Aspects of interacting CFTs and the C-Theorem

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Based on:

- K.S.: Nucl. Phys. **B880** (2014) 225, 1312.4560.
- Full generalization: arXiv:1809.03522 (with G. Georgiou).
- on C-theorem: Phys. Lett. **B782** (2018) 613, 1805.03731, (with G. Georgiou, P. Panopoulos, E. Sagkrioti & K. Siampos).

Other related works with:
G. Itsios, K. Siampos, G. Georgiou, D. Thompson, S. Demulder

Some motivation

- Systematic construction of new deformations of (interacting) CFT's having explicit Lagrangian descriptions.
- A subset of them is integrable.
- Exact beta-function and anomalous dimensions in QFTs.
- It is a rare occasion to be able to compute them exactly as traditionally are computed perturbatively, e.g.

$$\text{QCD}: \qquad \mu\frac{dg}{d\mu} = -\frac{7g^3}{16\pi^2} + \cdots \ ,$$

  where μ is the energy scale.
- Smooth RG flows (UV to IR) between CFTs.
- Type-II supergravity embeddings
- Applications/developments in an AdS/CFT context.

Outline

- The theories of interest
- Construction of effective actions (self- and mutual-interacting)
- A truncation to a two parameter integrable model.
- Beta-functions and anomalous dimensions.

  Perturbative info + symmetry + analyticity $\Longrightarrow$ exact info

- Interplay between CFT and gravitational methods.
- Exact C-function obeying Zamolodchikov's c-theorem (the first example in literature).

The theories of interest

Let any two 2-dim CFT with some action and an underlying group G structure having currents $J_i^a(z)$ & $\bar{J}_i^a(\bar{z})$, $i = 1, 2$, obeying the usual algebras at levels k_1, k_2

$$J_i^a(z) J_i^b(w) = \frac{\delta_{ab}}{(z-w)^2} + \frac{f_{abc}}{\sqrt{k_i}} \frac{J_i^c(w)}{z-w} , \qquad i = 1, 2 .$$

We are interested in deformations by the four current bilinears:

- Current self-interactions

$$J_1^a \bar{J}_1^b , \qquad J_2^a \bar{J}_2^b .$$

- Current mutual-interactions

$$J_1^a \bar{J}_2^b , \qquad J_2^a \bar{J}_1^b .$$

- Simultaneously self-interactions and mutual-interactions

$$J_1^a \bar{J}_1^b , \qquad J_2^a \bar{J}_2^b , \qquad J_1^a \bar{J}_2^b , \qquad J_2^a \bar{J}_1^b .$$

These relevant operators generates a RG flow towards the IR.

We are aiming at:

- Computing the RG flow eqs

$$\frac{1}{2}\mu\frac{d\lambda_{ab}}{d\mu} = \cdots \ .$$

- Anomalous dimensions of currents and primary operators.
- Search for new fixed points under the RG flow towards the IR. New fixed points in the IR when $k_1 \neq k_2$.
- Identity the CFT in the IR; Compute the C-function.
- We would like these exactly in λ, unlike traditional approaches.
- Construct effective, all loop in λ, actions.

General effective actions

The starting point is the action

$$S = S_{\mathrm{WZW}} + S_{\mathrm{PCM}} ,$$

- The WZW part is

$$S_{\mathrm{WZW}} = S_{k_1}(g_1) + S_{k_2}(g_2) , \qquad k_i \in \mathbb{Z} .$$

  Two copies of the $G_{\mathrm{L,cur}} \times G_{\mathrm{R,cur}}$ current algebra at levels k_i.
- The PCM part is

$$S_{\mathrm{PCM}} = -\frac{1}{\pi} \int \mathrm{Tr}\Big(\tilde{g}_1^{-1} \partial_+ \tilde{g}_1 E_1 \tilde{g}_1 \partial_- \tilde{g}_1 + \tilde{g}_1^{-1} \partial_+ \tilde{g}_1 E_2 \tilde{g}_2^{-1} \partial_- \tilde{g}_2 + \tilde{g}_2^{-1} \partial_+ \tilde{g}_2 E_3 \tilde{g}_1^{-1} \partial_- \tilde{g}_1 + \tilde{g}_2^{-1} \partial_+ \tilde{g}_2 E_4 \tilde{g}_2^{-1} \partial_- \tilde{g}_2 \Big) .$$

  It represents two PCMs mutually interacting with real coupling matrices E_i's. It has two copies with global G_L symmetry.

