Generalized Geometry, A/B-models and topological M-theory

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based on 1802.04581 and 1805.11485
with A. Sinkovics and R. J. Szabo

We reformulate AKSZ constructions of topological sigma-models (in topological string theory) on doubled space and relate them to generalized complex structures.

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We describe the known membrane theories related to topological M-theory with an AKSZ 3-brane, which has the higher Courant bracket on $\mathcal{T} \oplus \wedge^2 \mathcal{T}^*$ as its underlying derived bracket.

[1805.11485]
Topological string theory

- Where does it come from?
  \[ \mathcal{N} = 2 \text{ sigma-model} \]
  & coupled to gravity
  \} \rightarrow \text{bosonic string theory}
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\begin{align*}
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\end{align*}
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- Procedure to get the topological sigma-model: \( \sim \) topological twisting
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- Two non-equivalent twists:

  \( \sim \) A- and B-models
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- Where does it appear in physical string theory?

  a) IIA or IIB compactifications \( \sim \) superpotential

  b) IIA compactification \( \sim \) entropy of BPS black hole
AKSZ sigma-models

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  [Alexandrov, Kontsevich, Schwartz, Zaboronsky]
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  Well known examples are Poisson sigma-model, A/B-models, Chern-Simons theory, Courant sigma-model, etc.
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  **Source manifold**: $T[1]\Sigma_d$ graded worldvolume with $\dim \Sigma_d = d$
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  - **Source manifold**: $T[1]\Sigma_d$ graded worldvolume with $\dim\Sigma_d = d$
  - **Target manifold**: $(\mathcal{M}, Q_\gamma, \omega)$ QP-manifold of degree $d - 1$
    
    $\omega = d\vartheta$ is a graded symplectic structure
    
    $Q_\gamma = \{\gamma, .\}$ is a cohomological vector field
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- **The space of fields** is the mapping space  
  $\mathcal{M} = \text{Map}(T[1] \Sigma_d, \mathcal{M})$

  $\implies$ An arbitrary coordinate $\phi \in \mathcal{M}$ of degree $|\phi|$ corresponds to a field $\phi \in \mathcal{M}$ of ghost number $|\phi|$
AKSZ sigma-models

- The **AKSZ action** is determined by the symplectic potential $\vartheta$ and the Hamiltonian $\gamma$

\[ S = S_{\text{kin}} + S_{\text{int}} \]

1. **Gauge fixing:**
   - Assign fields and antifields (paired in the BV symplectic structure $\omega$)
   - Gauge fix the antifields
   - Choose a Lagrangian submanifold $L \in M$ such that $\omega|_L = 0$. 
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It gives a solution to the master equation

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2) Gauge fix the antifields $\leftrightarrow$ Choose a Lagrangian submanifold $\mathcal{L} \in \mathcal{M}$ ($\omega|_{\mathcal{L}} = 0$).
A- and B-models

- **Poisson sigma-model or A-model** \((\mathcal{M} = T^*[1]M)\)

\[\omega = d\chi_i \wedge dX^i \quad \text{and} \quad \gamma = \frac{1}{2} \pi^{ij}(X) \chi_i \chi_j\]
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- **Complex structure sigma-model or B-model** \( (\mathcal{M} = T^*[1]T^*M \text{ doubled}) \)

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\omega = d\chi^i \wedge dX^i + d\tilde{\chi}^i \wedge d\tilde{X}_i \quad \text{and} \quad \gamma = J^i_j \chi_i \tilde{\chi}^j - \partial_j J^i_k \tilde{X}_i \tilde{\chi}^j \tilde{\chi}^k 
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S_J^{(2)} = \int_{T[1]\Sigma_2} \left( \chi_i DX^i - \tilde{X}_i D\tilde{\chi}^i + J^i_j \chi_i \tilde{\chi}^j + \partial_j J^i_k \tilde{X}_i \tilde{\chi}^j \tilde{\chi}^k \right)
\]

