A new construction of rational electromagnetic knots

Olaf Lechtenfeld **Gleb Zhilin**

Leibniz Universität Hannover, Germany

- Conformal equivalence of dS_4 to $\mathcal{I}\times S^3$ and two copies of $\mathbb{R}^{1,3}_{+}$
- The correspondence
- Construction of electromagnetic solutions
- Some properties of the solutions
- Examples
- Summary and discussion

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Conformal equivalence of dS_4 to $\mathcal{I} \times S^3$ and two copies of $\mathbb{R}^{1,3}_{+}$

Four-dimensional de Sitter space is a one-sheeted hyperboloid in $\mathbb{R}^{1,4}$ given by

$$-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 \;=\; \ell^2$$

metric: $$\mathrm{d}s^2 \;=\; -\mathrm{d}Z_0^2 + \mathrm{d}Z_1^2 + \mathrm{d}Z_2^2 + \mathrm{d}Z_3^2 + \mathrm{d}Z_4^2$$

Constant Z_0 slices are 3-spheres of varying radius, parametrization:

$$Z_0 \;=\; -\ell \cot\tau \qquad \text{and} \qquad Z_A \;=\; \frac{\ell}{\sin\tau}\,\omega_A \quad \text{for} \quad A = 1,\dots,4$$

with $\omega_A\omega_A = 1$ and $\tau \in \mathcal{I} := (0,\pi)$

metric: $$\mathrm{d}s^2 \;=\; \frac{\ell^2}{\sin^2\tau}\Big(-\mathrm{d}\tau^2 + \mathrm{d}\Omega_3^2\Big) \qquad \mathrm{d}\Omega_3^2 \quad \text{for} \quad S^3$$

Hence, dS_4 is conformally equivalent to a finite cylinder $\mathcal{I} \times S^3$

The $Z_0 + Z_4 < 0$ half of dS_4 is also conformally equivalent to future Minkowski space:

$$Z_0 = \frac{t^2 - r^2 - \ell^2}{2\,t}\,, \quad Z_1 = \ell\,\frac{x}{t}\,, \quad Z_2 = \ell\,\frac{y}{t}\,, \quad Z_3 = \ell\,\frac{z}{t}\,, \quad Z_4 = \frac{r^2 - t^2 - \ell^2}{2\,t}$$

with $\quad x, y, z \in \mathbb{R} \quad$ and $\quad r^2 = x^2 + y^2 + z^2 \quad$ but $\quad t \in \mathbb{R}_+$

since $\quad t \in [0, \infty] \quad$ corresponds to $\quad Z_0 \in [-\infty, \infty] \quad$ but $\quad Z_0 + Z_4 < 0$

metric: $\quad \mathrm{d}s^2 = \frac{\ell^2}{t^2}\left(-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)$

Can cover whole $\mathbb{R}^{1,3}$ by gluing a second dS_4 copy and using patch $Z_0 + Z_4 > 0$

Direct relation between cylinder and Minkowski coordinates:

$$\cot\tau = \frac{r^2-t^2+\ell^2}{2\,\ell\,t}, \quad \omega_1 = \gamma\,\frac{x}{\ell}, \quad \omega_2 = \gamma\,\frac{y}{\ell}, \quad \omega_3 = \gamma\,\frac{z}{\ell}, \quad \omega_4 = \gamma\frac{r^2-t^2-\ell^2}{2\,\ell^2}$$

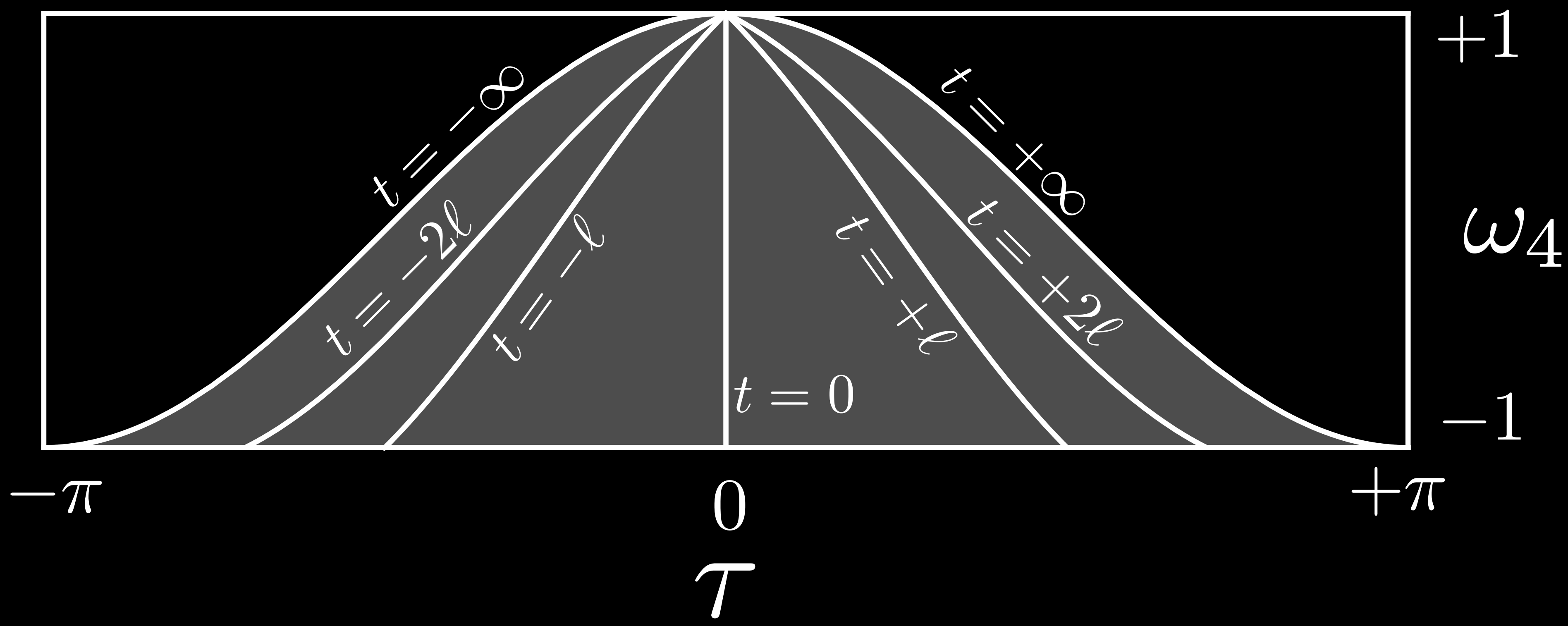
with the convenient abbreviation $\gamma = \dfrac{2\,\ell^2}{\sqrt{4\,\ell^2 t^2 + (r^2 - t^2 + \ell^2)^2}}$

$t = -\infty, 0, \infty$ corresponds to $\tau = -\pi, 0, \pi$ so the cylinder is doubled to $2\mathcal{I} \times S^3$