The construction of the action

We gauge the global symmetry acting as

$$g_i \to \Lambda_i^{-1} g_i \Lambda_i \ , \qquad \tilde{g}_i \to \Lambda_i^{-1} \tilde{g}_i \ , \qquad i = 1, 2 \ .$$

- Introduce gauge fields $A_\pm$ and $B_\pm$ in the Lie algebra of G.
- The condition
$$\tilde{g}_i = \mathbb{1} \ , \qquad i = 1, 2 \ ,$$
completely fixes the gauge.

Then the gauged fixed action becomes

$$
\begin{aligned}
S = {} & S_{k_1}(g_1) + S_{k_2}(g_2) \\
& + \frac{k_1}{\pi}\int \mathrm{Tr}\left(A_-\partial_+ g_1 g_1^{-1} - A_+ g_1^{-1}\partial_- g_1 + A_- g_1 A_+ g_1^{-1}\right) \\
& + \frac{k_2}{\pi}\int \mathrm{Tr}\left(B_-\partial_+ g_2 g_2^{-1} - B_+ g_2^{-1}\partial_- g_2 + B_- g_2 B_+ g_2^{-1}\right) \\
& - \frac{\sqrt{k_1 k_2}}{\pi}\int \mathrm{Tr}\left(B_+\lambda_1^{-1}A_- + A_+\lambda_2^{-1}B_- + B_+\lambda_3^{-1}B_- + A_+\lambda_4^{-1}A_-\right) ,
\end{aligned}
$$

where the λ's depend on the E's.

- The σ-model is obtained by integrating out the gauge fields.
- Restricting to just $g_1, A_\pm$ (or $g_2, B_\pm$) we get the usual λ-deformed models (self-interactions of currents).
- Interactions between $A_\pm$'s and $B_\pm$'s.
- The σ-model will have both self and mutual interactions.

A two parameter integrable truncation

To simplify remove some interactions:

- $\lambda_2, \lambda_3 \to \infty$ removes $A_+ B_-$ & $A_+ A_-$, keeps $B_+ A_-$ & $A_+ A_-$.
- Also let the remaining λ's diagonal.

The action turns out to be:

$$
\begin{aligned}
S = S_{k_1}(g_1) + \frac{k_1}{\pi} \int d^2\sigma \; J_{1+} (\lambda^{-1} \mathbb{1} - D_1^T)^{-1} J_{1-} \\
+ S_{k_2}(g_2) + \frac{k_2}{\pi} \tilde{\lambda} \lambda^{-1} \int d^2\sigma \; J_{2+} (\lambda^{-1} \mathbb{1} - D_1^T)^{-1} J_{1-} \; .
\end{aligned}
$$

with λ and $\tilde{\lambda}$ the deformation parameters and

$$
J_+^a = -i \,\mathrm{Tr}(t^a \partial_+ g g^{-1}), \quad J_-^a = -i \,\mathrm{Tr}(t^a g^{-1} \partial_- g) \,, \quad D_{ab} = \mathrm{Tr}(t_a g t_b g^{-1}) \,,
$$

where the t^a's are rep. matrices.

- For the $\tilde{\lambda} = 0$ the usual λ-deformed model (and a WZW).

Important properties

- For small λ and $\tilde{\lambda}$

$$S = S_{k_1}(g_1) + S_{k_2}(g_2) + \frac{k_1}{\pi}\lambda \int d^2\sigma\, J_{1+}J_{1-} + \frac{k_2}{\pi}\tilde{\lambda}\int d^2\sigma\, J_{2+}J_{1-} \;+\; \mathcal{O}(\lambda^2, \lambda\tilde{\lambda}) \ .$$

- Euclidean signature: λ and $\tilde{\lambda}$ obey

$$\lambda^2 + \frac{k_2}{k_1}\,\tilde{\lambda}^2 < 1 \ .$$

- Small curvatures: $k_1, k_2 \gg 1$ and staying within the ellipsis.
- Near the ellipsis a zoom-in limit $\rightarrow$ non-Abelian T-dual.
- A remarkable non-perturbative symmetry

$$g_1 \rightarrow g_1^{-1}\ , \quad \lambda \rightarrow \frac{1}{\lambda}\ , \quad k_1 \rightarrow -k_1\ , \quad \tilde{\lambda} \rightarrow \frac{\tilde{\lambda}}{\lambda}\ .$$

- Equations of motion admit a Lax pair: Classically integrable.