[Ikeda, Tokunaga]
Doubled Poisson sigma-model for A/B-models

- Poisson sigma-model on doubled targetspace $\mathcal{M} = T^*[1] T^* M$

$$\omega = d\chi_I \wedge dX^I \quad \text{and} \quad \gamma = \frac{1}{2} \Omega^{IJ}(X) \chi_I \chi_J$$

with $X^I = \left( \begin{array}{c} X^i \\ \tilde{X}_i \end{array} \right)$ and $\chi_I = \left( \begin{array}{c} \chi_i \\ \tilde{\chi}_i \end{array} \right)$

$$S^{(2)}_\Omega = \int_{T[1] \Sigma_2} \left( \chi_I D X^I + \frac{1}{2} \Omega^{IJ}(X) \chi_I \chi_J \right).$$
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$$S^{(2)}_\Omega = \int_{\Sigma_2} \left( \chi_i D X^i + \frac{1}{2} \Omega^{ij}(X) \chi_i \chi_J \right).$$

- Observation: it gives the AKSZ formulation of A- or B-models

$$\Omega^{ij} = \begin{pmatrix} \pi^{ij} & 0 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \text{A-model}$$

$$\Omega^{ij} = \begin{pmatrix} 0 & J^i_j \\ -j^i_j & -2 \partial_{[i} J^k_{j]} \tilde{X}_k \end{pmatrix} \quad \Rightarrow \quad \text{B-model}$$
AKSZ membrane formulation for A/B-models

- We lift up the doubled Poisson sigma-model to an AKSZ membrane (contravariant Courant sigma-model [Bessho, Heller, Ikeda, Watamura] on $\mathcal{M} = T^*[2] T[1] T^* \mathcal{M}$)

\[
\omega = dF_I \wedge dX^I + d\chi_I \wedge d\psi^I
\]

\[
\mathcal{S}_{\Omega, \mathcal{R}}^{(3)} = \int_{T[1]\Sigma_3} \left( F_I DX^I - \chi_I D\psi^I + \Omega^{IJ} F_I \chi_J 
- \frac{1}{2} \partial_I \Omega^{JK} \psi^I \chi_J \chi_K + \frac{1}{3!} \mathcal{R}^{IJK} \chi_I \chi_J \chi_K \right)
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$$\omega = \frac{dF_I}{2} \wedge dX^I_0 + dX^I_1 \wedge d\psi^I_1$$

$$S^{(3)}_{\Omega,R} = \int_{T[1] \Sigma_3} \left( F_I DX^I - \chi_I D\psi^I + \Omega^{IJ} F_I \chi_J - \frac{1}{2} \partial_I \Omega^{JK} \psi^I \chi_J \chi_K + \frac{1}{3!} R^{IJK} \chi_I \chi_J \chi_K \right)$$

- It allows the definition of fluxes

$$R^{IJK} \rightarrow H_{ijk}, F^i_{jk}, Q^{ij}_{k}, R^{ij}_{k}$$

and the master equation gives them Bianchi identities $[\Omega, R]_S = 0$. 
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- In the partial gauge

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F_I = D\chi_I \quad \text{and} \quad \psi^I = -DX^I
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\]

it gives back the doubled Poisson sigma-model.
Relation to generalized complex structure and topological S-duality

Halving degree 1 fields with DFT projection [Chatzistavrakidis,Jonke,Khoo,Szabo] and imposing the master equation (it has the role of a section condition) we get the Courant sigma-model

$$S_{\pi,J}^{(3)} = \int_{T[1]\Sigma_3} \left( F_i D\chi^i - \chi_i D\psi^i + \pi^{ij} F_i \chi_j + J^i_j F_i \psi^j - \frac{1}{2} \partial_i \pi^{jk} \psi^i \chi_j \chi_k + \partial_i J^i_j \psi^i \psi^j \chi_k \right)$$
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- The master equation gives the integrability condition for the generalized complex structure

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- It gives back the AKSZ formulations of A- or B-models on the boundary after a full gauge fixing on the bulk.
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- Topological S-duality between A- and B-models arises \((g_A \leftrightarrow 1/g_B)\) from \(\mathcal{J}\), as a canonical transformation:

\[ S^{(2)}_B \leftarrow S^{(3)}_{\pi,J} \rightarrow S^{(2)}_A \]
Different reductions and connections between the AKSZ string and membrane sigma-models related to the topological A- and B-models.
Topological membranes on $G_2$-manifold

