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Classification of non-Riemannian generalized metrics in Double Field Theory

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Dualities and Generalized Geometries, Corfu

Talk based on:

K. M. and J-H. Park
"**Classification of non-Riemannian doubled-yet-gauged spacetime**."
Eur. Phys. J. C 77, no. 10, 685 (2017)
arXiv:1707.03713

K. Cho, K. M. and J-H. Park
"**Kaluza-Klein reduction on a maximally non-Riemannian space is moduli-free**."
arXiv:1808.10605

(Partial) motivation

- Generalized geometry naturally accommodates **Riemannian** geometry $g_{\mu\nu}$
- Recently, some **non-Riemannian** (nR) geometries have known a surge of interest:
  - **Newton-Cartan geometry** $h^{\mu\nu}\psi_{\nu} = 0$

Cartan (23'), Friedrichs (27') , Trautman (63'),Havas (64')
Dombrowski, Horneffer (68'), Künzle (72'), Ehlers (81')

✓ **Quantum Hall effect** Son (13')
✓ **Lifshitz/Schrödinger holography** Hartong et al. (13') Bergshoeff et al. (14')
✓ **Hořava-Lifshitz gravity** Hartong et al. (15'), Afshar et al. (15')
✓ **Non-relativistic supergravity** Andringa et al. (13')

  - **Carroll geometry** $\gamma_{\mu\nu}\xi^{\nu} = 0$

Henneaux (79'), Duval et al. (14')

✓ **BMS group** Duval et al. (14')
✓ **Flat holography** Hartong (15'), Bagchi et al. (16')
✓ **Hydrodynamics** Ciambelli et al. (18')

- Double Field Theory may provide a framework to account for these nR geometries.

Double Field Theory

Stringy geometry

- Spacetime is doubled, with coordinates X^A where $A \in \{1, \dots, 2D\}$
- Section condition $\partial^A \phi \, \partial_A \psi = 0$
- **O(D,D)** invariant metric $\mathcal{J}_{AB} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$
- Generalized Lie derivative $\hat{\mathcal{L}}_X Y^A = X^C \partial_C Y^A + (\partial^A X_C - \partial_C X^A) Y^C$
- C-bracket $[X, Y]^A_{\mathbf{C}} = X^B \partial_B Y^A - Y^B \partial_B X^A + \frac{1}{2} Y^B \partial^A X_B - \frac{1}{2} X^B \partial^A Y_B$

$$\left[\hat{\mathcal{L}}_X, \hat{\mathcal{L}}_Y\right] \sim \hat{\mathcal{L}}_{[X,Y]_{\mathbf{C}}}$$

$\sim$:up to section condition

Double Field Theory

Generalized metric $\mathcal{H}_{AB}$

- Defining conditions $\mathcal{H}_{AB} = \mathcal{H}_{BA}$, $\mathcal{H}_A{}^B \mathcal{H}_B{}^C = \delta_A{}^C$ where $\mathcal{H}_A{}^B := \mathcal{H}_{AC}\mathcal{J}^{CB}$

  Symmetric **O(D,D)**

- Under the split $X^A = (\tilde{x}_\mu, x^\mu)$: $\mathcal{H}_{AB} = \begin{pmatrix} \mathcal{H}^{\mu\nu} & \mathcal{H}^\mu{}_\lambda \\ \mathcal{H}_\kappa{}^\nu & \mathcal{H}_{\kappa\lambda} \end{pmatrix}$ $\begin{array}{l} A \in \{1, \ldots, 2D\} \\ \mu \in \{1, \ldots, D\} \end{array}$

  the above constraints read: $$\begin{array}{lll} \mathcal{H}^{\mu\nu} = \mathcal{H}^{\nu\mu} & , & \mathcal{H}^{(\mu}{}_\rho \mathcal{H}^{\nu)\rho} = 0 \\ \mathcal{H}_{\mu\nu} = \mathcal{H}_{\nu\mu} & , & \mathcal{H}_{\rho(\mu} \mathcal{H}^\rho{}_{\nu)} = 0 \\ \mathcal{H}_\mu{}^\nu = \mathcal{H}^\nu{}_\mu & , & \mathcal{H}^\mu{}_\rho \mathcal{H}^\rho{}_\nu + \mathcal{H}^{\mu\rho}\mathcal{H}_{\rho\nu} = \delta^\mu{}_\nu \end{array}$$

  Symmetric **O(D,D)**

Question: What is the most general solution to this set of equations?