Exact β-function and anomalous dims

Beta-function in the two parameter flow

Using the background field method one finds [Georgiou-KS, 18]

$$\beta_\lambda(\lambda,\tilde\lambda) = -c_G\lambda(1-\lambda)\frac{k_1\lambda(1-\lambda)-k_2\tilde\lambda^2(1+\lambda-\tilde\lambda)}{2(k_1(1-\lambda^2)-k_2\tilde\lambda^2)^2}\ ,$$

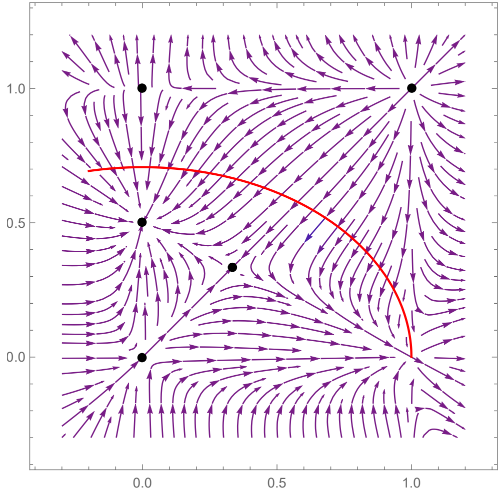
$$\beta_{\tilde\lambda}(\lambda,\tilde\lambda) = -c_G\tilde\lambda(1-\tilde\lambda)\frac{k_1(1-\lambda)\left(\tilde\lambda-\lambda(\lambda-\tilde\lambda)\right)-k_2\tilde\lambda^2}{2(k_1(1-\lambda^2)-k_2\tilde\lambda^2)^2}\ ,$$

where c_G is the quadratic Casimir in the adjoint.

- There are six points where they vanish simultaneously.
- Their location depends on the ratio of $\frac{k_1}{k_2}$.
- Some are IR stable.
- Others have one relevant and one irrelevant direction.

RG flows in the $(\lambda, \tilde{\lambda})$-plane ($\lambda$-horizontal) for $k_2 = 2k_1$:

- The ellipsis bounds the region of Euclidean signature.
- Arrows point towards the IR.



CFT and symmetry approach [Georgiou-Siampos-KS 15 & 16]

We may compute the beta-functions and anomalous dimensions by combining perturbation theory, symmetry arguments and analyticity.

Let $\tilde{\lambda} = 0$ and $k_1 = k_2 = k$ (single λ-deformation).

Using QFT perturbative methods one finds:

- The β-function

$$\beta_\lambda = \frac{1}{2}\mu\frac{d\lambda}{d\mu} = -\frac{c_G}{2k}\left(\lambda^2 - 2\lambda^3 + \mathcal{O}(\lambda^4)\right) \ .$$

- The anomalous dimension of the currents

$$\gamma^{(J)} = \mu\frac{d\ln Z^{1/2}}{d\mu} = \frac{c_G}{k}\left(\lambda^2 - 2\lambda^3 + \mathcal{O}(\lambda^4)\right) \ .$$

- Task: Extend these exactly in λ.

The exact β-function and anomalous dimensions are of the form

$$\beta_\lambda = -\frac{c_G}{2k}\frac{f(\lambda)}{(1+\lambda)^2}\ , \qquad \gamma^{(J)} = \frac{c_G}{k}\frac{g(\lambda)}{(1-\lambda)(1+\lambda)^3}\ ,$$

where $f(\lambda)$ and $g(\lambda)$ are analytic in λ.