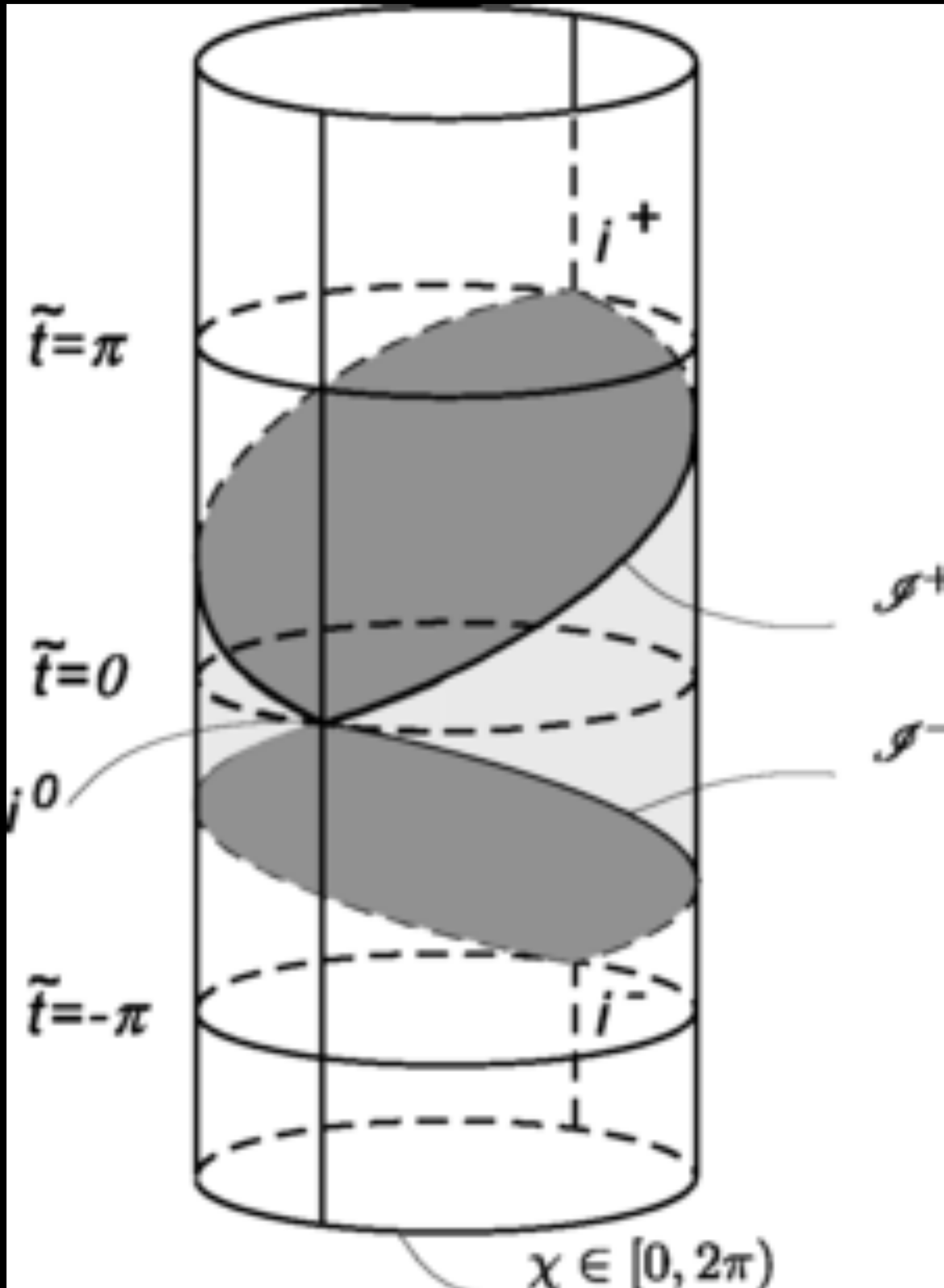
Full Minkowski space is covered by the cylinder patch $\omega_4 \leq \cos\tau$

Cylinder time τ becomes a regular smooth function of (t, x, y, z), but more useful is

$$\exp(\mathrm{i}\,\tau) = \frac{(\ell + \mathrm{i}t)^2 + r^2}{\sqrt{4\,\ell^2 t^2 + (r^2 - t^2 + \ell^2)^2}}$$



$t = -\infty$
$t = -2\ell$
$t = -\ell$
$t = 0$
$t = +\ell$
$t = +2\ell$
$t = +\infty$
$+1$
ω_4
-1
$-\pi$
0
$+\pi$
τ



$\tilde{t}=\pi$
$\tilde{t}=0$
$\tilde{t}=-\pi$
i^+
i^0
i^-
$\mathscr{I}^+$
$\mathscr{I}^-$
$\chi \in [0, 2\pi)$

The correspondence

Yang–Mills and Maxwell are conformally invariant in four spacetime dimensions

$\Rightarrow$ may solve on cylinder $2\mathcal{I} \times S^3$ rather than directly on Minkowski space $\mathbb{R}^{1,3}$

Why? S^3 enables manifestly SO(4)-covariant formalism!

Geometric ansatz for the gauge potential taking values in a Lie algebra $\mathfrak{g}$

$$A = \sum_{a=1}^{3} X_a(\tau,\omega)\, e^a \qquad \text{on} \quad 2\mathcal{I} \times S^3$$

where $X_a \in \mathfrak{g}$ and $\{e^a\}$ are the left-invariant one-forms on S^3, so $A_\tau = 0$

Translate YM or Maxwell solutions from $2\mathcal{I} \times S^3$ to $\mathbb{R}^{1,3}$ simply by coordinate change

Behavior at the boundary $\cos\tau = \omega_4$ yields fall-off properties at $t \to \pm\infty$

To become explicit, need the one-forms $e^0 \equiv \mathrm{d}\tau$ and e^a, subject to

$$\mathrm{d}e^a + \varepsilon^a{}_{bc}\, e^b \wedge e^c \;=\; 0 \qquad \text{and} \qquad e^a e^a \;=\; \mathrm{d}\Omega_3^2$$

In terms of S^3 coordinates $(a,i,j,k = 1,2,3)$:

$$e^a \;=\; -\eta^a_{BC}\,\omega_B\,\mathrm{d}\omega_C \qquad \text{where} \quad \eta^i_{jk} = \varepsilon^i{}_{jk} \quad \text{and} \quad \eta^i_{j4} = -\eta^i_{4j} = \delta^i_j$$

In terms of Minkowski coordinates (calculation!):

$$e^0 \;=\; \frac{\gamma^2}{\ell^3}\Bigl(\tfrac12(t^2+r^2+\ell^2)\,\mathrm{d}t - t\,x^k\mathrm{d}x^k\Bigr)$$

$$e^a \;=\; \frac{\gamma^2}{\ell^3}\Bigl(t\,x^a\mathrm{d}t - \bigl(\tfrac12(t^2-r^2+\ell^2)\,\delta^a_k + x^a x^k + \ell\,\varepsilon^a{}_{jk}x^j\bigr)\,\mathrm{d}x^k\Bigr)$$

with notation $(x^i) = (x,y,z)$ and (for later) $(x^\mu) = (x^0, x^i) = (t,x,y,z)$

Simplest YM solutions: impose SO(4) symmetry $\Rightarrow \quad X_a(\tau,\omega) = X_a(\tau)$

YM equations become ordinary matrix differential equations:

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} X_a \;=\; -4\,X_a + 3\,\varepsilon_{abc}\,[X_b, X_c] - \Big[X_b, [X_a, X_b]\Big] \qquad \text{and} \qquad \Big[\frac{\mathrm{d}}{\mathrm{d}\tau} X_a, X_a\Big] \;=\; 0$$

Known analytic non-Abelian solutions, for $\mathfrak{g} = su(2)$: $\qquad [T_a, T_b] = 2\varepsilon_{abc} T_c$

$$X_a(\tau) \;=\; q(\tau)\, T_a \qquad \text{with} \qquad \frac{\mathrm{d}^2 q}{\mathrm{d}\tau^2} \;=\; -\frac{\partial V}{\partial q} \qquad \text{for} \qquad V(q) \;=\; 2\, q^2 (q-1)^2$$