Main result

Classification of Generalized metrics

Proposition:
- Generalized metrics of DFT are classified by two non-negative integers $(n,\bar{n})$ such that $0\leq n+\bar{n}\leq D$
- The most general $(n,\bar{n})$ DFT Generalized metric takes the form:

$$\mathcal{H}_{AB}=\begin{pmatrix} H^{\mu\nu} & -H^{\mu\sigma}B_{\sigma\lambda}+Y_i^{\mu}X_{\lambda}^{i}-\bar{Y}_{\bar{i}}^{\mu}\bar{X}_{\lambda}^{\bar{i}} \\ B_{\kappa\rho}H^{\rho\nu}+X_{\kappa}^{i}Y_i^{\nu}-\bar{X}_{\kappa}^{\bar{i}}\bar{Y}_{\bar{i}}^{\nu} & K_{\kappa\lambda}-B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda}+2X_{(\kappa}^{i}B_{\lambda)\rho}Y_i^{\rho}-2\bar{X}_{(\kappa}^{\bar{i}}B_{\lambda)\rho}\bar{Y}_{\bar{i}}^{\rho} \end{pmatrix}$$

➡ **Symmetries** $\quad H^{\mu\nu}=H^{\nu\mu}\quad,\quad K_{\mu\nu}=K_{\nu\mu}\quad,\quad B_{\mu\nu}=-B_{\nu\mu}$

➡ **Kernels**

$$\begin{array}{lll} H^{\mu\nu}X_{\nu}^{i}=0 \quad, & K_{\mu\nu}Y_i^{\nu}=0 & i\in\{1,\ldots,n\} \\ H^{\mu\nu}\bar{X}_{\nu}^{\bar{i}}=0 \quad, & K_{\mu\nu}\bar{Y}_{\bar{i}}^{\nu}=0 & \bar{i}\in\{1,\ldots,\bar{n}\} \end{array}$$

Both $H^{\mu\nu}$ and $K_{\mu\nu}$ have kernel of dimension $n+\bar{n}$

➡ **Completeness** $\quad H^{\mu\rho}K_{\rho\nu}+Y_i^{\mu}X_{\nu}^{i}+\bar{Y}_{\bar{i}}^{\mu}\bar{X}_{\nu}^{\bar{i}}=\delta^{\mu}{}_{\nu}$

Main result

Classification of Generalized metrics

Proposition:

- Generalized metrics of DFT are classified by two non-negative integers $(n,\bar{n})$ such that $0\leq n+\bar{n}\leq D$
- The most general $(n,\bar{n})$ DFT Generalized metric takes the form:

$$\mathcal{H}_{AB}=\begin{pmatrix}1&0\\B&1\end{pmatrix}\begin{pmatrix}H & Y_i(X^i)^T-\bar{Y}_{\bar{i}}(\bar{X}^{\bar{i}})^T\\ X^i(Y_i)^T-\bar{X}^{\bar{i}}(\bar{Y}_{\bar{i}})^T & K\end{pmatrix}\begin{pmatrix}1&-B\\0&1\end{pmatrix}$$

- The following transformation leaves the metric invariant:

$$\begin{aligned}
Y_i^{\mu} &\overset{V}{\mapsto} Y_i^{\mu}+H^{\mu\nu}V_{\nu i}\\
\bar{Y}_{\bar{i}}^{\mu} &\overset{V}{\mapsto} \bar{Y}_{\bar{i}}^{\mu}+H^{\mu\nu}\bar{V}_{\nu\bar{i}}\\
K_{\mu\nu} &\overset{V}{\mapsto} K_{\mu\nu}-2X^i_{(\mu}K_{\nu)\rho}H^{\rho\sigma}V_{\sigma i}-2\bar{X}^{\bar{i}}_{(\mu}K_{\nu)\rho}H^{\rho\sigma}\bar{V}_{\sigma\bar{i}}+(X^i_{\mu}V_{\rho i}+\bar{X}^{\bar{i}}_{\mu}\bar{V}_{\rho\bar{i}})H^{\rho\sigma}(X^j_{\nu}V_{\sigma j}+\bar{X}^{\bar{j}}_{\nu}\bar{V}_{\sigma\bar{j}})\\
B_{\mu\nu} &\overset{V}{\mapsto} B_{\mu\nu}-2X^i_{[\mu}V_{\nu]i}+2\bar{X}^{\bar{i}}_{[\mu}\bar{V}_{\nu]\bar{i}}+2X^i_{[\mu}\bar{X}^{\bar{i}}_{\nu]}\left(Y_i^{\rho}\bar{V}_{\rho\bar{i}}+\bar{Y}_{\bar{i}}^{\rho}V_{\rho i}+V_{\rho i}H^{\rho\sigma}\bar{V}_{\sigma\bar{i}}\right)
\end{aligned}$$

Two sets of local parameters $V_{\mu i}$ and $\bar{V}_{\mu\bar{i}}$

Note that $H^{\mu\nu}$, X^i_{μ} and $\bar{X}^{\bar{i}}_{\mu}$ are invariant.

Examples (0,0) case: Generalized Geometry

Denoting $H^{\mu\nu} \equiv g^{\mu\nu}$ and $K_{\mu\nu} \equiv g_{\mu\nu}$, the $(n,\bar{n}) = (0,0)$ generalized metric reads:

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\sigma} B_{\sigma\lambda} \\ B_{\kappa\rho}\, g^{\rho\nu} & g_{\kappa\lambda} - B_{\kappa\rho}\, g^{\rho\sigma} B_{\sigma\lambda} \end{pmatrix} \quad \text{where } g^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}{}_{\nu}$$

- **Conditions for O(D,D)**
- $g_{\mu\nu}$ **is a (pseudo)-Riemannian metric**
- $B_{\mu\nu}$ **is a 2-form**

- **Invariance**
- **None**

Only case including a Riemannian metric.
Other cases will be qualified as **non-Riemannian**.

Examples (1,0) case: Generalized Newton-Cartan geometry

Denoting $H^{\mu\nu} \equiv h^{\mu\nu}, X_\mu \equiv \psi_\mu, Y^\mu \equiv N^\mu$ and $K_{\mu\nu} \equiv \overset{N}{\gamma}_{\mu\nu}$,

the $(1,0)$ generalized metric reads:

$$\mathcal{H}_{AB} = \begin{pmatrix} h^{\mu\nu} & -h^{\mu\sigma}\overset{N}{B}_{\sigma\lambda} + N^\mu\psi_\lambda \\ \overset{N}{B}_{\kappa\rho}h^{\rho\nu} + \psi_\kappa N^\nu & \overset{N}{\gamma}_{\kappa\lambda} - \overset{N}{B}_{\kappa\rho}h^{\rho\sigma}\overset{N}{B}_{\sigma\lambda} + 2\psi_{(\kappa}\overset{N}{B}_{\lambda)\rho}N^\rho \end{pmatrix}$$

- **Conditions for O(D,D)**
  - $(h^{\mu\nu}, \psi_\mu)$ **is a Leibnizian metric structure**
  - N^μ **is a field of observers**
  - $\overset{N}{\gamma}_{\mu\nu}$ **is the transverse metric associated to** N^μ
  - $\overset{N}{B}_{\mu\nu}$ **is a 2-form**

where:
- $h^{\mu\nu}\psi_\nu = 0$
- $\overset{N}{\gamma}_{\mu\nu}N^\nu = 0$
- $\psi_\mu N^\mu = 1$
- $h^{\mu\nu}\overset{N}{\gamma}_{\mu\nu} + N^\mu\psi_\nu = \delta^\mu{}_\nu$

- **Invariance**
  - **Milne boost of** $(N^\mu, \overset{N}{\gamma}_{\mu\nu}, \overset{N}{B}_{\mu\nu})$

