Kerr-Schild DFT and Classical Double Copy

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Dualities and Generalized Geometries

- Double copy structure states that the scattering amplitudes of the Yang-Mills theory and gravity are related by exchanging the color and kinematic factors [Bern, Carrasco Johansson 2008,2010]

$$c_i \leftrightarrows n_i$$

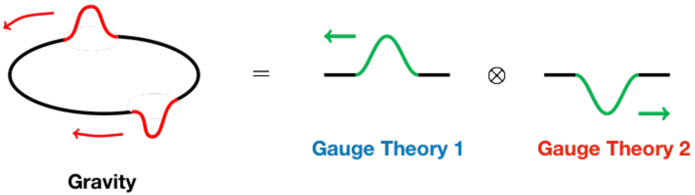
- Gravity amplitudes can be obtained by just replacing the color factor to the kinematic factor without any knowledge of the gravity action or Feynman rules.
- For tree level amplitude, it is equivalent to the field theory limit, $\alpha' \to 0$, of KLT relation.

- Tree level closed string and open string scattering amplitudes are related via the KLT relation [Kawai, Lewellen, Tye 1986]

$$M_n^{\text{tree}} = A_n^{\text{tree}} \mathcal{K}_n \tilde{A}_n^{\text{tree}}$$

where $\mathcal{K}_n$ is the KLT kernel.

- KLT relation provides the string theory origin of double copy structure.



- Spectrum

$$\text{graviton}^{\pm 2}(p_i) = \text{gluon}^{\pm 1}(p_i) \otimes \text{gluon}^{\pm 1}(p_i)$$

$$\left.\begin{array}{l}\text{dilaton}\\ \text{axion}\end{array}\right\} = \text{gluon}^{\pm 1}(p_i) \otimes \text{gluon}^{\mp 1}(p_i)$$

(perturbative) gravity = (Yang-Mills)^2

- Tree level scattering amplitude $\longrightarrow$ on-shell, no quantum effects
  It is natural to deduce its extension to the level of the classical equations of motion.
- Q: Can solutions of the Einstein field equations be represented by solutions of the Yang-Mills equations beyond the linearized level?

$$\text{Solution of GR} \underbrace{\Longleftrightarrow}_{?} \text{Solution of YM}$$

- Graviton $h_{\mu\nu}$ is given by the linearized perturbation of the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

  Recall the spectrum relation. Is it possible to represent $h_{\mu\nu} \sim A_\mu \tilde{A}_\nu$?
- One possible way is the so called classical double copy based on Kerr-Schild formalism in GR [Monteiro, O'Connell, White, 2014]

- The Kerr-Schild ansatz is an extension of linear perturbation around a background metric $\tilde{g}$.
- Einstein equation is nonlinear PDE $\Longrightarrow$ Hard to solve
- What is the condition

Einstein equation becomes linear?

- Kerr and Schild proposed a metric ansatz which makes Einstein equation a linear equation [Kerr 1963], [Kerr, Schild 1965] .
- Meyers-Perry BH, (A)dS Kerr, (A)dS Kerr-Newman, Black string, branes, Waves in flat and (A)dS spaces (PP-wave, Kundt wave, Shock wave) etc.

- Kerr-Schild ansatz

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + \kappa\varphi\ell_\mu\ell_\nu$$

$\tilde{g}_{\mu\nu}$: a background metric satisfying Einstein equation

ℓ_μ : null vector

$$\ell_\mu\tilde{g}^{\mu\nu}\ell_\nu = \ell_\mu g^{\mu\nu}\ell_\nu = 0$$

- The main advantage of the Kerr-Schild ansatz is that it preserves some features of the linearized perturbation

$$g^{\mu\nu} = \tilde{g}^{\mu\nu} - \kappa\varphi\ell^\mu\ell^\nu\,, \qquad \det(g) = \det(\tilde{g})$$

- Suppose that ℓ satisfies the geodesic equation

$$\ell^\mu\tilde{\nabla}_\mu\ell_\nu = \ell^\mu\partial_\mu\ell_\nu = 0$$

then the vacuum eom reduces to the linear equation

$$\kappa\tilde{\nabla}_\rho\Big(\tilde{\nabla}_{(\mu}\big(\varphi\ell_{\nu)}\ell^\rho\big) - \frac{1}{2}\tilde{\nabla}^\rho\big(\varphi\ell_\mu\ell_\nu\big)\Big) = 0$$