Double-well potential: vacua $q = 0$ or 1, sphaleron $q = \frac{1}{2}$, bounce, ...

$$\Rightarrow \qquad A \;=\; q(\tau)\, g^{-1} \mathrm{d} g \qquad \text{for} \quad g : S^3 \to \mathrm{SU}(2)$$

Nontrivial static homogeneous solution (on cylinder) is $\quad A = \frac{1}{2} T_a e^a = \frac{1}{2} g^{-1} \mathrm{d} g$

Translates to finite-action homogeneous color-magnetic YM solution on dS_4

Known analytic Abelian solutions, $\mathfrak{g} = \mathbb{R}$:

$$X_a(\tau) \;\Rightarrow\; X_a(\tau)\,T_3 \qquad \text{with} \qquad \frac{\mathrm{d}^2 X_a}{\mathrm{d}\tau^2} \;=\; -4\,X_a$$

Harmonic motion: oscillations $X_a(\tau) \;=\; c_a \cos\big(2(\tau-\tau_a)\big)$

From now on focus on Maxwell's equations $\Leftrightarrow$ $X_a(\tau,\omega)$ are real-valued functions

Task: transfer oscillatory cylinder solutons to Minkowski space

Helpful: $\exp(2\mathrm{i}\tau)$ is a rational function of t and r $\qquad x \equiv \{x^\mu\}$

$$A \;=\; X_a(\tau(x))\,e^a(x) \;=\; A_\mu(x)\,\mathrm{d}x^\mu \qquad \text{yields } A_\mu(x) \qquad A_t \neq 0$$

$$\mathrm{d}A \;=\; \frac{\mathrm{d}}{\mathrm{d}\tau}X_a\,e^0 \wedge e^a - \varepsilon^a{}_{bc}X_a\,e^b \wedge e^c \;=\; \tfrac{1}{2}F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \qquad \text{yields } F_{\mu\nu}(x)$$

and hence electric and magnetic fields $E_i \;=\; F_{i0}$ and $B_i \;=\; \tfrac{1}{2}\varepsilon_{ijk}F_{jk}$

An example (putting $\ell = 1$):

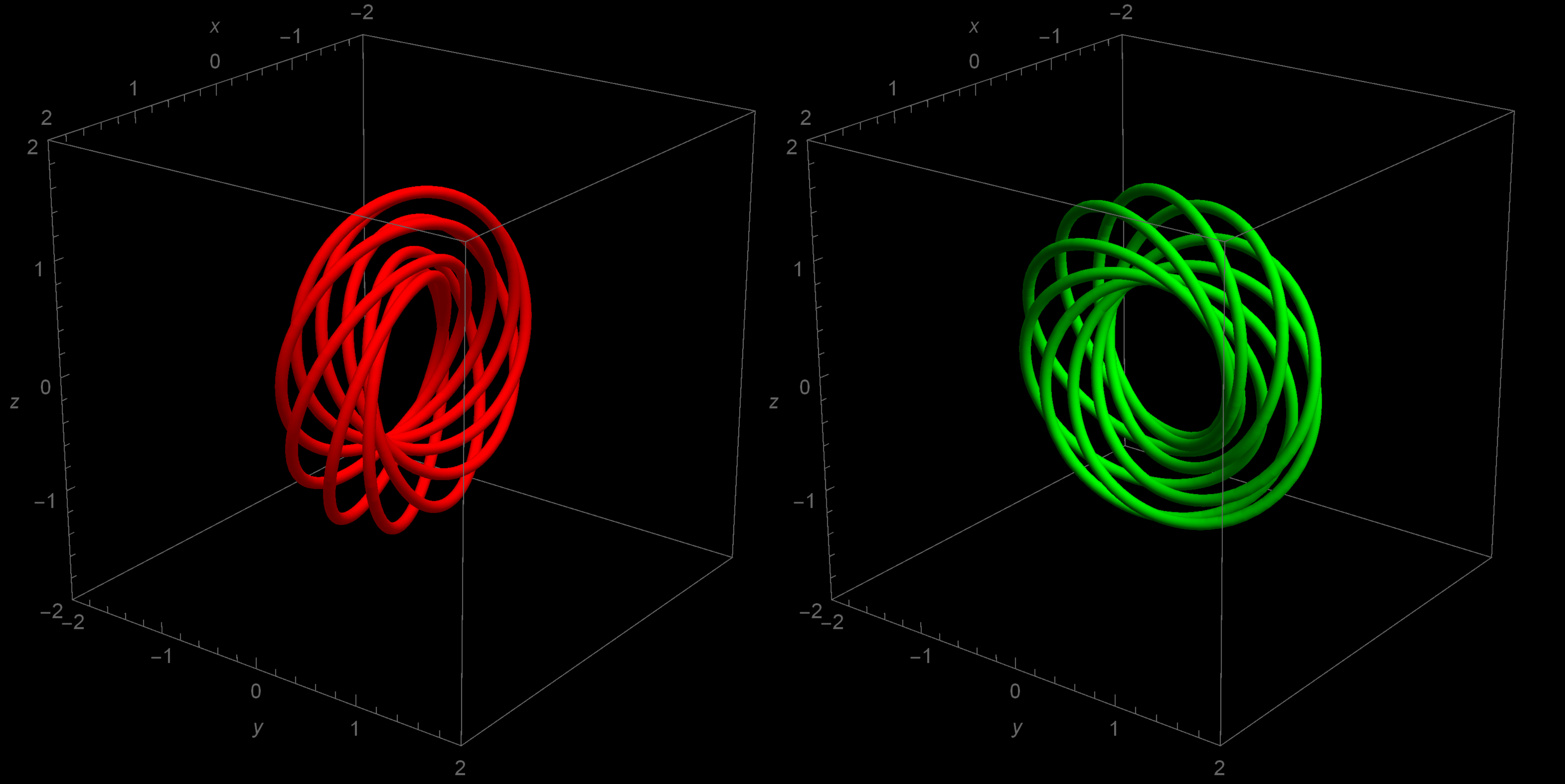
$$X_1(\tau) \;=\; -\tfrac{1}{8}\sin 2\tau\,, \qquad X_2(\tau) \;=\; -\tfrac{1}{8}\cos 2\tau\,, \qquad X_3(\tau) \;=\; 0$$

Result of short computation:

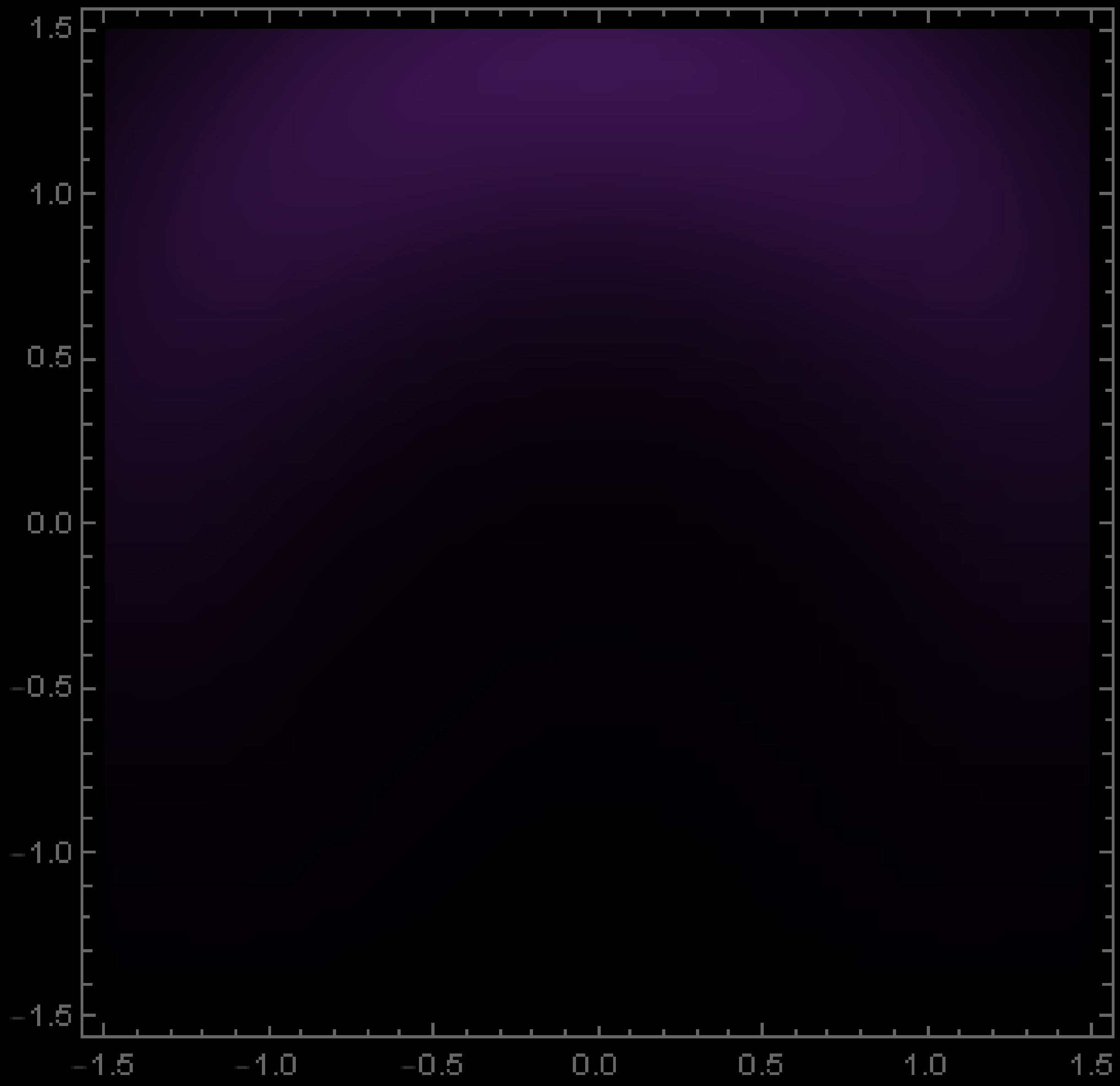
$$\vec{E} + \mathrm{i}\vec{B} \;=\; \frac{1}{\big((t-\mathrm{i})^2 - r^2\big)^3} \begin{pmatrix} (x-\mathrm{i}y)^2 - (t-\mathrm{i}-z)^2 \\ \mathrm{i}(x-\mathrm{i}y)^2 + \mathrm{i}(t-\mathrm{i}-z)^2 \\ -2\,(x-\mathrm{i}y)\,(t-\mathrm{i}-z) \end{pmatrix}$$

This is the celebrated Hopf–Rañada electromagnetic knot.

Our approach also yields its gauge potential.



Some magnetic (red) and electric (green) field lines at $t=0$



Energy density in $y=0$ plane, changing with time

Construction of electromagnetic solutions

Admit arbitrary SO(4)-non-symmetric solutions $\Rightarrow$ $X_a = X_a(\tau,\omega)$

But capture the ω-dependence in an SO(4)-covariant fashion!

Ingredients:

Left-invariant right multiplication: $R_a = -\eta^a_{BC}\,\omega_B \dfrac{\partial}{\partial \omega_C}$ $\Rightarrow$ $[R_a, R_b] = 2\,\varepsilon_{abc}\,R_c$

Right-invariant left multiplication: $L_a = -\tilde{\eta}^a_{BC}\,\omega_B \dfrac{\partial}{\partial \omega_C}$ $\Rightarrow$ $[L_a, L_b] = 2\,\varepsilon_{abc}\,L_c$

Dual to the corresponding one-forms, e.g. $e^a(R_b) = \delta^a_b$, and commute, $[R_a, L_b] = 0$

An arbitrary function Φ on S^3 obeys $\mathrm{d}\Phi \;=\; e^a\,R_a\Phi$

Space of functions on S^3 decomposes into irreps of $su(2)_L \oplus su(2)_R$ labelled by a common spin $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$

Define hermitian "angular momenta" $I_a := \frac{\mathrm{i}}{2} L_a$ and $J_a := \frac{\mathrm{i}}{2} R_a$

A basis of hyperspherical harmonics

$Y_{j;m,n}(\omega)$ with $m, n = -j, -j+1, \ldots, +j$ and $2j = 0, 1, 2, \ldots$

is specified by the relations

$$I^2 \, Y_{j;m,n} = J^2 \, Y_{j;m,n} = j(j+1) \, Y_{j;m,n} \quad ,$$

$$I_3 \, Y_{j;m,n} = m \, Y_{j;m,n} \quad \text{and} \quad J_3 \, Y_{j;m,n} = n \, Y_{j;m,n}$$

For an explicit construction, introduce two complex coordinates

$\alpha = \omega_1 + \mathrm{i}\omega_2$ and $\beta = \omega_3 + \mathrm{i}\omega_4$ subject to $\bar\alpha\alpha + \bar\beta\beta = 1$

Angular momenta generators read

$$
\begin{aligned}
I_+ &= (\bar\beta\partial_{\bar\alpha} - \alpha\partial_\beta)/\sqrt{2}\,, & J_+ &= (\beta\partial_{\bar\alpha} - \alpha\partial_{\bar\beta})/\sqrt{2} \\
I_3 &= (\alpha\partial_\alpha + \bar\beta\partial_{\bar\beta} - \bar\alpha\partial_{\bar\alpha} - \beta\partial_\beta)/2\,, & J_3 &= (\alpha\partial_\alpha + \beta\partial_\beta - \bar\alpha\partial_{\bar\alpha} - \bar\beta\partial_{\bar\beta})/2 \\
I_- &= (\bar\alpha\partial_{\bar\beta} - \beta\partial_\alpha)/\sqrt{2}\,, & J_- &= (\bar\alpha\partial_\beta - \bar\beta\partial_\alpha)/\sqrt{2}
\end{aligned}
$$

Normalized hyperspherical harmonics are represented as

$$
Y_{j;m,n} = \sqrt{\frac{2j+1}{2\pi^2}}\sqrt{\frac{2^{j-m}(j+m)!}{(2j)!\,(j-m)!}\frac{2^{j-n}(j+n)!}{(2j)!\,(j-n)!}}\,(I_-)^{j-m}(J_-)^{j-n}\,\alpha^{2j}
$$

$\Rightarrow$ homogenous polynomials of degree $2j$ in $\{\alpha, \bar\alpha, \beta, \bar\beta\}$

Set up a left-invariant formulation

General Maxwell gauge potential on $2\mathcal{I} \times S^3$:

$$A \;=\; X_0(\tau,\omega)\,\mathrm{d}\tau + X_a(\tau,\omega)\,e^a$$

Coulomb gauge: $X_0(\tau,\omega) = 0$ and $J_a\,X_a(\tau,\omega) = 0$

Maxwell's equations are coupled wave equations:

$$-\tfrac14\,\partial_\tau^2 X_a \;=\; (J^2{+}1)\,X_a + \mathrm{i}\,\varepsilon_{abc}J_b\,X_c$$

