# Gravity in three dimensions as a noncommutative gauge theory

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Workshop on the Standard Model and Beyond

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# *Gravity in three dimensions as a gauge theory*

*The algebra* *Witten '88*

- 3-d Gravity: gauge theory of $\mathfrak{iso}(1,2)$ (Poincaré - isometry of $\mathrm{M}^3$)
- Presence of $\Lambda$: dS or AdS algebras, i.e. $\mathfrak{so}(1,3), \mathfrak{so}(2,2)$
- Corresponding generators: $\mathrm{P}_a, \mathrm{J}_{ab}, a = 1,2,3$ (translations, LT)
- Satisfy the following CRs:

$$[J_{ab}, J_{cd}] = 4\eta_{[a[c}J_{d]b]} \,, \quad [P_a, J_{bc}] = 2\eta_{a[b}P_{c]} \,, \quad [P_a, P_b] = \Lambda J_{ab}$$

- CRs valid in *arbitrary* dim; particularly in 3-d:

$$[J_a, J_b] = \epsilon_{abc}J^c \,, \quad [P_a, J_b] = \epsilon_{abc}P^c \,, \quad [P_a, P_b] = \Lambda\epsilon_{abc}J^c$$

- After the redefinition: $J^a = \frac{1}{2}\epsilon^{abc}J_{bc}$

*The gauging procedure*

- Intro of a gauge field for each generator: ${e_\mu}^a, {\omega_\mu}^a$(transl, LT)
- The Lie-valued 1-form gauge connection is:

$$A_\mu = {e_\mu}^a(x) P_a + {\omega_\mu}^a(x) J_a$$

- Transforms in the adjoint rep, according to the rule:

$$\delta A_\mu = \partial_\mu \epsilon + [A_\mu, \epsilon]$$

- The gauge transformation parameter is expanded as:

$$\epsilon = \xi^a(x) P_a + \lambda^a(x) J_a$$

- *Combining* the above $\rightarrow$ transformations of the fields:

$$\delta {e_\mu}^a = \partial_\mu \xi^a - \epsilon^{abc}(\xi_b \omega_{\mu c} + \lambda_b e_{\mu c})$$
$$\delta {\omega_\mu}^a = \partial_\mu \lambda^a - \epsilon^{abc}(\lambda_b \omega_{\mu c} + \Lambda \xi_b e_{\mu c})$$

## *Curvatures and action*

- Curvatures of the fields are given by:

$$R_{\mu\nu}(A) = 2\partial_{[\mu}A_{\nu]} + [A_\mu, A_\nu]$$

- Tensor $R_{\mu\nu}$ is also Lie-valued:

$$R_{\mu\nu}(A) = T_{\mu\nu}{}^a P_a + R_{\mu\nu}{}^a J_a$$

- *Combining* the above → curvatures of the fields:

$$T_{\mu\nu}{}^a = 2\partial_{[\mu}e_{\nu]}{}^a + 2\epsilon^{abc}\omega_{[\mu b}e_{\nu]c}$$
$$R_{\mu\nu}{}^a = 2\partial_{[\mu}\omega_{\nu]}{}^a + \epsilon^{abc}(\omega_{\mu b}\omega_{\nu c} + \Lambda e_{\mu b}e_{\nu c})$$

- The Chern-Simons action functional of the Poincaré, dS or AdS algebra is found to be *identical* to the 3-d E-H action:

$$\mathcal{S}_{CS} = \frac{1}{16\pi G}\int \epsilon^{\mu\nu\rho}(e_\mu{}^a(\partial_\nu\omega_{\rho a} - \partial_\rho\omega_{\nu a}) + \epsilon_{abc}e_\mu{}^a\omega_\nu{}^b\omega_\rho{}^c + \frac{\Lambda}{3}\epsilon_{abc}e_\mu{}^c e_\nu{}^b e_\rho{}^c) \equiv S_{EH}$$

*3-d gravity is a Chern-Simons gauge theory.*

*Remarks on 4-d gravity*

*Utiyama '56, Kibble '61*
*MacDowell-Mansouri '77*
*Kibble-Stelle '85*

- Vielbein formulation of GR: Gauging Poincaré algebra $\mathfrak{iso}(1,3)$
- Comprises ten generators: $\mathrm{P}_a, \mathrm{J}_{ab}, a = 1, \ldots 4$ (transl, LT)
- Satisfy the aforementioned CRs (for $\Lambda = 0$)
- Gauging in the same way leading to field transformations
- Curvatures are obtained accordingly
- Dynamics follow from the E-H action:

$$\mathcal{S}_{EH4} = \frac{1}{2} \int \mathrm{d}^4 x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} e_\mu{}^a e_\nu{}^b R_{\rho\sigma}{}^{cd}$$

- Form of Einstein action: $A^2(dA + A^2)$
- Such action does not exist in gauge theories
- In that sense, 4-d gravity is not equivalent to a gauge theory.