- **Topological M-theory:** Hitchin-type form theory of $G_2$-manifolds
  
  It is a unification of the form theory of A- and B-models.
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  1) Using Mathai-Quillen formalism

\[ \omega = dX_i^0 \wedge dF_i^3 + d\chi_i^2 \wedge d\psi_i^1 \]
\[ \gamma = F_i^1 \psi_i^1 S(4) \]
\[ G_2 = \int_{\Sigma^4} (F_i^1 D X_i^0 + \psi_i^1 D \chi_i^2 + F_i^1 \psi_i^1) \]

[Anguelova, Medeiros, Sinkovics]
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- Two topological membrane has been proposed to be the worldvolume formulation:
  
  1) Using Mathai-Quillen formalism

  [Anguelova, Medeiros, Sinkovics]

  2) BRST gauge fixed version of the associative 3-form

  [Bonelli, Tanzini, Zabzine]
Topological membranes on $G_2$-manifold

- **Topological M-theory**: Hitchin-type form theory of $G_2$-manifolds
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- Two topological membrane has been proposed to be the worldvolume formulation:
  1) Using Mathai-Quillen formalism
     - [Anguelova,Medeiros,Sinkovics]
  2) BRST gauge fixed version of the associative 3-form
     - [Bonelli,Tanzini,Zabzine]

- They can be unified within one single AKSZ 3-brane ($\mathcal{M} = T^*[3] T[1]M$)
  
  \[
  \omega = dX^i_0 \wedge dF_i + d\chi^i_2 \wedge d\psi^i_1 \quad \text{and} \quad \gamma = F_i \psi^i 
  \]

  \[
  S_{G_2}^{(4)} = \int_{T[1]\Sigma_4} \left( F_i DX^i + \psi^i D\chi_i + F_i \psi^i \right) 
  \]

  Worldvolume dimension reductions of $S_{G_2}^{(4)}$ gives both topological membranes in different gauges.
The master equation defines a higher analog of the Courant algebroid:

\textbf{Lie algebroid up to homotopy}

[Ikeda, Uchino]
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[Ikeda, Uchino]

Degree two functions correspond to elements in \( TM \oplus \bigwedge^2 T^*M \)

\[
A^i(X)\chi_i + \frac{1}{2} \alpha_{ij}(X) \psi^i \psi^j \quad \longleftrightarrow \quad A^i(X) \frac{\partial}{\partial X^i} + \frac{1}{2} \alpha_{ij}(X) dX^i \wedge dX^j,
\]

and the derived bracket

\[
[A + \alpha, B + \beta]_C = \{\{\gamma, A + \alpha\}, B + \beta\} = [A, B] + \mathcal{L}_A\beta - \mathcal{L}_B\alpha + \frac{1}{2} d(\iota_B \alpha - \iota_A \beta)
\]

is the higher Courant bracket on \( TM \oplus \bigwedge^2 T^*M \).
Relation to higher Courant bracket

- The master equation defines a higher analog of the Courant algebroid:

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  \[\text{[Ikeda,Uchino]}\]

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\]

\[
= [A, B] + \mathcal{L}_A \beta - \mathcal{L}_B \alpha + \frac{1}{2} d(\iota_B \alpha - \iota_A \beta)
\]

is the higher Courant bracket on \( TM \oplus \Lambda^2 T^*M \).

- This construction allows the definition of geometric fluxes.
Summary

- We reformulated the AKSZ constructions of A- and B-models on doubled space and introduced a Courant sigma-model for generalized complex structure, which reduces to the A- and B-models on the boundary.
- As an application, we derived topological S-duality from generalized complex structure.
- Our approach led to new classes of Courant algebroids associated to (generalized) complex geometry.
- We constructed an AKSZ 3-brane, which unifies the known topological membranes on $G_2$-manifold.
- We showed that its derived bracket gives the higher Courant bracket on $TM \oplus \bigwedge^2 T^*M$. 
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Outlook

- Explore the relation of our Courant sigma-model for generalized complex structure to topological string theory.
- Investigate the AKSZ 3-brane theory in the context of non-geometric flux backgrounds.
Thank you for your attention!