More transparent rewriting: $X_\pm = (X_1 \pm \mathrm{i}X_2)/\sqrt{2}$

$$\begin{aligned}
-\tfrac14\,\partial_\tau^2 X_+ &= (J^2 - J_3 + 1)\,X_+ + J_+X_3 \\
-\tfrac14\,\partial_\tau^2 X_3 &= (J^2 + 1)\,X_3 - J_+X_- + J_-X_+ \\
-\tfrac14\,\partial_\tau^2 X_- &= (J^2 + J_3 + 1)\,X_- - J_-X_3 \\
0 &= J_3X_3 + J_+X_- + J_-X_+
\end{aligned}$$

Expand in our basis of hyperspherical harmonics:

$$X_a(\tau,\omega) \;=\; \sum_{jmn} X_a^{j;m,n}(\tau)\, Y_{j;m,n}(\alpha,\beta)$$

From the form of the equations it is obvious that

- the equations are diagonal in j and m, so these may be kept fixed
- they only couple triplets $(X_3^{j;m,n}, X_+^{j;m,n+1}, X_-^{j;m,n-1})$,

  so $X_\pm \propto J_\pm X_3$ for $X_3 \propto Y_{j;m,n}$
- the ansatz $X_a^{j;m,n}(\tau) \propto \mathrm{e}^{\mathrm{i}\Omega_a^{j;n}\tau} c_a^{j;n}$ gives a linear system for $\Omega_a^{j;n}$ and $c_a^{j;n}$

The frequencies turn out to be integral: $\Omega_a^{j;n} \;=\; \pm 2(j{+}1)$ or $\pm 2j$

Two types of solutions:

- type I : $j \geq 0\,, \quad m = -j, \ldots, +j\,, \quad n = -j-1, \ldots, j+1\,, \quad \Omega^j = \pm 2(j+1)$

$$
\begin{aligned}
X_+ &= \sqrt{(j-n)(j-n+1)/2}\, \mathrm{e}^{\pm 2(j+1)\mathrm{i}\tau}\, Y_{j;m,n+1} \\
X_3 &= \sqrt{(j+1)^2 - n^2}\, \mathrm{e}^{\pm 2(j+1)\mathrm{i}\tau}\, Y_{j;m,n} \\
X_- &= -\sqrt{(j+n)(j+n+1)/2}\, \mathrm{e}^{\pm 2(j+1)\mathrm{i}\tau}\, Y_{j;m,n-1}
\end{aligned}
$$

- type II : $j \geq 1\,, \quad m = -j, \ldots, +j\,, \quad n = -j+1, \ldots, j-1\,, \quad \Omega^j = \pm 2j$

$$
\begin{aligned}
X_+ &= -\sqrt{(j+n)(j+n+1)/2}\, \mathrm{e}^{\pm 2j\,\mathrm{i}\tau}\, Y_{j;m,n+1} \\
X_3 &= \sqrt{j^2 - n^2}\, \mathrm{e}^{\pm 2j\,\mathrm{i}\tau}\, Y_{j;m,n} \\
X_- &= \sqrt{(j-n)(j-n+1)/2}\, \mathrm{e}^{\pm 2j\,\mathrm{i}\tau}\, Y_{j;m,n-1}
\end{aligned}
$$

Some properties of the solutions

Each complex solution yields two real ones (real part and imaginary part)

Count for j fixed:

$2(2j{+}1)(2j{+}3)$ type-I solutions and $2(2j{+}1)(2j{-}1)$ type-II solutions ($j{>}0$)

together: $4(2j{+}1)^2$ solutions for $j{>}0$ and 6 solutions for $j{=}0$

Constant solutions ($\Omega = 0$) are not allowed; simplest are $j{=}0$ type I (Hopf–Rañada)

Spin j type I $\longleftarrow$ parity $(L \leftrightarrow R, m \leftrightarrow n)$ $\longrightarrow$ spin $j{+}1$ type II

Electromagnetic duality: shifting $|\Omega^j|\tau$ by $\pm\frac{\pi}{2}$ yields a dual solution A_D

Main technical task:

transform a chosen solution on $2\mathcal{I} \times S^3$ to Minkowski coordinates (t, x, y, z),

straightforward due to explicit formulæ for all ingredients $\Rightarrow$ only rational functions

Helicity $h = \frac{1}{2} \int\limits_{\mathbb{R}^3} \left(A \wedge F + A_D \wedge F_D \right)$ is conserved

Energy $E = \frac{1}{2} \int\limits_{\mathbb{R}^3} \mathrm{d}^3x \left(\vec{E}^2 + \vec{B}^2 \right)$ is conserved and related to h

Best computed in "sphere frame" at $t = \tau = 0$: $F = \mathcal{E}_a \, e^a \wedge e^0 + \frac{1}{2} \mathcal{B}_a \, \varepsilon^a{}_{bc} \, e^b \wedge e^c$

$$\int\limits_{\mathbb{R}^3} \mathrm{d}^3x \, \vec{E}^2 = \frac{1}{\ell} \int\limits_{S^3} \mathrm{d}^3\Omega_3 \, (1-\omega_4) \, \mathcal{E}_a \mathcal{E}_a \quad \text{and} \quad \int\limits_{\mathbb{R}^3} \mathrm{d}^3x \, \vec{B}^2 = \frac{1}{\ell} \int\limits_{S^3} \mathrm{d}^3\Omega_3 \, (1-\omega_4) \, \mathcal{B}_a \mathcal{B}_a$$

exploiting orthogonality properties of the hyperspherical harmonics

Examples

Example 1: $(j;m,n)=(1,0,0)$, type I, combine $\mathrm{e}^{4\mathrm{i}\tau}+\mathrm{e}^{-4\mathrm{i}\tau}=2\cos 4\tau$

$$X_+ \;=\; -\tfrac{\sqrt{3}}{\pi}\,\alpha\beta\;\cos 4\tau\,, \quad X_3 \;=\; \tfrac{\sqrt{6}}{\pi}\,(\beta\bar\beta-\alpha\bar\alpha)\;\cos 4\tau\,, \quad X_- \;=\; -\tfrac{\sqrt{3}}{\pi}\,\bar\alpha\bar\beta\;\cos 4\tau$$

$$\Rightarrow \qquad h \;=\; 12 \qquad \text{and} \qquad E \;=\; 48/\ell$$

Example 2: $(j;m,n)=(2;1,-1)$, type I $\qquad E=6\,h/\ell$

Example 3: $(j;m,n)=(\frac{5}{2};\frac{3}{2},\frac{1}{2})$, type I $\qquad E=7\,h/\ell$

