SKT and Generalised Geometry

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Complex Geometry

Manifold (M^{2d}, J)

Complex structure: $J \in End(TM) \qquad J^2 = -1$

Projectors: $\quad \pi_\pm := \frac{1}{2}(\mathbf{1} \pm iJ)$

These define an involutive distribution if

$$\pi_\mp[\pi_\pm u, \pi_\pm v] = 0 \quad \Longleftrightarrow \quad \mathcal{N}(J) = 0 \text{ (Nijenhuis)}$$

This is integrability of J.
Local holomorphic coordinates, $M^{2d} \supset \mathcal{O} \approx \mathbb{C}^{2k}$, and holomorphic transition functions.

Hermitean

Hermitean Metric: $J^t g J = g$

$(\tilde{g} \to g = \tilde{g} + J^t \tilde{g} J)$

Symplectic 2-form: $\omega := gJ$

Kähler: $d\omega = 0, \quad \nabla J = 0, \qquad g_{z\bar{z}} = \partial_z \partial_{\bar{z}} K(z, \bar{z})$

Generalized Complex Geometry

Complex structure:

N. Hitchin 2002
M. Gualtieri 2003

$$\mathcal{J} \in End(TM \oplus T^*M), \qquad \mathcal{J}^2 = -1$$

$$\Pi_\pm := \tfrac{1}{2}(\mathbf{1} \pm i\mathcal{J})$$

“Nijenhuis”:

$$\mathcal{N}_C(\mathcal{J}) = 0 \quad \Longleftrightarrow \quad \Pi_\mp[\Pi_\pm U, \Pi_\pm V]_C = 0$$

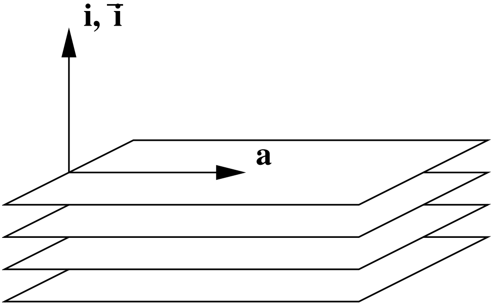
where

$$U = \begin{pmatrix} u \\ \xi \end{pmatrix}, \quad V = \begin{pmatrix} v \\ \rho \end{pmatrix}$$

$$[U, V]_C = [u, v] + \mathcal{L}_u\rho - \mathcal{L}_v\xi - \tfrac{1}{2}d(\imath_u\rho - \imath_v\xi)$$

"Newlander-Nirenberg"

When $\mathcal{J}$ is integrable there are local holomorphic and Darboux coordinates such that M^{2d} looks like $\mathbb{C}^k \times \mathbb{R}^{2d-k}$.



The automorphisms of this courant bracket are diffeomorphisms and *b-transforms*:

$$e^b \begin{pmatrix} u \\ \xi \end{pmatrix} = \begin{pmatrix} u \\ \xi + \imath_u b \end{pmatrix} , \quad db = 0 \; .$$

In a coordinate basis (∂_x, dx) a b-transform acts on $\mathcal{J}$ as follows:

$$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mathcal{J} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} ,$$

In such a basis, the natural pairing

$$< (u, \xi), (v, \rho) >= \frac{1}{2}(\imath_u \rho + \imath_v \xi)$$

is represented by the matrix

$$\mathcal{I} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

A final requirement of GCG is that

$$\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I}$$

Generalized Kähler Geometry

Two commuting generalized complex structures

$$\mathcal{J}^2_{(1,2)} = -\mathbf{1}\ , \qquad [\mathcal{J}_{(1)}, \mathcal{J}_{(2)}] = 0\ ,$$

$$\mathcal{J}^t_{(1,2)}\mathcal{I}\mathcal{J}_{(1,2)} = \mathcal{I}\ , \qquad \mathcal{G} := -\mathcal{J}_{(1)}\mathcal{J}_{(2)}$$

Ex. Kähler $(\omega = gJ)$:

$$\mathcal{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix} \quad \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \quad \mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

Relation to $2D$ sigma models

$$\mathcal{S} = \int d^2x \mathbb{D}_+ \mathbb{D}_- \left(\mathbb{D}_+ \varphi^i (g_{ij} + B_{ij}) \mathbb{D}_- \varphi^j \right)$$

Ansatz for the extra supersymmetries:

$$\delta\varphi^i = \epsilon^+ J^i_{(+)j} \mathbb{D}_+ \varphi^j + \epsilon^- J^i_{(-)j} \mathbb{D}_- \varphi^j$$

Gates-Hull-Roček=Bihermitean=Generalised Kähler.