- Consider KS ansatz on a flat background, $\tilde{g} = \eta$

$$g_{\mu\nu} = \eta_{\mu\nu} + \varphi \ell_\mu \ell_\nu$$

- Identify the null vector ℓ and φ with gauge field and the biadjoint scala field [Monteiro, O'Connell, White, 2014]

$$A^a_\mu = \varphi \ell_\mu c^a \qquad \Phi^{aa'} = \varphi c^a c'^{a'}$$

- Assume that spacetime is stationary and choose ℓ^μ as $\ell^0 = 1$

$$R_{00} = \frac{1}{2}\nabla^2\varphi$$
$$R_{0i} = \frac{1}{2}\partial^j\big(\partial_i(\varphi\ell_j) - \partial_j(\varphi\ell_i)\big) = -\frac{1}{2}\partial^j F_{ij}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$

Questions

- How can we include Kalb-Ramond field $B_{\mu\nu}$ and dilaton ϕ in Kerr-Schild formalism?

$$\square \otimes \square = \square\square \oplus \begin{array}{c}\square\\ \square\end{array} \oplus \bullet$$

- Curved background generalization - It is not clear how to define scattering amplitude in curved background in general. (time-dependent backgrounds, nonasymptotic flat spaces)
- Classical double coupy for non Kerr-Schild type geometries?
- Nonlinear level?

- Generalized Kerr-Schild method in DFT

  $\Longrightarrow$ Novel solution generating technique for supergravity.

  $\Longrightarrow$ Arbitrary on-shell background
- Classical double copy for entire massless NSNS sector
- Classical double copy in Killing spinor equation

  $\Longrightarrow$ From the Killing spinor equation for gravitino, Yang-Mills BPS equation can be derived

- DFT is the best framework for describing the double copy structure.
- Double copy $\iff$ Left-right decomposition of closed string theory
- Generalized metric is represented by the coset

$$\mathcal{H} \to \frac{O(d,d)}{O(d-1,1) \times O(1,1-d)}$$

and this implies there are two local Lorentz groups

- These are related with local Lorentz groups for left-right sectors of closed string theory. [Arkani-Hamed,Kaplan, 2008], [Hohm, 2011]

$$\eta_{\mu\nu} + h_{\mu\nu} \to h_{m\bar{n}}$$

- Cheung and Remmen derived perturbative DFT action (without dilaton and $B_{\mu\nu}$) around an arbitrary curved background from Einstein-Hilbert action by assuming the two local Lorentz groups. [Cheung, Remmen, 2016]

1. Introduction
2. Generalized Kerr-Schild ansatz
3. Field equations and quasi-linear structure
4. Classical double copy
5. Examples
6. Conclusion

Generalized Kerr-Schild ansatz

- First, we analyze the properties of linear perturbations of generalized metric around an on-shell background generalized metric $\mathcal{H}_0$ satisfying

$$\mathcal{H}_{0MN}\mathcal{J}^{NP}\mathcal{H}_{0PQ} = \mathcal{J}_{MQ}$$

- Split $\mathcal{H}$ into the background part and perturbation parts

$$\mathcal{H}_{MN} = \mathcal{H}_{0MN} + \kappa\hat{\gamma}_{MN}\,,$$

  where $\hat{\gamma}$ describes perturbation and κ is a small expansion parameter.

- The $\hat{\gamma}$ is not arbitrary, but constrained by $O(d,d)$ constraint

$$\kappa\mathcal{H}_0\mathcal{J}\hat{\gamma} + \kappa\hat{\gamma}\mathcal{J}\mathcal{H}_0 + \kappa^2\hat{\gamma}\mathcal{J}\hat{\gamma} = 0\,.$$

- One may solve the constraint recursively

$$\hat{\gamma} = \hat{\gamma}^{(0)} + \kappa\hat{\gamma}^{(1)} + \kappa^2\hat{\gamma}^{(2)} + \cdots$$

- If we truncate the higher order terms in κ, we get linearized $O(d,d)$ constraint

$$\mathcal{H}_0 \mathcal{J} \hat{\gamma} + \hat{\gamma} \mathcal{J} \mathcal{H}_0 = 0 \,, \qquad \hat{\gamma} = \hat{\gamma}^{(0)}$$