## *Gauge theories on noncommutative spaces*

### *The nc framework* 

*Szabo '01*

- Late 40's: Nc structure of spacetime at small scales for an effective ultraviolet cutoff $\to$ control of divergences in qfts *Snyder '47*
- Ignored $\to$ success of renormalization programme
- Inspiration from qm: Operators instead of variables
- Nc spacetime defined by replacing coords $x^i$ by Herm generators $X^i$ of a nc algebra of functions, $\mathcal{A}$, obeying: $[X^i, X^j] = i\theta^{ij}$ *Connes '94, Madore '99*
- 80's: nc geometry revived after the generalization of diff structure *Connes '85, Woronowicz '87*
- Along with the definition of a generalized integration $\to$ Yang-Mills gauge theories on nc spaces *Connes-Rieffel '87*

- Operators $X_\mu \in \mathcal{A}$ satisfy the CR: $[X_\mu, X_\nu] = i\theta_{\mu\nu}$, $\theta_{\mu\nu}$ arbitrary
- Lie-type nc: $[X_\mu, X_\nu] = iC_{\mu\nu}{}^\rho X_\rho$
- Natural intro of nc gauge theories through *covariant nc coordinates*: $\mathcal{X}_\mu = X_\mu + A_\mu$ *Madore-Schraml-Schupp-Wess '00*
- Obeys a covariant gauge transformation rule: $\delta\mathcal{X}_\mu = i[\epsilon, \mathcal{X}_\mu]$
- $A_\mu$ transforms in analogy with the gauge connection: $\delta A_\mu = -i[X_\mu, \epsilon] + i[\epsilon, A_\mu]$, ($\epsilon$ - the gauge parameter)
- Definition of a (Lie-type) nc *covariant field strength tensor*: $F_{\mu\nu} = [\mathcal{X}_\mu, \mathcal{X}_\nu] - iC_{\mu\nu}{}^\rho \mathcal{X}_\rho$

## *Non-Abelian case*

- Gauge theory could be Abelian or non-Abelian:
  - Abelian if $\epsilon$ is a function in $\mathcal{A}$
  - Non-Abelian if $\epsilon$ is matrix valued $(\mathrm{Mat}(\mathcal{A}))$

▷ *In non-Abelian case, where are the gauge fields valued?*

- Let us consider the CR of two elements of an algebra:

$$[\epsilon, A] = [\epsilon^A T^A, A^B T^B] = \frac{1}{2}\{\epsilon^A, A^B\}[T^A, T^B] + \frac{1}{2}[\epsilon^A, A^B]\{T^A, T^B\}$$

- *Not* possible to restrict to a matrix algebra:
  last term neither *vanishes* in nc nor is an *algebra element*

- There are two options to overpass the difficulty:
  - Consider the universal enveloping algebra
  - Extend the generators and/or fix the rep so that the anticommutators close

▷ *We employ the second option*

## *3-d fuzzy spaces based on SU(2) and SU(1,1)*

### *The Euclidean case*

- Euclidean case: 3-d fuzzy space based on SU(2)
- Fuzzy sphere, $S_F^2$: Matrix approximation of ordinary sphere, $S^2$

*Hoppe '82, Madore '92*
*For higher-dim $S_F$ see:*
*Kimura '02, Dolan - O'Connor '03,*
*Sperling - Steinacker '17*