J.Gates, C. Hull, M Roček 1984

$$\boxed{\begin{array}{c} J^2_{(\pm)} = -\mathbf{1}\ , \quad J^t_{(\pm)} g J_{(\pm)} = g\ , \quad \nabla^{(\pm)} J_{(\pm)} = 0 \\[2ex] \Gamma^{(\pm)} = \Gamma^0 \pm \tfrac{1}{2} g^{-1} H\ , \quad H = dB\ . \end{array}}$$

$$(M, g, J_{(\pm)}, H)$$

The G-map

$$\mathcal{J}^{(1,2)} =$$

$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\begin{pmatrix} J^{(+)} \pm J^{(-)} & -(\omega_{(+)}^{-1} \mp \omega_{(-)}^{-1}) \\ \omega_{(+)} \mp \omega_{(-)} & -(J^{t(+)} \pm J^{t(-)}) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}$$

M. Gualtieri 2003

SKT from $(2,0)$ and $(2,1)$ sigma models

C.Hull 1986
P.Howe and G. Papadopoulos 1996

$$\mathcal{S} = \int d^2x \mathbb{D}_+\mathbb{D}_- \left(\mathbb{D}_+\varphi^i (g_{ij} + B_{ij})\mathbb{D}_-\varphi^j\right)$$

Ansatz for the extra supersymmetry:

$$\delta\varphi^i = \epsilon J^i{}_j \mathbb{D}_+\varphi^j$$

$$\boxed{\begin{array}{c} J^2 = -\mathbf{1}\ , \quad J^t g J = g\ , \quad \nabla^{(+)} J = 0 \\ \Gamma^{(+)} = \Gamma^0 + \tfrac{1}{2} g^{-1} H\ , \quad H = dB\ . \end{array}}$$

(M,g,J, H)

Generalised metric: G. Cavalcanti 2012

$$\mathcal{G} \in \mathbb{T} = End(TM \oplus T^*M), \qquad \mathcal{G}^2 = +1$$

$$\Pi^G_{\pm} := \tfrac{1}{2}\left(\mathbf{1} \pm \mathcal{G}\right)$$

$$\mathcal{G} \to \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

Metric split into ± 1 orthogonal eigenspaces:

$$\Pi^G_{\mp}\mathbb{T}_{\pm} = 0\ , \quad \mathbb{T} = \mathbb{T}_{+} \oplus \mathbb{T}_{-}\ ,$$

Focus on $\mathbb{T}_+$ and consider a map $\mathcal{J}_+$ acting on $\mathbb{T}_+$, and satisfying

$$\begin{aligned}
&\mathcal{J}_+^2 = -\Pi_+^G \\
&[\mathcal{J}_+, \mathcal{G}] = 0 \\
&(\Pi_+^G)^t \mathcal{J}_+^t \mathcal{I} \mathcal{J}_+ \Pi_+^G = (\Pi_+^G)^t \mathcal{I} \Pi_+^G \ .
\end{aligned} \tag{0.1}$$

$\mathcal{J}_+$ may be extended to an almost complex structure on all of $\mathbb{T}$ by requiring that it vanishes on $\mathbb{T}_-$. Since then $\mathcal{J}_+\Pi_-^G = 0$, we write it as

$$\mathcal{J}_+ = \mathcal{J}\Pi_+^G \ ,$$

where $\mathcal{J}$ is an almost complex structure on $\mathbb{T}$. The second condition in (0.1) implies the generalised Hermiticity condition

$$[\mathcal{J}, \mathcal{G}] = 0 \ ,$$

and the first and third conditions in (0.1) then also hold.