- From $O(d,d)$ constraint for $\mathcal{H}_0$, one can define a background chirality and the corresponding projection operators

$$P_0 = \frac{1}{2}(\mathcal{J} + \mathcal{H}_0) \,, \qquad \bar{P}_0 = \frac{1}{2}(\mathcal{J} - \mathcal{H}_0) \,,$$

which satisfy

$$P_0^2 = P_0 \,, \qquad \bar{P}_0^2 = \bar{P}_0 \,, \qquad P_0 \bar{P}_0 = \bar{P}_0 P_0 = 0 \,.$$

- One can show that $\hat{\gamma}$ has mixed chirality

$$\hat{\gamma} = P_0 \hat{\gamma} \bar{P}_0 + \bar{P}_0 \hat{\gamma} P_0 \,, \qquad P_0 \hat{\gamma} P_0 = \bar{P}_0 \hat{\gamma} \bar{P}_0 = 0$$

- Following the conventional Kerr-Schild ansatz, we now assume that $\hat{\gamma}$ is a finite perturbation and κ is a formal finite parameter. The chirality condition is no longer a linearized approximation, but an exact relation.
- This implies $\hat{\gamma}$ is a nilpotent matrix

$$\hat{\gamma}\mathcal{J}\hat{\gamma} = 0$$

- Denote $E = P_0\hat{\gamma}\bar{P}_0$, then $\hat{\gamma} = E + E^t$. The nilpotency condition of $\hat{\gamma}$ is rewritten by

$$E_{MN}E^{t\,N}{}_{P} = E^{t}{}_{MN}E^{N}{}_{P} = 0\,.$$

- By definition, E has mixed background chirality

$$E_{MN} = P_{0MP}E^{P}{}_{N} = E_{MP}\bar{P}_{0}{}^{P}{}_{N}\,.$$

- We assume that the rank of the matrix E is n. Any $2d \times 2d$ matrix with rank n can be recast in terms of n-pairs of some $O(d,d)$ vectors K^a_M and $\bar{K}^a_M$

$$E_{MN} = \sum_{a=1}^{n} \varphi^a K_M{}^a \bar{K}^t{}_N{}^a \,,$$

where φ^a are scalar functions.

- For the purpose of generalizing the Kerr-Schild ansatz in GR, we set $n = 1$

$$E_{MN} = \varphi K_M \bar{K}^t{}_N \,.$$

Since E is nilpotent, K and $\bar{K}$ should be null vectors

$$K_M K^M = 0 \,, \qquad \bar{K}_M \bar{K}^M = 0 \,,$$

- From the chirality condition on E_{MN}, K^M and $\bar{K}^M$ satisfy

$$P_{0MN} K^N = K_M \,, \qquad \bar{P}_{0MN} \bar{K}^N = \bar{K}_M \,, \qquad K_M \bar{K}^M = 0 \,,$$

- Taking all the results together, the generalized metric can be written as

$$\mathcal{H}_{MN} = \mathcal{H}_{0MN} + \kappa\varphi\big(K_M\bar{K}_N + \bar{K}_M K_N\big)\,,$$

- We refer this form as generalized Kerr-Schild ansatz. This ansatz satisfies the $O(d,d)$ constraint automatically without any approximation or truncation.
- Chirality condition $\Longrightarrow$ the K_M and $\bar{K}_M$ are parametrized in terms of the d-dimensional vectors l^μ and $\bar{l}^\mu$

$$K_M = \frac{1}{\sqrt{2}}\begin{pmatrix} l^\mu \\ (\tilde{g}+\tilde{B})_{\mu\nu}l^\nu \end{pmatrix}\,, \qquad \bar{K}_M = \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{l}^\mu \\ (-\tilde{g}+\tilde{B})_{\mu\nu}\bar{l}^\nu \end{pmatrix}\,.$$

- Null condition $\Longrightarrow$ l and $\bar{l}$ are null vectors

$$l^\mu\tilde{g}_{\mu\nu}l^\nu = l^\mu l_\mu = 0\,, \qquad \bar{l}^\mu\tilde{g}_{\mu\nu}\bar{l}^\nu = \bar{l}^\mu\bar{l}_\mu = 0\,, \qquad l\cdot\bar{l}\neq 0$$