Examples (D-1,0) case: Generalized Carroll geometry

Denoting $H^{\mu\nu} \equiv \xi^\mu \xi^\nu$, $K_{\mu\nu} \equiv A_\mu A_\nu$, $\gamma_{\mu\nu} \equiv X^i_\mu Y^j_\nu \delta_{ij}$ and $\overset{A}{h}{}^{\mu\nu} \equiv Y^\mu_i Y^\nu_j \delta^{ij}$
the $(D-1,0)$ generalized metric reads:

$$\mathcal{H}_{AB} = \begin{pmatrix} \xi^\mu \xi^\nu & -\xi^\mu \xi^\sigma \overset{A}{B}_{\sigma\lambda} + \overset{A}{h}{}^{\mu\sigma} \gamma_{\sigma\lambda} \\ \overset{A}{B}_{\kappa\rho} \xi^\rho \xi^\nu + \gamma_{\kappa\sigma} \overset{A}{h}{}^{\sigma\nu} & A_\kappa A_\lambda - \overset{A}{B}_{\kappa\rho} \xi^\rho \xi^\sigma \overset{A}{B}_{\sigma\lambda} + 2\,\gamma_{\sigma(\kappa} \overset{A}{B}_{\lambda)\rho} \overset{A}{h}{}^{\rho\sigma} \end{pmatrix}$$

- **Conditions for O(D,D)**
- $(\gamma_{\mu\nu}, \xi^\mu)$ **is a Carrollian metric structure**
- A_μ **is an Ehresmann connection**
- $\overset{A}{h}{}^{\mu\nu}$ **is the transverse cometric associated to** A_μ
- $\overset{A}{B}_{\mu\nu}$ **is a 2-form**

where:
- $\overset{A}{h}{}^{\mu\nu} A_\nu = 0$
- $\gamma_{\mu\nu} \xi^\nu = 0$
- $A_\mu \xi^\mu = 1$
- $\overset{A}{h}{}^{\mu\lambda} \gamma_{\lambda\nu} + \xi^\mu A_\nu = \delta^\mu{}_\nu$

- **Invariance**
- **Carroll boost of** $(A, \overset{A}{h}{}^{\mu\nu}, \overset{A}{B}_{\mu\nu})$

c.f. J-H. Park's talk

Applications

Particle and string dynamics $(n,\bar{n})$

Free particle DFT action

Ko, Park, Suh (16')

$$S_{\text{particle}} = \int \mathrm{d}\tau \; e^{-1}\,\mathrm{D}_\tau x^A\,\mathrm{D}_\tau x^B \mathcal{H}_{AB}(x) - \frac{1}{4}m^2 e \quad \text{with} \quad \mathrm{D}_\tau x^A := \dot{x}^A - \mathcal{A}^A_\tau$$

- **Two Lagrange multipliers** $\lambda_i := Y_i^\mu \mathcal{A}_{\tau\mu} \;,\; \bar{\lambda}_{\bar{i}} := \bar{Y}_{\bar{i}}^\mu \mathcal{A}_{\tau\mu}$
- **Freezing of « timelike » directions** $X^i_\mu \dot{x}^\mu = 0 \quad , \quad \bar{X}^{\bar{i}}_\mu \dot{x}^\mu = 0$

Free string DFT action

Hull (06'), Lee, Park (13')

$$S_{\text{string}} = -\tfrac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \; \frac{1}{2}\sqrt{-h}h^{\alpha\beta}\mathrm{D}_\alpha x^A \mathrm{D}_\beta x^B \mathcal{H}_{AB}(x) + \epsilon^{\alpha\beta}\mathrm{D}_\alpha x^A \mathcal{A}_{\beta A} \,,\, \mathrm{D}_\alpha x^A := \partial_\alpha x^A - \mathcal{A}^A_\alpha$$

- **Chirality of « timelike » directions** $\lambda_i := Y_i^\mu \mathcal{A}_{\tau\mu} \;,\; \bar{\lambda}_{\bar{i}} := \bar{Y}_{\bar{i}}^\mu \mathcal{A}_{\tau\mu}$

$$X^i_\mu \left(\partial_\alpha x^\mu + \tfrac{1}{\sqrt{-h}}\epsilon_\alpha{}^\beta \partial_\beta x^\mu\right) = 0 \quad , \quad \bar{X}^{\bar{i}}_\mu \left(\partial_\alpha x^\mu - \tfrac{1}{\sqrt{-h}}\epsilon_\alpha{}^\beta \partial_\beta x^\mu\right) = 0$$

n **Chiral** $\qquad$ $\bar{n}$ **Anti-chiral**

Conclusion

- Full classification of DFT Generalized metrics
  - Examples include:
    - ✓ (0,0) Generalized geometry
    - ✓ (1,0) Newton-Cartan geometry
    - ✓ (D-1,0) Carroll geometry
    - ✓ (D,0) Siegel's string
    - ✓ (1,1) Gomis-Ooguri Non-Relativistic string
  - Particle and string dynamics
  - Freezing/chirality of *timelike* directions
  - Double vielbein formulation for $(n, \bar{n})$