Solving the constraints

We parametrise the complex structure as:

$$\mathcal{J} = \begin{pmatrix} I & P \\ L & K \end{pmatrix}$$

The conditions $\mathcal{J}^2 = -1$, $\mathcal{J}^t\mathcal{I} = -\mathcal{I}\mathcal{J}$ and $[\mathcal{J}, \mathcal{G}] = 0$, give nine matrix equations in I , P , L and P.
Their solution may be summarised by saying that $\mathcal{J}$ can be written as follows:

$$\mathcal{J} = \begin{pmatrix} J \mp Pg & P \\ gPg & -J^t \pm gP \end{pmatrix}$$

where P is antisymmetric and J is an almost complex structure on $\mathcal{M}$ that preserves the metric:

$$P = -P^t \ , \quad J = I \pm Pg \ , \quad J^2 = -1 \ , \quad J^t g J = g \ .$$

It follows that the complex structure $\mathcal{J}_+$ on $\mathbb{T}_+$ is

$$\mathcal{J}_+ = \mathcal{J}\Pi_+^G = \Pi_+^G \mathcal{J}\Pi_+^G = \tfrac{1}{2}\begin{pmatrix} J & -(\omega)^{-1} \\ \omega & -J^t \end{pmatrix}$$

where $\omega := gJ$. This is the geometric field content of SKT.

Projection operators corresponding to $\mathcal{J}_+$ are

$$\Pi_\pm := \tfrac{1}{2}(1 \pm i\mathcal{J}_+) \ .$$

Using them we get a further split

$$\mathbb{T}_+ = \mathbb{T}_+^{(1,0)} \oplus \mathbb{T}_+^{(0,1)}$$

where

$$\mathbb{T}_+^{(1,0)} = \Pi_+ \mathbb{T}_+ \ , \quad \mathbb{T}_+^{(0,1)} = \Pi_- \mathbb{T}_+$$

are $\pm i$-eigenspaces.

We demand integrability on $\mathbb{T}_+^{(1,0)}$ using the H-twisted Courant bracket

$$\llbracket U_1, U_2 \rrbracket_H = [u_1, u_2] + \mathcal{L}_{u_1} d\xi_2 - i_{u_2} d\xi_1 + i_{u_1} i_{u_2} H$$

$$:= \left(\begin{array}{c} [u_1, u_2] \\ \mathcal{L}_{u_1} d\xi_2 - i_{u_2} d\xi_1 + i_{u_1} i_{u_2} H \end{array} \right)$$

with $dH = 0$.

When $U_i \in \mathbb{T}_+^{(1,0)}$,

$$U_i = \Pi_+ \Pi_+^G U_i \ , \ \iff \ U_i = \tfrac{1}{2} \left(\begin{array}{c} p_+ \hat{u}_i \\ g p_+ \hat{u}_i \end{array} \right) .$$

where

$$p_+ := \tfrac{1}{2}(1 + iJ)$$

$$\hat{u}_i := u_i + g^{-1} \xi_i$$

Involution conditions:

$$\Pi_{-}^{G}[\![U_1, U_2]\!]_H = 0 \ ,$$

and

$$\Pi_{-}[\![U_1, U_2]\!]_H = 0$$

to stay in $\mathbb{T}_{+}^{(1,0)}$.

These conditions are equivalent to the SKT conditions

$$\boxed{\begin{array}{c} \mathcal{N}(J) = 0 \ , \quad , \quad \nabla^{(+)} J = 0 \\ \Gamma^{(+)} = \Gamma^{0} + \frac{1}{2} g^{-1} H \ , \quad H = dB \ . \end{array}}$$

$$\mathcal{L}k_\alpha \neq 0\ , \quad L := V\tfrac{\partial}{\partial\bar\varphi^{\bar\alpha}}\ , \quad i_\xi H = du$$

$$\begin{aligned}
&\mathbb{D}_+\bar{\mathbb{D}}_+ \\
&i\left(k_\alpha D_-\varphi^\alpha - \bar k_{\bar\alpha} D_-\bar\varphi^{\bar\alpha}\right)(\varphi,\bar\varphi) - (A_- - \tfrac{i}{2}D_-V)X(\varphi,\tilde\varphi) \\
&+ V\frac{e^L - 1}{L}\left[(u_\alpha - \tfrac{i}{2}X_{,\alpha})D_-\varphi^\alpha + (\bar u_{\bar\alpha} + \tfrac{i}{2}X_{,\bar\alpha})D_-\bar\varphi^{\bar\alpha}\right] \\
&- \left(\Theta + \bar\Theta\right)A_- - iD_-\bar\Theta V\ .
\end{aligned}$$