- Using the parametrization of generalized metric, we have

$$
\begin{aligned}
(g^{-1})^{\mu\nu} &= (\tilde{g}^{-1})^{\mu\nu} + \kappa\varphi l^{(\mu}\bar{l}^{\nu)}\,, \\
g_{\mu\nu} &= \tilde{g}_{\mu\nu} - \frac{\kappa\varphi}{1+\frac{1}{2}\kappa\varphi(l\cdot\bar{l})} l_{(\mu}\bar{l}_{\nu)}\,, \\
B_{\mu\nu} &= \tilde{B}_{\mu\nu} + \frac{\kappa\varphi}{1+\frac{1}{2}\kappa\varphi(l\cdot\bar{l})} l_{[\mu}\bar{l}_{\nu]}\,, \\
\det g &= (\det\tilde{g})\left(1+\frac{1}{2}\kappa\varphi(l\cdot\bar{l})\right)^{-2}
\end{aligned}
$$

- Though $\mathcal{H}$ is linear in κ, g and B are nonlinear.
- If we identify l^μ and $\bar{l}^\mu$ and ignore the B field, then it reduces to the conventional Kerr-Schild ansatz,

$$
g^{\mu\nu} = \tilde{g}^{\mu\nu} + \kappa\varphi l^\mu l^\nu\,, \qquad g_{\mu\nu} = \tilde{g}_{\mu\nu} - \kappa\varphi l_\mu l_\nu\,.
$$

Field equations and quasi-linear structure

- For simplicity consider a flat background,

$$\mathcal{H}_{0\,MN} = \begin{pmatrix} \eta^{\mu\nu} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}, \qquad d_0 = \text{const.}$$

- Generalized Kerr-Schild ansatz

$$\mathcal{H}_{MN} = \mathcal{H}_{0\,MN} + \kappa\varphi\big(K_M \bar{K}_N + \bar{K}_M K_N\big)$$

$$d = d_0 + \kappa f\,.$$

- Identities

$$K^M \partial_N K_M = 0\,, \qquad \bar{K}^M \partial_N \bar{K}_M = 0\,,$$

$$K^M \partial_N \bar{K}_M = 0\,, \qquad \bar{K}^M \partial_N K_M = 0\,.$$

- DFT connection satisfies

$$K^P \Gamma_{PMN} \bar{K}^N = 0\,, \qquad \bar{K}^P \Gamma_{PMN} K^N = 0\,, \qquad \Gamma^P{}_{PM} K^M = \Gamma^P{}_{PM} \bar{K}^M = 0$$

  and this implies

$$K^M \nabla_M \bar{K}_N = K^M \partial_M \bar{K}_N\,, \qquad \bar{K}^M \nabla_M K_N = \bar{K}^M \partial_M K_N$$
$$K^M \bar{K}^N \nabla_M \partial_N f = K^M \bar{K}^N \partial_M \partial_N f$$

- A constraint from the DFT equations of motion, $\mathcal{S}_{KL} = 0$,

$$K^K \bar{K}^L S_{KL} = 2 K^K \bar{K}^L \partial_K \partial_L f - \frac{1}{2} \varphi \big(K^K \partial_K \bar{K}_M\big)\big(K^L \partial_L \bar{K}^M\big) + \frac{1}{2} \varphi \big(\bar{K}^K \partial_K K_M\big)\big(\bar{K}^L \partial_L K^M\big) = 0\,.$$

- DFT connection satisfies

$$K^P \Gamma_{PMN} \bar{K}^N = 0\,, \qquad \bar{K}^P \Gamma_{PMN} K^N = 0\,, \qquad \Gamma^P{}_{PM} K^M = \Gamma^P{}_{PM} \bar{K}^M = 0$$

and this implies

$$K^M \nabla_M \bar{K}_N = K^M \partial_M \bar{K}_N = 0\,, \qquad \bar{K}^M \nabla_M K_N = \bar{K}^M \partial_M K_N = 0$$

$$K^M \bar{K}^N \nabla_M \partial_N f = K^M \bar{K}^N \partial_M \partial_N f = 0$$

- A constraint from the DFT equations of motion, $\mathcal{S}_{KL} = 0$,

$$K^K \bar{K}^L S_{KL} = 2K^K \bar{K}^L \partial_K \partial_L f - \frac{1}{2}\varphi\big(K^K \partial_K \bar{K}_M\big)\big(K^L \partial_L \bar{K}^M\big) + \frac{1}{2}\varphi\big(\bar{K}^K \partial_K K_M\big)\big(\bar{K}^L \partial_L K^M\big) = 0\,.$$