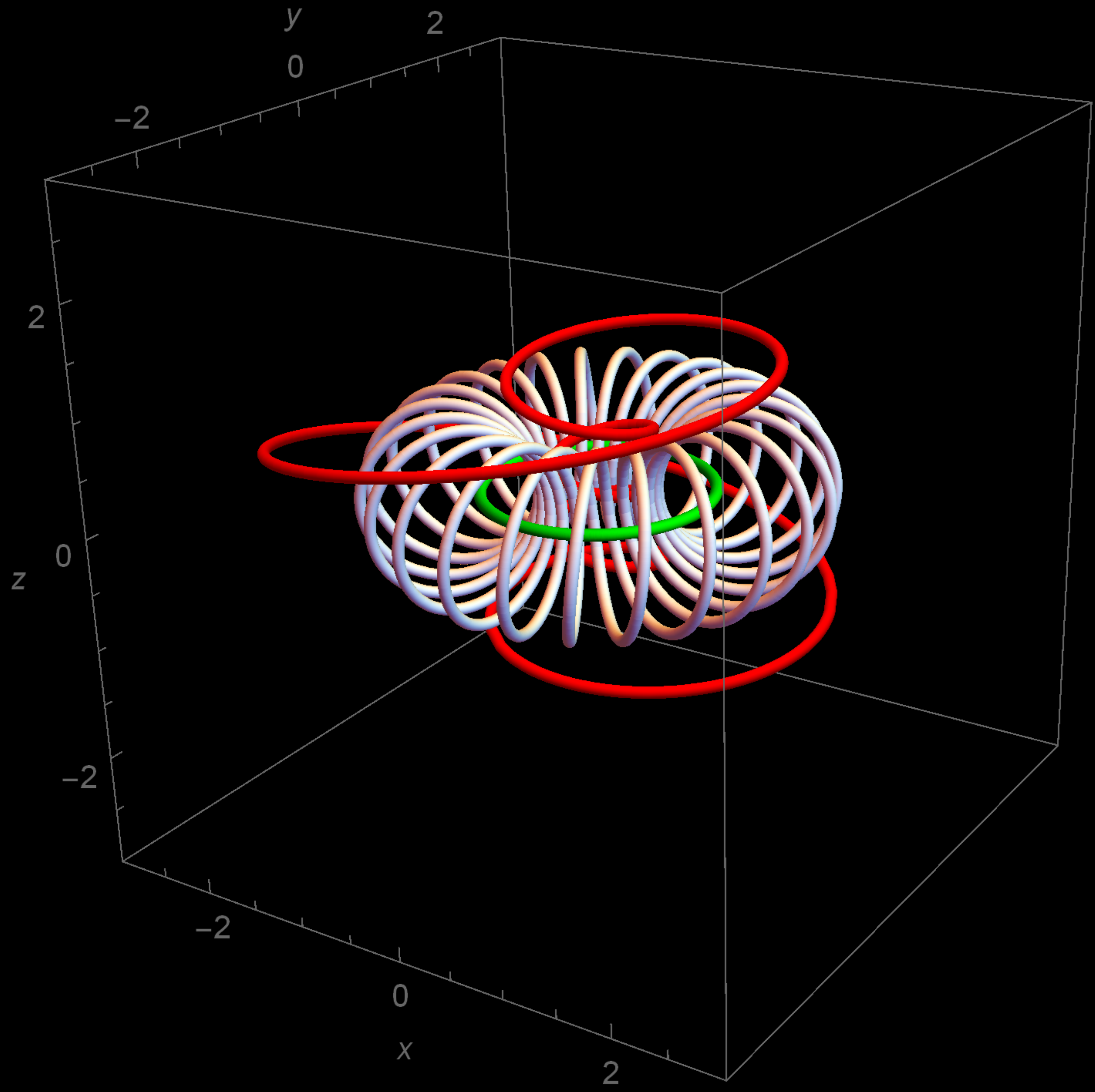
Example 4: $(j;m,n)=(2;1,2)$, type I $\qquad E=6\,h/\ell$

Example 1

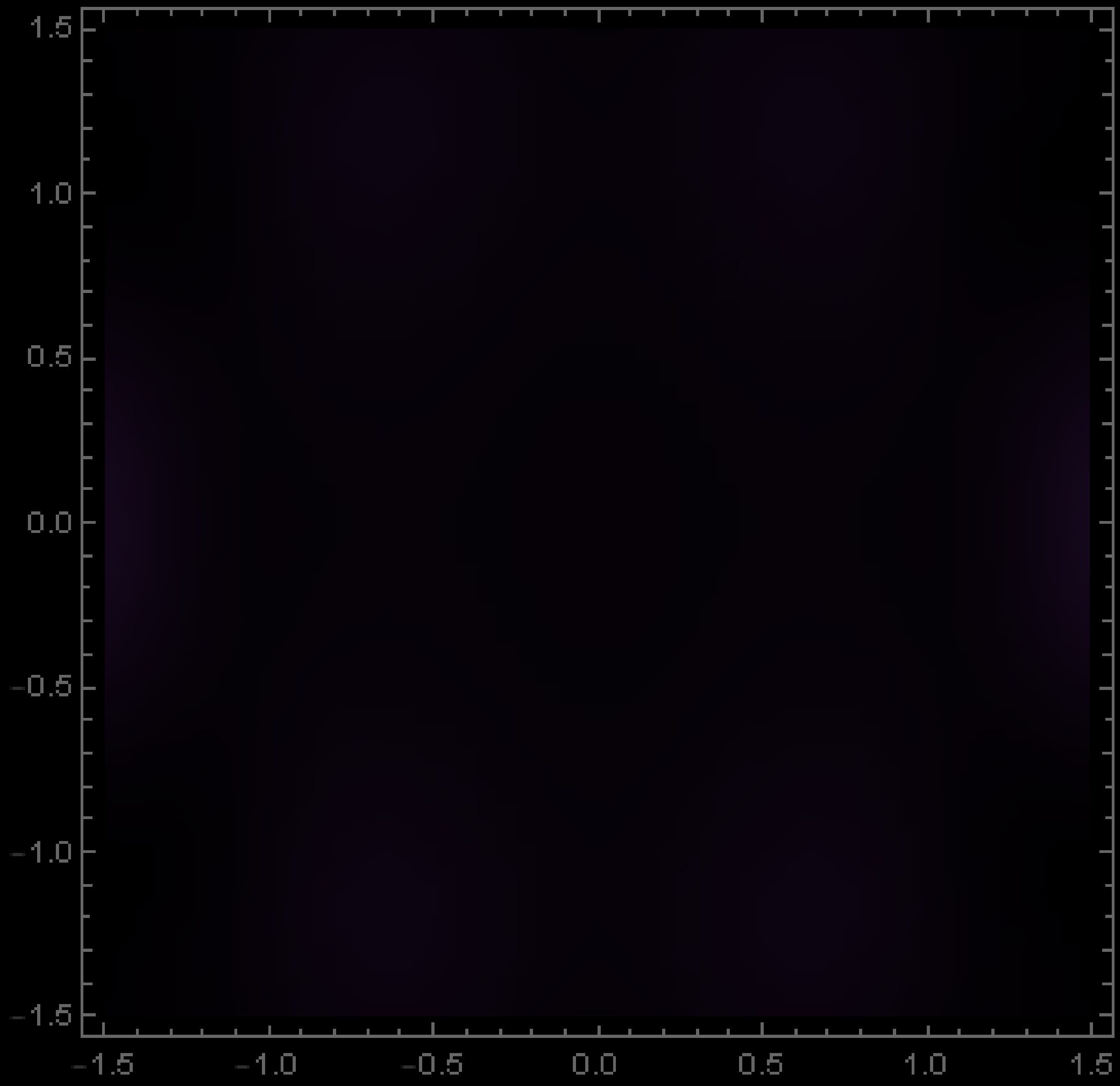
$$(E+\mathrm{i}B)_x = \frac{-2\mathrm{i}}{((t-\mathrm{i})^2-x^2-y^2-z^2)^5} \times$$
$$\times \Big\{2y+3\mathrm{i}ty-xz+2t^2y+2\mathrm{i}txz-8x^2y-8y^3+4yz^2$$
$$+\ 4\mathrm{i}t^3y-6t^2xz-8\mathrm{i}tx^2y-8\mathrm{i}ty^3+4\mathrm{i}tyz^2+10x^3z+10xy^2z-2xz^3$$
$$+\ 2(\mathrm{i}txz+x^2y+y^3+yz^2)(-t^2+x^2+y^2+z^2)+(\mathrm{i}ty-xz)(-t^2+x^2+y^2+z^2)^2\Big\}$$