- Demand well defined non-Abelian and pseudodual limits as $\lambda \to \pm 1$ and $k \to \infty$.
- Due to the symmetry

  $$k \to -k\ , \qquad \lambda \to \frac{1}{\lambda}\ ,$$

  we have

  $$\lambda^4 f(1/\lambda) = f(\lambda)\ , \qquad \lambda^4 g(1/\lambda) = g(\lambda)\ .$$

- $f(\lambda)$ and $g(\lambda)$ are polynomials of degree four, fixed by the above symmetry and the two-loop perturbative result.

- The final result for the beta-function is

$$\beta_\lambda = -\frac{c_G}{2k}\frac{\lambda^2}{(1+\lambda)^2} \leqslant 0 \ ,$$

in agreement with [Kutasov 89] and [Gerganov-LeClair-Moriconi 01].

- For the anomalous dimension

$$\gamma^{(J)} = \frac{c_G}{k}\frac{\lambda^2}{(1-\lambda)(1+\lambda)^3} \geqslant 0 \ .$$

Agreement with perturbation theory to $\mathcal{O}(\lambda^3)$ and $\mathcal{O}(\lambda^4)$.

- Similarly for current 3-point functions and correlators of primary fields [Georgiou-KS-Siampos 16].
- Complete agreement [Itsios-KS-Siampos, KS-Siampos 14, KS-Siampos-Sagkrioti 18] using the gravitational one-loop β-functions [Ecker-Honerkamp 71, Friedan 80, Fridling-van de Ven 86]

$$\frac{\mathrm{d}G_{\mu\nu}}{\mathrm{d}t} + \frac{\mathrm{d}B_{\mu\nu}}{\mathrm{d}t} = R^-_{\mu\nu} + \nabla^-_\nu \xi_\mu \ , \quad t = \ln \mu^2 \ .$$

Mutually-interacting theories [Georgiou-KS 16,17]

Consider a different truncation sending $\lambda_{3,4} \to \infty$.
Then only mutual interactions

$$\lambda_1 J_1^a \bar{J}_2^a \ , \qquad \lambda_2 J_2^a \bar{J}_1^a \ .$$

- Deformation is integrable.
- Beta-function

$$\frac{1}{2}\mu\frac{d\lambda}{d\mu} = -\frac{c_G}{2\sqrt{k_1 k_2}}\frac{\lambda^2(\lambda-\lambda_0)(\lambda-\lambda_0^{-1})}{(1-\lambda^2)^2} \ ,$$

  where λ is either λ_1 or λ_2.

- A new fixed point in the IR at

$$\lambda = \lambda_0 = \sqrt{\frac{k_1}{k_2}} < 1 \ .$$

- Using path integrals arguments one may show

$$\underbrace{G_{k_1} \times G_{k_2}}_{\text{At the UV, }\lambda=0} \quad \overset{\text{RG flow}}{\Longrightarrow} \quad \underbrace{\frac{G_{k_1} \times G_{k_2-k_1}}{G_{k_2}} \times G_{k_2-k_1}}_{\text{At the IR, }\lambda=\lambda_0} ,$$

- $c_{\mathrm{IR}} < c_{\mathrm{UV}}$, i.e. in accordance with Zamolodchikov's c-theorem.

$\boxed{\text{Exact Zamolodchikov's } C\text{-function, i.e. } C = C(\lambda, \lambda_0)?}$

The C-function

The central charge in a CFT

An important quantity that appears in many places:

- Density of states: In a CFT with central charge c [Cardy 86]

$$n(N) \approx e^{2\pi\sqrt{\frac{N}{6}c}} , \qquad N \gg 1 .$$

  Measures the degrees of freedom.

- Trace anomaly: Classically, the trace of the stress energy tensor $\Theta = 0$. But, when a CFT couples to curved background

$$\langle \Theta \rangle = -\frac{c R^{(2)}}{12} .$$

- Virasoro algebra of the Virasoro constraints

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n} .$$

Beyond the CFT

- QFTs at the UV have more degrees of freedom than in the IR.
- UV information is lost irreversibly towards the IR.
- We need a quantitative measure of the above statements.

Zamolodchikov's c-theorem [A.B. Zamolodchikov 86]

Let a 2-dim QFT with couplings λ^i and β-functions β^i.