- Using the parametrization of K and $\bar{K}$ on a flat background,

$$K_M = \frac{1}{\sqrt{2}} \begin{pmatrix} l^\mu \\ l_\mu \end{pmatrix} , \qquad \bar{K}_M = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{l}^\mu \\ -\bar{l}_\mu \end{pmatrix}$$

we have generalized geodesic equation

$$l^\mu \partial_\mu \bar{l}_\nu = 0 \,, \qquad \bar{l}^\mu \partial_\mu l_\nu = 0 \,,$$
$$l^\mu \partial_\mu f = 0 \,, \qquad \bar{l}^\mu \partial_\mu f = 0 \,,$$

- If we identify l^μ and $\bar{l}^\mu$ and ignore f, it becomes the conventional geodesic equation

$$l^\mu \nabla_\mu l^\nu = l^\mu \partial_\mu l^\nu = 0 \,.$$

- Assume that K_M and $\bar{K}_M$ satisfy the generalized geodesic equation.
- Equations of motion in the flat backgrounds

$$S = -2\kappa\partial_K\partial_L(\varphi K^K\bar{K}^L) + 4\kappa\mathcal{H}_0^{KL}\partial_K\partial_L f - 4\kappa^2\mathcal{H}_0^{KL}\partial_K f\partial_L f = 0\,.$$

$$\begin{aligned} P_{0(K}{}^M\bar{P}_{0L)}{}^N\mathcal{S}_{MN} = \kappa\Big[&-\frac{1}{2}\mathcal{H}_0^{MN}\partial_M\partial_N\big(\varphi K_{(K}\bar{K}_{L)}\big) + \partial_M\partial_N\big(\varphi K^N\bar{K}_{(K}\big)P_{0L)}{}^M \\ &- \partial_M\partial_N\big(\varphi K_{(K}\bar{K}^N\big)\bar{P}_{0L)}{}^M + 4P_{0(K}{}^M\bar{P}_{0L)}{}^N\partial_M\partial_N f\Big] \\ &+\kappa^2\Big[\mathcal{H}_0^{MN}\partial_M f\partial_N\big(\varphi K_{(K}\bar{K}_{L)}\big) - 2P_{0(K}{}^M\partial_{|M|}\big(\varphi K^N\bar{K}_{L)}\partial_N f\big) \\ &+2\bar{P}_{0(K}{}^M\partial_{|M|}\big(\varphi K_{L)}\bar{K}^N\partial_N f\big)\Big] = 0\,. \end{aligned}$$

- Interestingly, one can show that the first two chiral equations are redundant

$$P_{0K}{}^M P_{0L}{}^N\mathcal{S}_{MN} = -K_K\bar{K}^M P_{0(L}{}^P\bar{P}_{0M)}{}^Q\mathcal{S}_{PQ}\,,$$

$$\bar{P}_{0K}{}^M\bar{P}_{0L}{}^N\mathcal{S}_{MN} = -\bar{K}_K K^M P_{0(L}{}^P\bar{P}_{0M)}{}^Q\mathcal{S}_{PQ}\,.$$

- Field equation for $\mathcal{H}$: $\mathcal{R}_{KL} = P_{0(K}{}^M\bar{P}_{0L)}{}^N\mathcal{S}_{MN}$.

- In terms of d-dimensional vector indices,

$$\mathcal{R} = \kappa\Big[\,\partial_\mu\partial_\nu\big(\varphi l^\mu \bar{l}^\nu\big) - 4\Box f\,\Big] + 4\kappa^2\partial_\mu f\partial^\mu f = 0\,,$$

$$\begin{aligned}\mathcal{R}_{\mu\nu} = \kappa\Big[\,&\Box\big(\varphi l_\mu \bar{l}_\nu\big) - \partial^\rho\partial_\mu\big(\varphi l_\rho \bar{l}_\nu\big) - \partial^\rho\partial_\nu\big(\varphi l_\mu \bar{l}_\rho\big) + 4\partial_\mu\partial_\nu f\,\Big]\\ &- 2\kappa^2\Big[\partial_\rho f\partial^\rho\big(\varphi l_\mu \bar{l}_\nu\big) - \partial_\mu\big(\varphi l^\rho \bar{l}_\nu\partial_\rho f\big) - \partial_\nu\big(\varphi l_\mu \bar{l}^\rho\partial_\rho f\big)\,\Big] = 0\,.\end{aligned}$$

- Note that $\mathcal{R}_{\mu\nu}$ is not symmetric tensor:
  - symmetric part $\rightarrow$ eom of g
  - antisymmetric part $\rightarrow$ eom of B
- It is interesting that the generalized KS ansatz for $g_{\mu\nu}$ and $B_{\mu\nu}$ is not linear in φ, l^μ and $\bar{l}^\mu$, but the field equations are linear in these fields.
- However, unlike the conventional KS formalism in GR, the above equations are quadratic in κ due to the presence of f.
- If we set $f = 0$, field equations reduce to linear equations.