$$(E+\mathrm{i}B)_y = \frac{2\mathrm{i}}{((t-\mathrm{i})^2-x^2-y^2-z^2)^5} \times$$
$$\times \Big\{2x+3\mathrm{i}tx+yz+2t^2x-2\mathrm{i}tyz-8x^3-8xy^2+4xz^2$$
$$+\ 4\mathrm{i}t^3x+6t^2yz-8\mathrm{i}tx^3-8\mathrm{i}txy^2+4\mathrm{i}txz^2-10x^2yz-10y^3z+2yz^3$$
$$+\ 2(-\mathrm{i}tyz+x^3+xy^2+xz^2)(-t^2+x^2+y^2+z^2)+(\mathrm{i}tx+yz)(-t^2+x^2+y^2+z^2)^2\Big\}$$

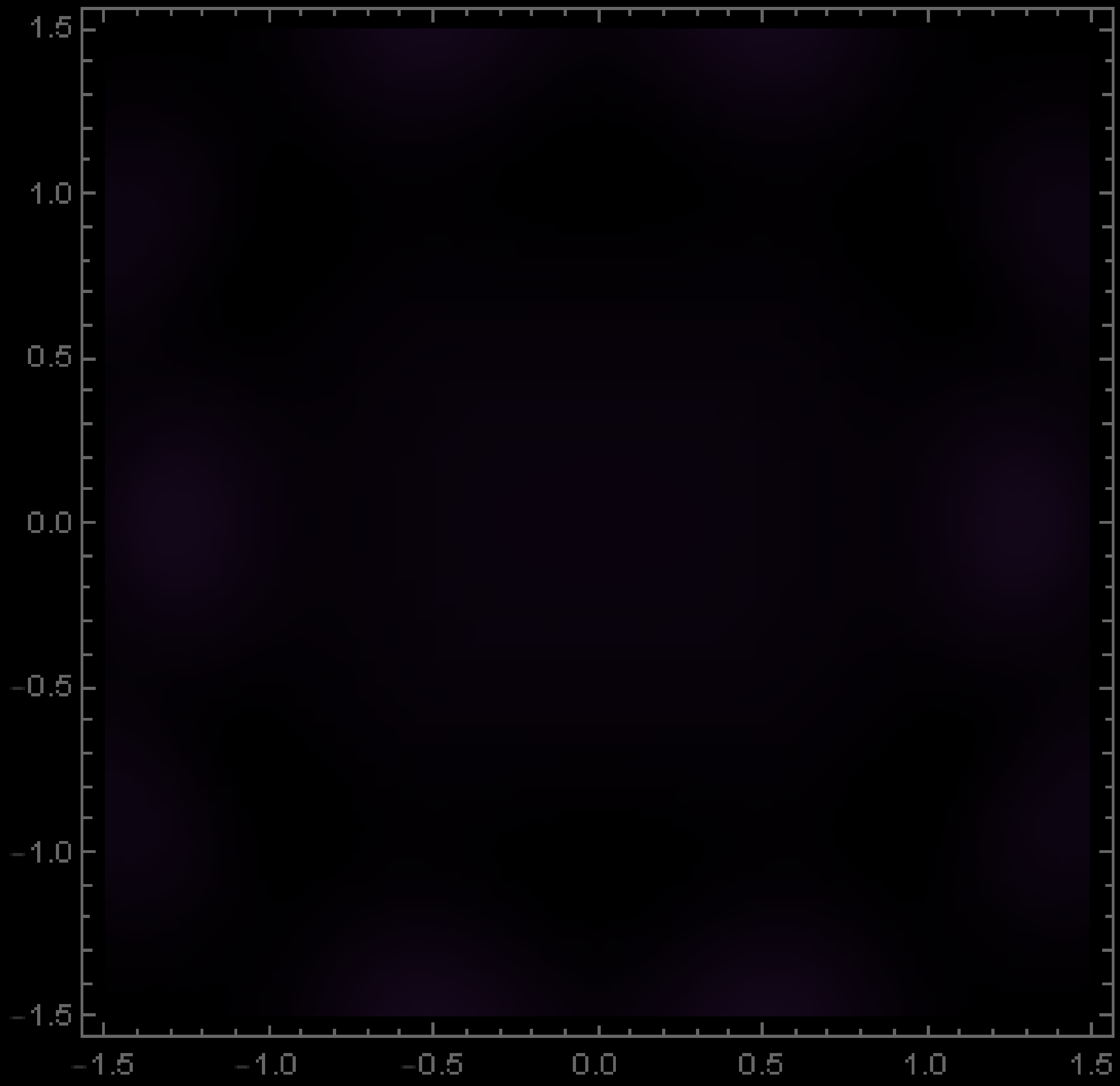
$$(E+\mathrm{i}B)_z = \frac{\mathrm{i}}{((t-\mathrm{i})^2-x^2-y^2-z^2)^5} \times$$
$$\times \Big\{1+2\mathrm{i}t+t^2-11x^2-11y^2+3z^2+4\mathrm{i}t^3-16\mathrm{i}tx^2-16\mathrm{i}ty^2+4\mathrm{i}tz^2$$
$$-\ t^4-2t^2x^2-2t^2y^2-2t^2z^2+11x^4+22x^2y^2+10x^2z^2+11y^4-10y^2z^2+3z^4$$
$$+\ 2\mathrm{i}t(t^2-3x^2-3y^2-z^2)(t^2-x^2-y^2-z^2)-(t^2+x^2+y^2-z^2)(-t^2+x^2+y^2+z^2)^2\Big\}$$