This yields the corresponding relations for the dual metric and *b*-field

$$g^D_{\Theta\bar\Theta} = \frac{1}{g_{0\bar 0}}\ , \quad b^D_{\Theta\mu} = \frac{g_{\bar 0\mu}}{g_{0\bar 0}}$$

$$g^D_{\mu\bar\Theta} = \frac{1}{g_{0\bar 0}}[b_{\mu 0} + i\vartheta_\mu] = \frac{-iu_\mu}{g_{0\bar 0}}\ , \quad b^D_{\bar\Theta\bar\mu} = \frac{g_{0\bar\mu}}{g_{0\bar 0}}$$

$$g^D_{\bar\mu\Theta} = \frac{1}{g_{0\bar 0}}[b_{\bar\mu\bar 0} - i\bar\vartheta_{\bar\mu}] = \frac{i\bar u_\mu}{g_{0\bar 0}}\ , \quad b^D_{\mu\nu} = b_{\mu\nu} + \frac{2i}{g_{0\bar 0}} g_{\bar 0[\mu}u_{\nu]}$$

$$g^D_{\mu\bar\mu} = g_{\mu\bar\mu} - \frac{1}{g_{0\bar 0}}\left[g_{\mu\bar 0}g_{\bar\mu 0} - u_\mu\bar u_{\bar\mu}\right]$$

$$b^D_{\bar\mu\bar\nu} = b_{\bar\mu\bar\nu} - \frac{2i}{g_{0\bar 0}} g_{0[\bar\mu}\bar u_{\bar\nu]}$$

T-duality 101

The bosonic nonlinear sigma model

$$\phi^i : \Sigma \to \mathcal{T} \quad : \quad x^\mu \mapsto \phi^i(x)$$

$$\int d^2x \left(g_{ij}(\phi)\partial_{+\!\!+}\phi^i\partial_{=}\phi^j \right)$$

Isometry

$$\partial g_{ij}/\partial\phi^0 = 0$$

$$\int d^2x\,\big(g_{ab}(\phi)\partial_{\#}\phi^a\partial_{=}\phi^b + 2g_{a0}(\phi)\partial_{(\#}\phi^a A_{=)} + g_{00}(\phi)A_{(\#}A_{=)}\big)$$

$$+\int d^2x\tilde{\phi}\partial_{[\#}A_{=]}$$

The dual action

$$\delta A_{\pm} \Rightarrow A_{\pm} = \frac{1}{g_{00}}(\partial_{\pm}\tilde{\phi} - g_{a0}\partial_{\pm}\phi^a)$$

$$\int d^2x \left(g^D_{ij}(\tilde{\phi})\partial_{+\!\!\!+}\tilde{\phi}^i\partial_{=}\tilde{\phi}^j + b^D_{ij}(\tilde{\phi})\partial_{[+\!\!\!+}\tilde{\phi}^i\partial_{=]}\tilde{\phi}^j \right)$$

$$\tilde{\phi}^0 = \tilde{\phi},\ \tilde{\phi}^a = \phi^a$$

The geometries are related by the Buscher rules.
In this case:

$$g^D_{00} = \frac{1}{g_{00}} \; , \quad g^D_{0a} = 0 \; , \quad g^D_{ab} = g_{ab} - \frac{g_{0a} g_{0b}}{g_{00}}$$

$$b^D_{0a} = \frac{g_{0a}}{g_{00}} \; , \quad b^D_{ab} = 0$$

Enter SUSY

Duality in Superspace is interesting because the target space geometry of supersymmetric nonlinear sigma models is, and duality relates different such geometries.