- Recall that the generalized KS ansatz is linear in κ, but there is no a priori restriction on f

$$d = d_0 + \kappa f\,, \qquad f = \sum_{n=0}^{\infty} \kappa^n f^{(n)}$$

- By substituting the series expansion of f into the field equations, we get

$$\mathcal{S} = \sum_{n=1}^{\infty} \kappa^n \mathcal{S}^{(n)}\,, \qquad \mathcal{R}_{MN} = \sum_{n=1}^{\infty} \kappa^n \mathcal{R}^{(n)}{}_{MN}\,,$$

- Linear equations

$$\partial_\mu \partial_\nu \left(\varphi l^\mu \bar{l}^\nu\right) - 4\Box f^{(0)} = 0\,,$$

$$\Box\left(\varphi l_\mu \bar{l}_\nu\right) - \partial^\rho \partial_\mu \left(\varphi l_\rho \bar{l}_\nu\right) - \partial^\rho \partial_\nu \left(\varphi l_\mu \bar{l}_\rho\right) + 4\partial_\mu \partial_\nu f^{(0)} = 0\,.$$

- Higher order equations

$$\Box f^{(n+1)} - \sum_{p+q=n} \partial_\mu f^{(p)} \partial^\mu f^{(q)} = 0\,, \qquad n \geq 0$$

$$4\partial_\mu \partial_\nu f^{(n+1)} - 2\partial_\rho f^{(n)} \partial^\rho \left(\varphi l_\mu \bar{l}_\nu\right) + \cdots = 0\,, \qquad n \geq 0$$

- Note that φ, l^μ, $\bar{l}^\mu$ and $f^{(0)}$ can be completely determined from the linear equations only, and the higher order equations define recursion relations with respect to $f^{(n)}$, for $n > 0$.
- This means that the metric and the Kalb-Ramond field are determined from the linear equations only.
- Instead of taking the approach using the recursion relations, we substitute φ, l^μ and $\bar{l}^\mu$ into the full equations of motion. Since these equations are linear with respect to $\mathcal{F} = e^{-2\kappa f}$ and f, one can determine f completely.
- The solutions of the linear equations are the solutions of the full field equations automatically, but the converse is not true. This is because the linear equations does not capture the full nonperturbative effects.
- However, the linear equations are remarkably simple compare to the supergravity equations of motion, and it provides a useful tool for finding exact solutions of supergravities.

- The Killing spinor equation reduce the supergravity field equations to first order in derivatives. Combined with the generalized KS ansatz, it will lead to linear equations.
- The SUSY variation of fermions provides the Killing spinor equations, which are

$$\delta\rho = -\gamma^p \mathcal{D}_p \varepsilon = -\gamma^p V_p{}^M \partial_M \varepsilon - \frac{1}{4} V^M{}_p \Phi_{Mmn} \gamma^{pmn} \varepsilon - \frac{1}{2} V^{Mm} \Phi_{Mmn} \gamma^n \varepsilon = 0\,,$$

$$\delta\psi_{\bar{p}} = \bar{V}^M{}_{\bar{p}} \mathcal{D}_M \varepsilon = \bar{V}^M{}_{\bar{p}} \partial_M \varepsilon + \frac{1}{4} \bar{V}^M{}_{\bar{p}} \Phi_{Mmn} \gamma^{mn} \varepsilon = 0\,,$$

- For simplicity, let us choose ε as a Killing spinor for the background geometry satisfying

$$\partial_p \phi \gamma^p \varepsilon_0 + \frac{1}{12} \tilde{H}_{mnp} \gamma^{mnp} \varepsilon_0 = 0\,,$$

$$\tilde{D}^+_{\bar{p}} \varepsilon_0 = 0\,,$$

where ε_0 is the background Killing spinor.