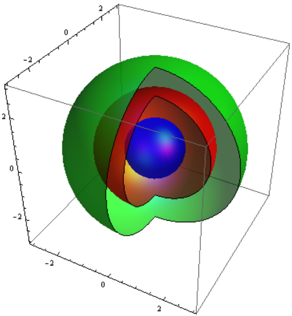
- $S^2$ defined by coordinates of $\mathbb{R}^3$ modulo $\sum_{a=1}^{3} x_a x^a = r^2$
- $S_F^2$ defined by three rescaled angular momentum operators, $X_i = \lambda J_i$, $J_i$ the Lie algebra generators in a UIR of SU(2). The $X_i$s satisfy:

$$[X_i, X_j] = i\lambda\epsilon_{ijk}X_k \,, \quad \sum_{i=1}^{3} X_i X_i = \lambda^2 j(j+1) := r^2 \,, \lambda \in \mathbb{R} \,, 2j \in \mathbb{N}$$

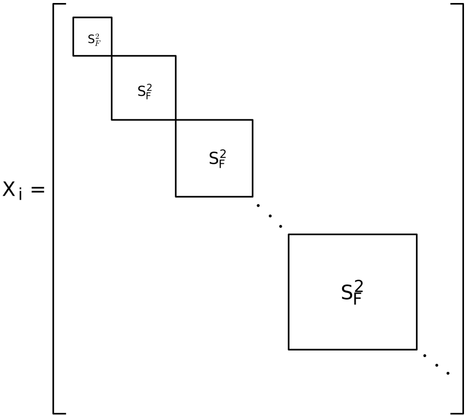
- Allowing $X_i$ to live in *reducible* rep: obtain the nc $\mathbb{R}^3_\lambda$, viewed as direct sum of $S_F^2$ *with all possible radii* (determined by $2j$) - a discrete foliation of $\mathbb{R}^3$ by multiple $S_F^2$

*Hammou-Lagraa-Sheikh Jabbari '02*
*Vitale-Wallet '13, Vitale '14*

# *The fuzzy space* $\mathbb{R}^3_\lambda$




$\mathbb{R}^3_\lambda$*: Foliation of* $\mathbb{R}^3$ *by fuzzy spheres (onion-like construction)*




*Matrix (coordinate) of* $\mathbb{R}^3_\lambda$ *as a block diagonal form - each block is a fuzzy sphere*

*The Lorentzian case*

- In analogy: Lorentzian case: 3-d fuzzy space based on $\mathrm{SU}(1,1)$

  *Grosse - Prešnajder '93*

  *Jurman-Steinacker '14*

- Fuzzy hyperboloids, $\mathrm{dS}^2_\mathrm{F}$, defined by three rescaled operators, $X_i = \lambda J_i$, (in appropriate irreps) satisfying:

$$[X_i, X_j] = i\lambda {C_{ij}}^k X_k \,, \quad \sum_{i,j} \eta^{ij} X_i X_j = \lambda^2 j(j-1)\,,$$

- where ${\mathrm{C}_{ij}}^k$ are the structure constants of $\mathfrak{su}(1,1)$
- Again, letting $\mathrm{X}_i$ live in (infinite-dim) *reducible* reps: Block diagonal form - each block being a ${\mathrm{dS}_\mathrm{F}}^2$
- 3-d Minkowski spacetime foliated with leaves being ${\mathrm{dS}_\mathrm{F}}^2$ of different radii

## *Gravity as gauge theory on 3-d fuzzy spaces*

*The Lorentzian case* — *Aschieri-Castellani '09*

- Consideration of the foliated $\mathrm{M}^3$ with $\Lambda > 0$
- Natural symmetry of the space: $\mathrm{SO}(1,3)$ ($\mathrm{SO}(4)$ for the Eucl.)

*Kováčik - Presnajder '13*

- Consider the corresponding spin group: $SO(1,3) \cong Spin(1,3) = SL(2,\mathbb{C})$
- Anticommutators *do not close* → Fix at spinor rep generated by: $\sum_{AB} = \frac{1}{2}\gamma_{AB} = \frac{1}{4}[\gamma_A, \gamma_B]\,, A = 1, \dots 4$
- Satisfying the CRs and aCRs:

$$[\gamma_{AB}, \gamma_{CD}] = 8\eta_{[A[C}\gamma_{D]B]}\,, \quad \{\gamma_{AB}, \gamma_{CD}\} = 4\eta_{C[B}\eta_{A]D}\mathbb{1} + 2i\epsilon_{ABCD}\gamma_5$$

- Inclusion of $\gamma_5$ and identity in the algebra → extension of $SL(2,\mathbb{C})$ to $GL(2,\mathbb{C})$ with set of generators: $\{\gamma_{AB}, \gamma_5, i\mathbb{1}\}$