B. Zumino 1979
L. Alvarez-Gaume and D. Freedman 1980
J. Gates, C. Hull and M. Roček 1984
C. Hull and E. Witten 1985

Susy	(1,1)	(2,2)	(2,2)	(4,4)	(4,4)
E=G+B	G, B	G	G, B	G	G, B
Geom	Riem.	Kähler	biherm.	hyperk.	bihyperc.

N.B. I. Bakas three papers from. 1995

Chiral twisted chiral duality for (Generalised) Kähler geometry

$$\{\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm\} = i\partial_{\substack{+\\=}}$$

$$\bar{\mathbb{D}}_\pm \phi = 0 \; , \quad \bar{\mathbb{D}}_+ \chi = \mathbb{D}_- \chi = 0$$

$$K(\phi + \bar{\phi} + V, \varphi, \bar{\varphi}) - (\chi + \bar{\chi})V$$

The dual Lagrangian

$$\delta V : K_\phi - (\chi + \bar{\chi}) = 0$$

$$g.f. \Rightarrow V = V(\chi + \bar{\chi}, \varphi, \bar{\varphi})$$

$$K^D(\chi + \bar{\chi}, \varphi, \bar{\varphi}) = K(V(\chi + \bar{\chi}, ..), ..) - (\chi + \bar{\chi})V(\chi + \bar{\chi}, ..)$$

$(2,2)$

No spectators

$$g_{\phi\bar{\phi}} = K,_{\phi\bar{\phi}} \ , \quad \Rightarrow \quad g^D,_{\chi\bar{\chi}} = -K^D,_{\chi\bar{\chi}} = \frac{1}{g_{\phi\bar{\phi}}}$$

Double Buscher rules? In general, the geometry becomes transparent after reduction $(1,1)$ superspace.

Down to (2, 1)

To reduce to (2, 1) we write

$$\mathbb{D}_- = D_- - iQ_- \ , \quad Q_-\phi = iD_-\phi \ , \quad Q_-\chi = -\tfrac{1}{2} iD_-\Theta$$

$$\tilde{\phi} = \bar{\phi} - V \ , \quad Q_- V_| = 2A_- - iD_- V$$

$$\nabla_-\phi = D_-\phi + iA_- \ , \quad \nabla_-\tilde{\phi} = D_-\tilde{\phi} + iA_-$$

$$i\left[K_0\nabla_-\phi - K_0\nabla_-\tilde{\phi} + K_Z D_- Z - K_{\bar{Z}} D_-\bar{Z}\right] + \bar{\Theta} iD_- V - (\Theta + \bar{\Theta})A_-$$

M. Abou-Zeid, C. M. Hull, 1997, 1998

M. Abou-Zeid, C. M. Hull, U. L. , and M. Roček, in prep.

The gauged Lagrangian

$$i\left[k_\alpha(\varphi,\tilde\varphi)\nabla_-\varphi^\alpha - \tilde k_{\bar\alpha}(\varphi,\tilde\varphi)\nabla_-\tilde\varphi^{\bar\alpha}\right] + \bar\Theta i D_- V - \left(\Theta+\bar\Theta\right) A_-$$

.

$$\bar{\mathbb{D}}_+\varphi^\alpha = 0\ , \quad i\frac{\partial k_\alpha}{\partial\varphi^0} - i\frac{\partial k_\alpha}{\partial\bar\varphi^0} = 0$$

$$\tilde\varphi^0 = \bar\varphi^0 + V\ , \quad \tilde\varphi^\mu = \bar\varphi^\mu$$

The dual Lagrangian

$$\delta A_- :\Rightarrow X + \Theta + \bar{\Theta} := k_0 + \bar{k}_{\bar{0}} + \Theta + \bar{\Theta} = 0$$

$$\delta V :\Rightarrow A_- = \ldots$$

$$L^D = i\left(k^D_\Theta D_-\Theta - \bar{k}^D_{\bar{\Theta}} D_-\bar{\Theta} + k^D_\mu D_-\varphi^\mu - \bar{k}^D_{\bar{\mu}} D_-\bar{\varphi}^{\bar{\mu}}\right)$$

With $X := k_0 + \bar{k}_{\bar{0}}$, $Z := k_0 - \bar{k}_{\bar{0}}$, the vector potentials are related by

$$k^D_{\ominus} = -\frac{1}{2}\left[V + \frac{Z}{g_{0\bar{0}}}\right] , \quad k^D_{\bar{\ominus}} = -\frac{1}{2}\left[V - \frac{Z}{g_{0\bar{0}}}\right]$$

$$k^D_{\mu} = \left[k_{\mu} - \frac{1}{2}\frac{ZX_{,\mu}}{g_{0\bar{0}}}\right] , \quad k^D_{\bar{\mu}} = \left[\bar{k}_{\bar{\mu}} + \frac{1}{2}\frac{ZX_{,\bar{\mu}}}{g_{0\bar{0}}}\right]$$

.