- Kaluza-Klein ansatz for DFT c.f. J-H. Park's talk
  - Applications include:
    - ✓ DFT avatar of the light-like reduction formalism of Duval et al. (85')
    - ✓ Full DFT formulation of Newton-Cartan and Carroll particle dynamics
    - ✓ Reduction of DFT action to Heterotic DFT
    - ✓ Reduction of DFT string action to heterotic string

- Perspectives
  - ➔ Dynamical DFT equations for $(n, \bar{n})$
  - ➔ Application to string compactification

In memory of Christian Duval

Thank You

Kaluza-Klein ansatz for DFT

c.f. J-H. Park's talk

K. M., J-H. Park (17')
K. Cho, K. M., J-H. Park (18')

The DFT Kaluza-Klein ansatz splits the ambient doubled coordinates:

$$\hat{D} = D' + D$$

and metrics:

$$\hat{\mathcal{J}} = \begin{pmatrix} \mathcal{J}' & 0 \\ 0 & \mathcal{J} \end{pmatrix} \qquad \hat{\mathcal{H}} = \exp\left[\hat{W}\right] \begin{pmatrix} \mathcal{H}' & 0 \\ 0 & \mathcal{H} \end{pmatrix} \exp\left[\hat{W}^{\mathbf{T}}\right]$$

- **Ambient** $\hat{\mathcal{H}}\ ,\ \hat{\mathcal{J}}$
- **Internal** $\mathcal{H}'\ ,\ \mathcal{J}'$
- **External** $\mathcal{H}\ ,\ \mathcal{J}$

where $\hat{W} = \begin{pmatrix} 0 & -W^{\mathbf{T}} \\ W & 0 \end{pmatrix}$ is the *graviphoton* satisfying:

$$W_A{}^{A'} W^{AB'} = 0\ ,\ \ W^A{}_{A'} \partial_A = 0$$

(sets half of the components to zero)

c.f. J-H. Park's talk

Application

Heterotic DFT action

K. Cho, K. M., J-H. Park (18')

Plugging into the $\hat{D} = D' + D$ DFT action $\hat{S}_{(0)}$

- **Maximally non-Riemannian internal space** $\mathcal{H}'_{A'B'} = \mathcal{J}'_{A'B'}$
- **DFT Scherk-Schwarz twist** M. Grana, D. Marques (12'), W. Cho *etal*. (15')

$$\begin{aligned}\hat{S}_{(0)} = S_{(0)} &- \tfrac{1}{4}\mathcal{H}^{AC}\mathcal{H}^{BD}F_{AB}{}^{\dot{A}}F_{CD\dot{A}} \\ &- \tfrac{1}{12}\mathcal{H}^{AD}\mathcal{H}^{BE}\mathcal{H}^{CF}\omega_{ABC}\omega_{DEF} \\ &+ \tfrac{1}{2}\mathcal{H}^{AD}\mathcal{H}^{BE}\mathcal{H}^{CF}\omega_{ABC}\mathcal{H}_{[D}{}^{G}\partial_E\mathcal{H}_{F]G}\end{aligned}$$

DFT Heterotic action

Upon choosing the Riemannian parameterization for $\mathcal{H}_{AB}$, $\hat{S}_{(0)}$ reduces to the heterotic supergravity action.

with
- $F_{AB}{}^{\dot{C}}$ Yang-Mills (non-abelian)
- ω_{ABC} Chern-Simons

- **Non-Riemannian internal space is *rigid*:** $\delta\mathcal{H}'_{A'B'} = 0$
- **Prevents the appearance of any graviscalar**
- **Moduli-free Kaluza-Klein reduction**