## *SO(3) notation*

- In SO(3) notation: $\gamma_{a4} \equiv \gamma_a$ and $\tilde{\gamma}^a \equiv \epsilon^{abc}\gamma_{bc}$, with $a = 1, 2, 3$
- The CRs and aCRs are now written:

$$[\tilde{\gamma}^a, \tilde{\gamma}^b] = -4\epsilon^{abc}\tilde{\gamma}_c\,,\ [\gamma_a, \tilde{\gamma}_b] = -4\epsilon_{abc}\gamma^c\,,\ [\gamma_a, \gamma_b] = \epsilon_{abc}\tilde{\gamma}^c\,,\ [\gamma^5, \gamma^{AB}] = 0$$
$$\{\tilde{\gamma}^a, \tilde{\gamma}^b\} = -8\eta^{ab}\mathbb{1}\,,\ \{\gamma_a, \tilde{\gamma}^b\} = 4i\delta_a^b\gamma_5\,,\ \{\gamma_a, \gamma_b\} = 2\eta_{ab}\mathbb{1}\,,$$
$$\{\gamma^5, \gamma^a\} = i\tilde{\gamma}_a\,,\ \{\tilde{\gamma}^5, \gamma^a\} = -4i\gamma_a$$

- Proceed with the gauging of $\mathrm{GL}(2, \mathbb{C})$
- Determine the covariant coordinate: $\mathcal{X}_\mu = X_\mu + \mathcal{A}_\mu$
  $\mathcal{A}_\mu = \mathcal{A}_\mu^i(X_a) \otimes T^i$ the $\mathfrak{gl}(2, \mathbb{C})$-valued gauge connection
- Gauge connection expands on the generators as:

$$\mathcal{A}_\mu = e_\mu{}^a(X) \otimes \gamma_a + \omega_\mu{}^a(X) \otimes \tilde{\gamma}_a + A_\mu(X) \otimes i\mathbb{1} + \tilde{A}_\mu(X) \otimes \gamma_5$$

*See also: Nair '03,'06, Abe - Nair '03*

- Gauge parameter, $\epsilon$, expands similarly:
  $\epsilon = \xi^a(X) \otimes \gamma_a + \lambda^a(X) \otimes \tilde{\gamma}_a + \epsilon_0(X) \otimes i\mathbb{1} + \tilde{\epsilon}_0(X) \otimes \gamma_5$

*Kinematics*

- Covariant transf rule: $\delta\mathcal{X}_\mu = [\epsilon, \mathcal{X}_\mu] \to$ transf of the gauge fields:

$$\delta e_\mu{}^a = -i[X_\mu + A_\mu, \xi^a] - 2\{\xi_b, \omega_{\mu c}\}\epsilon^{abc} - 2\{\lambda_b, e_{\mu c}\}\epsilon^{abc} + i[\epsilon_0, e_\mu{}^a] - 2i[\lambda^a, \tilde{A}_\mu] - 2i[\tilde{\epsilon}_0, \omega_\mu{}^a]$$

$$\delta\omega_\mu{}^a = -i[X_\mu + A_\mu, \lambda^a] + \tfrac{1}{2}\{\xi_b, e_{\mu c}\}\epsilon^{abc} - 2\{\lambda_b, \omega_{\mu c}\}\epsilon^{abc} + i[\epsilon_0, \omega_\mu{}^a] + \tfrac{i}{2}[\xi^a, \tilde{A}_\mu] + \tfrac{i}{2}[\tilde{\epsilon}_0, e_\mu{}^a]$$

$$\delta A_\mu = -i[X_\mu + A_\mu, \epsilon_0] - i[\xi_a, e_\mu{}^a] + 4i[\lambda_a, \omega_\mu{}^a] - i[\tilde{\epsilon}_0, \tilde{A}_\mu]$$

$$\delta\widetilde{A}_\mu = -i[X_\mu + A_\mu, \tilde{\epsilon}_0] + 2i[\xi_a, \omega_\mu{}^a] + 2i[\lambda_a, e_\mu{}^a] + i[\epsilon_0, \tilde{A}_\mu]$$

- Commutative limit: inner derivation becomes $[X_\mu, f] \to -i\partial_\mu f$:

$$\delta e_\mu{}^a = -\partial_\mu\xi^a - 4\xi_b\omega_{\mu c}\epsilon^{abc} - 4\lambda_b e_{\mu c}\epsilon^{abc}$$
$$\delta\omega_\mu{}^a = -\partial_\mu\lambda^a + \xi_b e_{\mu c}\epsilon^{abc} - 4\lambda_b\omega_{\mu c}\epsilon^{abc}$$