The geometries are calculated from this as

$$g_{\alpha\bar{\alpha}} = \bar{k}_{\bar{\alpha},\alpha} + k_{\alpha,\bar{\alpha}} , \quad b_{\alpha\beta} = k_{\beta,\alpha} - k_{\alpha,\beta}$$

.

This yields the corresponding relations for the dual metric and b-field

$$g^D_{\Theta\bar{\Theta}} = \frac{1}{g_{0\bar{0}}}\ , \quad g^D_{\mu\bar{\Theta}} = \frac{1}{g_{0\bar{0}}}[b_{\mu 0}]\ , \quad g^D_{\bar{\mu}\Theta} = \frac{1}{g_{0\bar{0}}}[b_{\bar{\mu}\bar{0}}]$$

$$g^D_{\mu\bar{\mu}} = g_{\mu\bar{\mu}} - \frac{1}{g_{0\bar{0}}}(g_{\mu\bar{0}}g_{\bar{\mu}0} - b_{\mu 0}b_{\bar{\mu}\bar{0}})$$

$$b^D_{\Theta\mu} = \frac{g_{\bar{0}\mu}}{g_{0\bar{0}}}\ , \quad b^D_{\bar{\Theta}\bar{\mu}} = \frac{g_{0\bar{\mu}}}{g_{0\bar{0}}}$$

$$b^D_{\mu\nu} = b_{\mu\nu} - \frac{2}{g_{0\bar{0}}}g_{\bar{0}[\mu}(b_{\nu]0})\ , \quad b^D_{\bar{\mu}\bar{\nu}} = b_{\bar{\mu}\bar{\nu}} - \frac{2}{g_{0\bar{0}}}g_{0[\bar{\mu}}(b_{\bar{\nu}]\bar{0}})$$

Reduction to $(1,1)$

As for the $(2,2)$ ciral to twisted chiral duality, this looks like a doubling of the Buscher rules.

Double Buscher rules? In general, the geometry becomes transparent after reduction $(1,1)$ superspace.

The (1, 1) Lagrangian

$$\mathbb{D}_+ = D_+ - iQ_+ \ , \quad Q_+\varphi = iD_+\varphi$$

$$Q_+V_| = 2A_+ \ , \quad Q_+A_{-|} = -i\left(\tfrac{1}{2}d + D_-A_+\right)$$

$$2\left[K_{00}(\nabla_+\phi\nabla_-\bar{\phi} + \nabla_+\bar{\phi}\nabla_-\phi) + K_{0Z}(D_+Z\nabla_-\bar{\phi} + \nabla_+\bar{\phi}D_-Z)\right.$$

$$\left.+K_{0\bar{Z}}(D_+\bar{Z}\nabla_-\bar{\phi} + \nabla_+\bar{\phi}D_-\bar{Z}) + K_{Z\bar{Z}}(D_+ZD_-\bar{Z} + D_+\bar{Z}D_-Z)\right]$$

$$\underbrace{\frac{id}{2}\left(2K_0 + \Theta + \bar{\Theta}\right)}_{2} + \underbrace{(\Theta - \bar{\Theta})2iD_{(+}A_{-)}}_{1}$$

.

