_3 + 48384\zeta_3^2 + 70416\zeta_4\right.$$
$$+144000\zeta_5 - 86400\zeta_6),$$
$$375373 + 323784\zeta_3 - 1130112\zeta_3^2 + 905904\zeta_4 - 672192\zeta_5 + 129600\zeta_6,$$
$$-864(431 - 1371\zeta_3 + 432\zeta_4 + 420\zeta_5),$$
$$\left.4(13709 + 394749\zeta_3 + 173664\zeta_3^2 - 379242\zeta_4 - 119232\zeta_5 + 162000\zeta_6)\right\}\ ,$$

γ_m, 5 loop cont'd

$$\gamma_{m41} = \left\{c_f^3, c_f^2, c_f d_1, c_f, d_1, d_2, 1\right\}.\Big\{-54(48797 - 247968\zeta_3 + 24192\zeta_4 + 444000\zeta_5 - 241920\zeta_7),$$
$$-18(406861 + 216156\zeta_3 - 190080\zeta_3^2 + 254880\zeta_4 - 606960\zeta_5 - 475200\zeta_6 + 362880\zeta_7),$$
$$-62208(11 + 154\zeta_3 - 370\zeta_5),$$
$$753557 + 15593904\zeta_3 - 3535488\zeta_3^2 - 6271344\zeta_4 - 17596224\zeta_5 + 1425600\zeta_6 + 1088640\zeta_7,$$
$$1728(3173 - 6270\zeta_3 + 1584\zeta_3^2 + 2970\zeta_4 - 13380\zeta_5),$$
$$1728(380 - 5595\zeta_3 - 1584\zeta_3^2 - 162\zeta_4 + 1320\zeta_5),$$
$$-2(4994047 + 11517108\zeta_3 - 57024\zeta_3^2 - 5931900\zeta_4 - 15037272\zeta_5 + 4989600\zeta_6 + 3810240\zeta_7)\Big\},$$

$$\gamma_{m40} = \left\{c_f^4, c_f^3, c_f^2, c_f d_2, c_f, d_2, d_3, 1\right\}.\Big\{972(50995 + 6784\zeta_3 + 16640\zeta_5),$$
$$-54(2565029 + 1880640\zeta_3 - 266112\zeta_4 - 1420800\zeta_5),$$
$$108(2625197 + 1740528\zeta_3 - 125136\zeta_4 - 2379360\zeta_5 - 665280\zeta_7),$$
$$373248(141 + 80\zeta_3 - 530\zeta_5),$$
$$-8(25256617 + 16408008\zeta_3 + 627264\zeta_3^2 - 812592\zeta_4 - 40411440\zeta_5 + 3920400\zeta_6 - 5987520\zeta_7),$$
$$-6912(9598 + 453\zeta_3 + 4356\zeta_3^2 + 1485\zeta_4 - 26100\zeta_5 - 1386\zeta_7),$$
$$5184(537 + 2494\zeta_3 + 5808\zeta_3^2 + 396\zeta_4 - 7820\zeta_5 - 1848\zeta_7),$$
$$4(22663417 + 10054464\zeta_3 + 1254528\zeta_3^2 - 1695276\zeta_4 - 41734440\zeta_5 + 7840800\zeta_6 + 5987520\zeta_7)\Big\}.$$

in full agreement with [Baikov, Chetyrkin, Kühn '14, Baikov, Chetyrkin, Kühn '17]

β function

We introduce the renormalization constants as

$$\begin{aligned} \psi_b &= \sqrt{Z_2}\psi_r\,, \quad A_b = \sqrt{Z_3}A_r\,, \quad c_b = \sqrt{Z_3^c}c_r\,, \\ m_b &= Z_m m_r\,, \quad g_b = \mu^\varepsilon Z_g g_r\,, \quad \xi_{L,b} = Z_\xi \xi_{L,r}\,, \end{aligned}$$

and for the vertices

$$Z_1^j \text{ where } j \in \{3g, 4g, ccg, \psi\psi g\}$$

The anomalous dimensions are related through Ward identities

$$\begin{aligned} \gamma_3 &= 2(\gamma_1^{ccg} - \gamma_3^c) - \beta\,, \quad & \gamma_1^{3g} &= 3(\gamma_1^{ccg} - \gamma_3^c) - \beta\,, \\ \gamma_1^{4g} &= 4(\gamma_1^{ccg} - \gamma_3^c) - \beta\,, \quad & \gamma_1^{\psi\psi g} &= \gamma_1^{ccg} - \gamma_3^c + \gamma_2\,, \end{aligned}$$

and thus we choose to evaluate

$$Z_1^{ccg} = \sqrt{Z_3}\, Z_3^c\, Z_g$$

β function

$$\partial_{\ln \mu^2} a = -a\Big[\varepsilon - \beta\Big] = -a\Big[\varepsilon + b_0\, a + b_1\, a^2 + b_2\, a^3 + b_3\, a^4 + b_4\, a^5 + \dots\Big]$$