- Then the Killing spinor equations are greatly simplified as

$$\Big(\partial_\mu \Psi + \frac{1}{2}\tilde{D}^+_\nu\big(\varphi' l_\mu \bar{l}^\nu\big)\Big)\gamma^\mu \varepsilon_0 = 0\,,$$

and

$$\Big(\tilde{D}_\mu\big(\varphi l_\nu \bar{l}_\rho\big) - \frac{1}{2}\tilde{H}_{\mu\rho\sigma}\big(\varphi l_\nu \bar{l}^\sigma\big)\Big)\gamma^{\mu\nu}\varepsilon_0 = 0\,.$$

where $\Psi = e^{-2\kappa f}$, $\varphi' = e^{-2\kappa f}\varphi$ and ε_0 is the background Killing spinor.

- For the flat background case

$$\Big(\partial_\mu \Psi + \frac{1}{2}\partial_\nu\big(\varphi' l_\mu \bar{l^\nu}\big)\Big)\gamma^\mu \varepsilon_0 = 0\,,$$

$$\partial_\mu\big(\varphi l_\nu \bar{l}_\rho\big)\gamma^{\mu\nu}\varepsilon_0 = 0\,,$$

where ε_0 is a constant spinor.

- These equations are remarkably simple, and much easier to solve than the full Killing spinor equations.

Cassical double copy

- The KLT and BCJ relations indicate that not only the Einstein field equation, but also the field equations of entire massless NS-NS sector should be related to the Maxwell equation.
- Suppose that the full geometry admits at least one Killing vector ξ^{μ},

$$\mathcal{L}_{\xi} g = 0\,, \qquad \mathcal{L}_{\xi} B = 0\,, \qquad \mathcal{L}_{\xi} \phi = 0\,.$$

- We can locally choose a coordinate system $x^{\mu} = \{x^{i}, y\}$ such that the Killing vector is a constant, $\xi^{\mu} = \partial x^{\mu}/\partial y = \delta_{y}^{\mu}$. The Killing vector ensures

$$\xi^{\rho}\partial_{\rho}\Big(\varphi l_{\mu}\bar{l}_{\nu}\Big) = \xi^{\mu}\partial_{\mu} f = 0$$

- We also normalize l_{μ} and $\bar{l}_{\mu}$ as follows:

$$\xi \cdot l = \xi \cdot \bar{l} = 1$$

- Classical double copy is achieved by contracting ξ^μ with the linear order equations of motion $\mathcal{R}^{(1)}_{\mu\nu} = 0$ in Kerr-Schild ansatz

$$\mathcal{R}^{(1)}_{\mu\nu} = \Box(\varphi l_\mu \bar{l}_\nu) - \partial^\rho \partial_\mu (\varphi l_\rho \bar{l}_\nu) - \partial^\rho \partial_\nu (\varphi l_\mu \bar{l}_\rho) + 4\partial_\mu \partial_\nu f^{(0)} = 0$$

- Zeroth copy

  Contracting ξ^μ with all the free indices of $\mathcal{R}^{(1)}_{\mu\nu}$, we make a scalar equation

$$\xi^\mu \xi^\nu \mathcal{R}^{(1)}_{\mu\nu} = \Box\varphi = 0\,,$$

- Monteiro, O'Connell and White identified φ as the biadjoint scalar field [Cachazo,He,Yuan,2013]

$$\Phi^{aa'} = \varphi c^a \bar{c}^{a'}$$

  where c^a and $\bar{c}^{\bar{a}}$ are color index vectors for Lie group G_1 and G_2.

- It can be understood as a linearized equation of motion for $\Phi^{aa'}$

$$\partial^2 \Phi^{aa'} - y f^{abc} f^{a'b'c'} \Phi^{bb'} \Phi^{cc'} = 0$$

- Single copy

  Contracting ξ^μ with the one of the free index of $\mathcal{R}_{\mu\nu}$, we have

$$\xi^\nu \mathcal{R}_{\mu\nu} = \Box(\varphi l_\mu) - \partial^\rho \partial_\mu (\varphi l_\rho) = 0\,,$$

$$\xi^\mu \mathcal{R}_{\mu\nu} = \Box(\varphi \bar{l}_\nu) - \partial^\rho \partial_\nu (\varphi \bar{l}_\rho) = 0\,.$$

- we identify φl_μ and $\varphi \bar{l}_\mu$ with gauge fields

$$A_\mu = \varphi l_\mu\,, \qquad \bar{A}_\mu = \varphi \bar{l}_\mu$$

- Then $\xi^\nu \mathcal{R}_{\mu\nu}$ and $\xi^\mu \mathcal{R}_{\mu\nu}$ reduce to a pair of Maxwell equations

$$\partial^\mu F_{\mu\nu} = 0\,, \qquad \partial^\mu \bar{F}_{\mu\nu} = 0\,,$$

  where $F_{\mu\nu}$ and $\bar{F}_{\mu\nu}$ are the field strengths of A_μ and $\bar{A}_\mu$ respectively,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \qquad \bar{F}_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu\,.$$