- There exist a function $C(\lambda) \geqslant 0$ such that

$$\frac{1}{2}\mu\frac{dC}{d\mu} = G_{ij}(\lambda)\beta^i\beta^j \geqslant 0 \ ,$$

  where G_{ij} is a positive definite metric in coupling space.
- Near a fixed point λ^* in which $\beta^i(\lambda^*) = 0$ we have that

$$\beta^i = G^{ij}\partial_j C + \cdots \ , \qquad G^{ij} = G_{ij}^{-1} \ .$$

- At the fixed points $C(\lambda^*)$ equals the CFT central charge.

The exact C-function: The one coupling case

On general grounds near a fixed point

$$G_{\lambda\lambda}\beta^{\lambda} \equiv \frac{\mu}{2}\partial_{\mu}\lambda = \frac{1}{24}\partial_{\lambda}C + \cdots \ ,$$

where the metric in the coupling space is $G_{\lambda\lambda} = \frac{\dim G}{(1-\lambda^2)^2}$.

- The beta-function near $\lambda = 0$ is

$$\beta^{\lambda}(\lambda, \lambda_0) = -\frac{c_G}{2\sqrt{k_1 k_2}}\lambda^2 + \cdots \ .$$

- At the UV CFT $G_{k_1} \times G_{k_2}$

$$c_{\rm UV} = 2\dim G - \frac{c_G \dim G}{2}\left(\frac{1}{k_1} + \frac{1}{k_2}\right) + \cdots \ .$$

- Therefore by a simple integration

$$C(\lambda, \lambda_0) = c_{\rm UV} - 4\frac{c_G \dim G}{\sqrt{k_1 k_2}}\lambda^3 + \mathcal{O}(\lambda^4)$$

Proceeding as before, i.e.

- Symmetry considerations, i.e. $k_{1,2} \mapsto -k_{2,1}$, $\lambda \to 1/\lambda$
- Demanding finite limiting behavior for $\lambda = \pm 1$ and $k_{1,2} \to \infty$
- Matching with the perturbative results

The exact in λ C-function is

$$\boxed{C(\lambda, \lambda_0) = 2 \dim G - \frac{c_G \dim G}{\sqrt{k_1 k_2}} \frac{(\lambda_0 + \lambda_0^{-1})(1 - 3\lambda^2 - 3\lambda^4 + \lambda^6) + 8\lambda^3}{2(1-\lambda^2)^3}} \, .$$

The first such example in literature.

Properties and checks

- It is positive, i.e. $C(\lambda, \lambda_0) > 0$.
- Near the IR fixed point at $\lambda = \lambda_0$

$$C = \underbrace{2\dim G - \frac{c_G \dim G}{k}\frac{1+\lambda_0^4}{2\lambda_0(1-\lambda_0^2)}}_{\text{CFT central charge at IR}} + \underbrace{\frac{6c_G \dim G}{k}\frac{\lambda_0}{(1-\lambda_0^2)^3}(\lambda-\lambda_0)^2}_{\text{obeys } \partial_\lambda C \sim \beta^\lambda} + \dots$$

- In addition

$$\mu\frac{dC}{d\mu} = 2\beta_\lambda \partial_\lambda C = 12\frac{c_G^2 \dim G}{k_1 k_2}\frac{\lambda^4(\lambda-\lambda_0)^2(\lambda-\lambda_0^{-1})^2}{(1-\lambda^2)^6} \ ,$$

Hence, C is monotonically increasing from the IR to the UV.

Concluding remarks

- New integrable σ-models, as all loop effective actions for current-current interactions of exact CFTs self and mutual.
- β-function and anomalous dimensions of operators computed, in particular, with leading perturbative results and symmetries.
- Non-trivial smooth flows between exact CFTs.
- Computation of Zamolodchikov's C-function as a complete function of the coupling.

Future directions:

- For $k_1 \neq k_2$ embed to type-II supergravity.
  As in [KS-Thompson 14, Demulder-KS-Thompson 15, Borsato-Tseytlin 15, Lunin 17] for the single λ-deformations.
- Identify all CFTs in the IR fixed points.
- For the two-parameter integrable model compute anomalous dimensions of operators.