1. is the usual Lagrange multiplier term imposing duality between the $(1,1)$ fields $\phi - \bar{\phi}$ and $\Theta - \bar{\Theta}$. In addition to this, there is the gauged Lagrangian and the the *d*-term 2. More explicitly, the latter imposes

$$X + \Theta + \bar{\Theta} := K_0(\phi + \bar{\phi}, \varphi^{\mu}, \bar{\varphi}^{\bar{\mu}}) + \Theta + \bar{\Theta} = 0$$

We can either solve this for $\Theta + \bar{\Theta} = (\Theta + \bar{\Theta})(\phi + \bar{\phi}, \varphi^{\mu}, \bar{\varphi}^{\bar{\mu}})$ or change coordinates to $(\Theta + \bar{\Theta}, \varphi^{\mu}, \bar{\varphi}^{\bar{\mu}})$. The $(2,2)$ and $(2,1)$ calculations correspond to the latter choice. In conclusion, the procedure leads to a dualisation of the imaginary part of ϕ to the imaginary part of Θ along with a coordinate transformation of their real parts.

Kähler gauge transformations

$$S = \int d^2x \mathbb{D}_+ \bar{\mathbb{D}}_+ \mathbb{D}_- \bar{\mathbb{D}}_- K(\phi, \bar{\phi})$$

$$K(\phi, \bar{\phi}) \to K + f(\phi) + \bar{f}(\bar{\phi})$$

$(2, 2)$

$$S = \int d^2x \mathbb{D}_+ \bar{\mathbb{D}}_+ D_- i \left[k_\alpha(\varphi, \varphi) D_- \varphi^\alpha - \bar{k}_{\bar{\alpha}}(\varphi, \varphi) D_- \varphi^{\bar{\alpha}} \right]$$

$$k_\alpha(\varphi, \bar{\varphi}) \to k_\alpha + i\partial_\alpha \chi(\varphi, \bar{\varphi}) + \vartheta_\alpha(\varphi)$$

$(2, 1)$

C. M. Hull, A. Karlhede, U. L. , M. Roček 1986.
M. Abou-Zeid, C. M. Hull, 1997, 1998

The gauged Lagrangian

$$\mathcal{L}_\xi k_\alpha = i\partial_\alpha \chi + \vartheta_\alpha$$

$$\left[i\left(k_\alpha D_- \varphi^\alpha - \bar{k}_{\bar\alpha} D_- \tilde{\varphi}^{\bar\alpha}\right) - A_- X\right](\varphi, \tilde{\varphi}) + \int_0^1 e^{tL} dt V \bar{\vartheta}_{\bar\mu}(\bar\varphi) D_- \bar{\varphi}^{\bar\mu}$$
$$+\bar{\Theta} i D_- V - (\Theta + \bar{\Theta}) A_-$$

where

$$L_{V \cdot \bar\xi} = i V \bar{\xi}^{\bar\alpha} \frac{\partial}{\partial \bar{\varphi}^{\bar\alpha}}$$

The dual vector potentials are

$$k^D_\Theta = -\tfrac{1}{2}\left[V + \frac{\hat{Z}}{g_{0\bar{0}}}\right] \;,\quad \bar{k}^D_\Theta = -\tfrac{1}{2}\left[V - \frac{\hat{Z}}{g_{0\bar{0}}}\right]$$

$$\bar{k}^D_\mu = \left[k_\mu - \tfrac{1}{2}\frac{\hat{Z}X_{,\mu}}{g_{0\bar{0}}}\right] \;,\quad k^D_{\bar{\mu}} = \left[\bar{k}_{\bar{\mu}} + i\int_0^1 dt e^{tL} V \bar{\vartheta}_{\bar{\mu}}(\bar{\varphi}) + \tfrac{1}{2}\frac{\hat{Z}X_{,\bar{\mu}}}{g_{0\bar{0}}}\right]$$

where

$$\widehat{Z} = Z + \chi$$

More geometry

The dual metric and B-field are related by the same “double” Buscher rules as before, with the same geometric understanding resulting from going to $(1,1)$. The only difference is that the B-field entering on the right is shifted compared to t the original one, e.g.,

$$g^D_{\mu\bar{\mu}} = g_{\mu\bar{\mu}} - \frac{1}{g_{0\bar{0}}}(g_{\mu\bar{0}}g_{\bar{\mu}0} - (b_{\mu 0} + i\vartheta_\mu)(b_{\bar{\mu}\bar{0}} - i\bar{\vartheta}_{\bar{\mu}}))$$

The shifted field are the hall mark of...