$$\begin{aligned}
3^1\, b_0 &= \Big[-4\Big] n_f + 11\ , \\
3^2\, b_1 &= \Big[-36 c_f - 60\Big] n_f + 102\ , \\
3^3\, b_2 &= \Big[132 c_f + 158\Big] n_f^2 + \Big[54 c_f^2 - 615 c_f - 1415\Big] n_f + 2857/2\ , \\
3^5\, b_3 &= \Big[1232 c_f + 424\Big] n_f^3 + 432(132\zeta_3 - 5) d_3 + (150653/2 - 1188\zeta_3) + \\
&\quad \Big[72(169 - 264\zeta_3) c_f^2 + 64(268 + 189\zeta_3) c_f + 1728(24\zeta_3 - 11) d_1 \\
&\quad + 6(3965 + 1008\zeta_3)\Big] n_f^2 + \Big[11178 c_f^3 + 36(264\zeta_3 - 1051) c_f^2 \\
&\quad + (7073 - 17712\zeta_3) c_f + 3456(4 - 39\zeta_3) d_2 + 3(3672\zeta_3 - 39143)\Big] n_f ,
\end{aligned}$$

[Tarasov,Vladimirov,Zharkov ’80; Larin,Vermaseren ’93]

[Ritbergen,Vermaseren,Larin ’97; Czakon ’04]

5-loop β function

$$
\begin{aligned}
3^5\, b_4 &= b_{44}\, n_f^4 + b_{43}\, n_f^3 + b_{42}\, n_f^2 + b_{41}\, n_f + b_{40}\ , \\
b_{44} &= \Big\{c_f, 1\Big\}.\Big\{-8(107+144\zeta_3), 4(229-480\zeta_3)\Big\}\ , \\
b_{43} &= \Big\{c_f^2, c_f, d_1, 1\Big\}.\Big\{-6(4961-11424\zeta_3+4752\zeta_4), -48(46+1065\zeta_3-378\zeta_4), \\
&\quad 1728(55-123\zeta_3+36\zeta_4+60\zeta_5), -3(6231+9736\zeta_3-3024\zeta_4-2880\zeta_5)\Big\}\ , \\
b_{42} &= \Big\{c_f^3, c_f^2, c_f d_1, c_f, d_2, d_1, 1\Big\}.\Big\{-54(2509+3216\zeta_3-6960\zeta_5), \\
&\quad 9(94749/2-28628\zeta_3+10296\zeta_4-39600\zeta_5), 25920(13+16\zeta_3-40\zeta_5), \\
&\quad 3(5701/2+79356\zeta_3-25488\zeta_4+43200\zeta_5), -864(115-1255\zeta_3+234\zeta_4+40\zeta_5), \\
&\quad -432(1347-2521\zeta_3+396\zeta_4-140\zeta_5), 843067/2+166014\zeta_3-8424\zeta_4-178200\zeta_5\Big\}\ , \\
b_{41} &= \Big\{c_f^4, c_f^3, c_f^2, c_f d_2, c_f, d_3, d_2, 1\Big\}.\Big\{-81(4157/2+384\zeta_3), 81(11151+5696\zeta_3-7480\zeta_5), \\
&\quad -3(548732+151743\zeta_3+13068\zeta_4-346140\zeta_5), -25920(3-4\zeta_3-20\zeta_5), \\
&\quad 8141995/8+35478\zeta_3+73062\zeta_4-706320\zeta_5, 216(113-2594\zeta_3+396\zeta_4+500\zeta_5), \\
&\quad 216(1414-15967\zeta_3+2574\zeta_4+8440\zeta_5), -5048959/4+31515\zeta_3-47223\zeta_4+298890\zeta_5\Big\}\ , \\
b_{40} &= \Big\{d_3, 1\Big\}.\Big\{-162(257-9358\zeta_3+1452\zeta_4+7700\zeta_5), \\
&\quad 8296235/16-4890\zeta_3+9801\zeta_4/2-28215\zeta_5\Big\}\ .
\end{aligned}
$$

all results in full agreement with the literature

[Baikov,Chetyrkin,Kühn; Falcioni,Herzog,Ruijl,Ueda,Vermaseren]

Conclusions

Presented results for

- conversion between the $\overline{\text{MS}}$ scheme and the on-shell scheme and various threshold mass schemes calculated at NNNLO
- top quark pole mass ambiguity
- wave function renormalization constant Z_2^{OS}
- five-loop results for anomalous quark mass dimension and β function
  - all results are available for a general gauge group and linear gauge dependence
  - full agreement with available results in the literature
  - completely independent calculation
  - $\Rightarrow$ five-loop running of strong coupling constant and quark masses