- On a flat background, Killing spinor equation for gravitino is given by

$$\kappa\partial_{[m}\left(\phi l_{n]}\bar{l}_{\mu}\right)\gamma^{mn}\varepsilon = 0$$

- contraction with a Killing vector ξ^{μ}

$$F_{\mu\nu}\gamma^{\mu\nu}\varepsilon = 0\,.$$

- This is the typical BPS equation of $N = 1$ SYM. This shows the classical double copy is still valid for supersymmetric backgrounds

Example

- A class of string backgrounds which have one conserved chiral null current on the world sheet. [Horowitz, Tseytlin, 1994]
- It is a generalization of the gravitational wave and fundamental string background and is exact in the α' expansion.
- In the target space they have a null Killing vector and unbroken supersymmetries.
- Special cases are the Taub-NUT geometry and rotating black holes.
- The explicit geometry is given by

$$\mathrm{d}s^2 = F(x^i)\mathrm{d}u\Big(\mathrm{d}v + K(u,x^i)\mathrm{d}u + 2V_i(u,x^i)\mathrm{d}x^i\Big) + \mathrm{d}x^i\mathrm{d}x^i\,,$$

$$B_{uv} = F(x^i)\,, \qquad B_{ui} = 2F(x^i)V_i(u,x^i)\,,$$

$$\phi = \phi(u) + \frac{1}{2}\log F(x^i)\,,$$

- This fits into the generalized Kerr-Schild ansatz in a flat background.

$$\mathrm{d}s^2 = \mathrm{d}u\mathrm{d}\tilde{v} + \mathrm{d}x^i\mathrm{d}x^i + (F-1)\mathrm{d}u\Big(\mathrm{d}\tilde{v} - \tilde{V}_i\tilde{V}^i\mathrm{d}u + \tilde{V}_i\mathrm{d}x^i\Big)\,,$$

where

$$V_i = \tilde{V}_i + \frac{1}{2}\partial_i X\,, \qquad v = \tilde{v} - X(x,u)\,,$$

$$X(x,u) = \int^u \Big(K + \frac{4F}{(F-1)}\tilde{V}_i\tilde{V}^i\Big)(\vec{x},u')\mathrm{d}u'\,,$$

- The associated φ and null vectors l and $\bar{l}$ can be easily read off

$$\kappa\varphi = F^{-1} - 1\,,$$

$$l_u = 1\,,$$

$$\bar{l}_u = -\Big(\frac{2F}{F-1}\Big)^2\tilde{V}_i\tilde{V}^i\,, \qquad \bar{l}_{\tilde{v}} = 1\,, \qquad \bar{l}_i = \frac{2F}{F-1}\tilde{V}_i\,,$$

and one can easily show that l and $\bar{l}$ are orthogonal with respect to the flat background metric.

- l and $\bar{l}$ satisfy the generalized geodesic constraint

$$l^\mu \partial_\mu \bar{l}_\nu = 0\,, \qquad \bar{l}^\mu \partial_\mu l_\nu = 0$$

- Equations of motion imply, $\kappa f = \phi(u)$

$$\begin{aligned} \partial_i \partial^i F^{-1} &= 0\,, \\ -\partial^i \partial_i K + 2\partial^i \partial_u V_i + 4F^{-1}\partial_u^2 \phi &= 0\,, \\ -4\partial^j \mathcal{F}_{ji} + 4\partial_u \phi \partial_i F^{-1} &= 0\,. \end{aligned}$$

  where $\mathcal{F}_{ij} = \partial_i V_j - \partial_j V_i$.

- This is the same exactly with the equation derived by Callan, Maldacena and Peet.

- A novel solution generating technique in supergravities via generalized Kerr-Schild method in DFT
- Classical double copy including $B_{\mu\nu}$ and dilaton
- Classical double copy in Killing spinor equation
- Heterotic DFT, including matters (RR sector, fermions), Introducing U(1) gauge fields using Kaluza-Klein reduction, Gauged supergravity extension via Scherk-Schwarz reduction, Extended Kerr-Schild method, Finding new supergravity solutions,....

Thank you