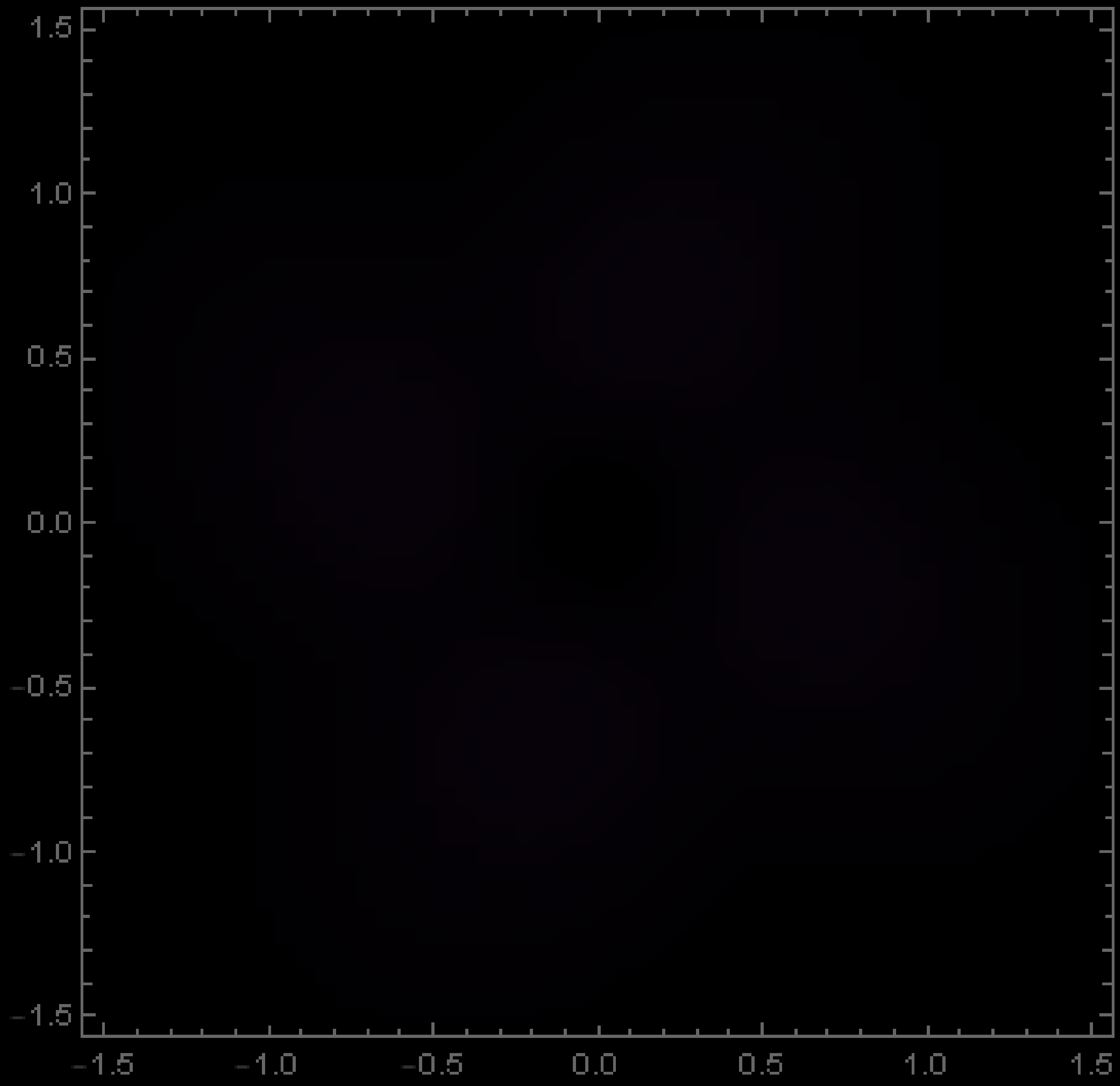
$(j;m,n)=(1;0,0)$: three particular magnetic field lines at $t=0$



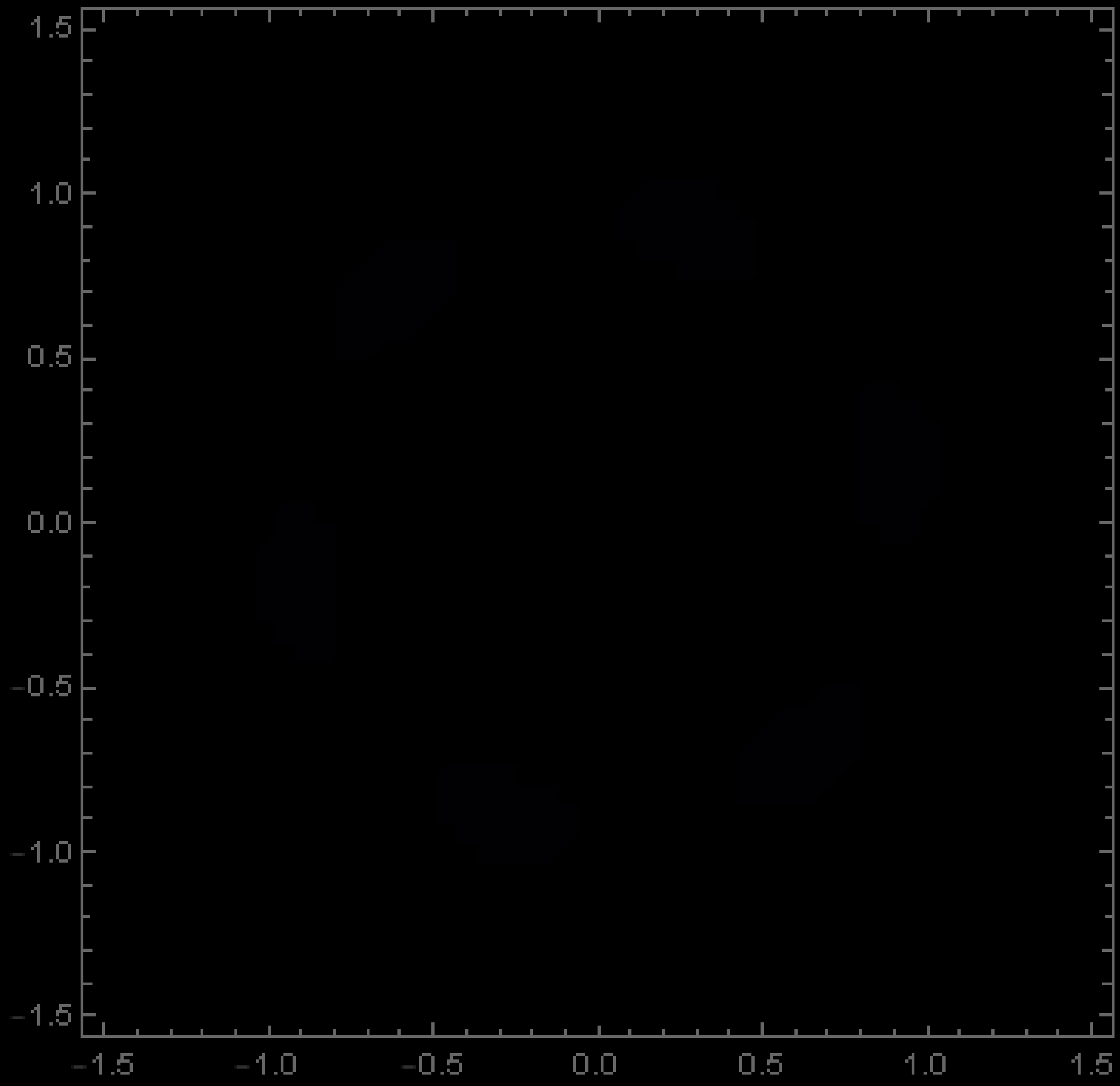
$(j; m, n) = (1; 0, 0)$: energy density in y=0 plane, changing with time



$(j; m, n) = (2; 1, -1)$: energy density in y=0 plane, changing with time



$(j;m,n) = (\frac{5}{2};\frac{3}{2},\frac{1}{2})$: energy density at t=0, scanning z = const planes



$(j;m,n) = (2;1,2)$: energy density at $t{=}0$, scanning z = const planes

Summary and discussion

- Rational electromagnetic fields with nontrivial topology investigated since 1989
- We introduced a new construction method based on two insights:
  - simplicity of solving Maxwell on a temporal cylinder over a three-sphere
  - conformal equivalence of a cylinder patch to four-dimensional Minkowski space
- $A = X_\nu(\tau,\omega)\, e^\nu = X_\nu(\tau(x),\omega(x))\, e^\nu{}_\mu(x)\, \mathrm{d}x^\mu$
- Only finite-time $\tau \in (-\pi, +\pi)$ dynamics is required on the cylinder
- Our solutions have finite energy and action, by construction
- A complete basis for sufficiently fast spatially and temporally decaying fields
- Non-Abelian extension couples different j components of $X_a \quad \Rightarrow \quad$ much harder
- Useful tool for numerical study of classical YM dynamics in Minkowski space

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THANK YOU !

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