- After the redefinitions: $\gamma_a \to \frac{2i}{\sqrt{\Lambda}}P_a$ , $\tilde{\gamma}_a \to -4J_a$ , $4\lambda^a \to \lambda^a$, $\xi^a\frac{2i}{\sqrt{\Lambda}} \to -\xi^a, e^a_\mu \to \frac{\sqrt{\Lambda}}{2i}e^a_\mu, \omega^a_\mu \to -\frac{1}{4}\omega^a_\mu$ → 3-d gravity

## *Curvatures*

- Definition of curvature:

$$\mathcal{R}_{\mu\nu} = [\mathcal{X}_\mu, \mathcal{X}_\nu] - i\lambda C_{\mu\nu}{}^{\rho}\mathcal{X}_\rho$$

- Curvature tensor can be expanded in the $\mathrm{GL}(2,\mathbb{C})$ generators:

$$\mathcal{R}_{\mu\nu} = T^a_{\mu\nu} \otimes \gamma_a + R^a_{\mu\nu} \otimes \tilde{\gamma}_a + F_{\mu\nu} \otimes i\mathbb{1} + \tilde{F}_{\mu\nu} \otimes \gamma_5$$

- The expressions of the various tensors are:

$$T^a_{\mu\nu} = i[X_\mu + A_\mu, e_\nu{}^a] - i[X_\nu + A_\nu, e_\mu{}^a] - 2\{e_{\mu b}, \omega_{\nu c}\}\epsilon^{abc} - 2\{\omega_{\mu b}, e_{\nu c}\}\epsilon^{abc} - 2i[\omega_\mu{}^a, \tilde{A}_\nu] + 2i[\omega_\nu{}^a, \tilde{A}_\mu] - i\lambda C_{\mu\nu}{}^{\rho} e_\rho{}^a$$

$$R^a_{\mu\nu} = i[X_\mu + A_\mu, \omega_\nu{}^a] - i[X_\nu + A_\nu, \omega_\mu{}^a] - 2\{\omega_{\mu b}, \omega_{\nu c}\}\epsilon^{abc} + \tfrac{1}{2}\{e_{\mu b}, e_{\nu c}\}\epsilon^{abc} + \tfrac{i}{2}[e_\mu{}^a, \tilde{A}_\nu] - \tfrac{i}{2}[e_\nu{}^a, \tilde{A}_\mu] - i\lambda C_{\mu\nu}{}^{\rho}\omega_\rho{}^a$$

$$F_{\mu\nu} = i[X_\mu + A_\mu, X_\nu + A_\nu] - i[e_\mu{}^a, e_{\nu a}] + 4i[\omega_\mu{}^a, \omega_{\nu a}] - i[\tilde{A}_\mu, \tilde{A}_\nu] - i\lambda C_{\mu\nu}{}^{\rho}(X_\rho + A_\rho)$$

$$\tilde{F}_{\mu\nu} = i[X_\mu + A_\mu, \tilde{A}_\nu] - i[X_\nu + A_\nu, \tilde{A}_\mu] + 2i[e_\mu{}^a, \omega_{\nu a}] + 2i[\omega_\mu{}^a, e_{\nu a}] - i\lambda C_{\mu\nu}{}^{\rho}\tilde{A}_\rho$$

- Commutative limit: *Coincidence* with the expressions of 3-d gravity after applying the redefinitions

*The action* 

Géré-Vitale-Wallet '14

- The action we propose is Chern-Simons type:

$$\mathcal{S} = \frac{1}{g^2}\mathrm{Trtr}\left(\frac{i}{3}C^{\mu\nu\rho}\mathcal{X}_\mu\mathcal{X}_\nu\mathcal{X}_\rho - \frac{\lambda}{2}\mathcal{X}_\mu\mathcal{X}^\mu\right)$$

- Tr: Trace over matrices $X$; tr: Trace over the algebra
- The action can be written as:

$$\mathcal{S} = \frac{1}{6g^2}\mathrm{Trtr}(iC^{\mu\nu\rho}\mathcal{X}_\mu\mathcal{R}_{\nu\rho}) + \mathcal{S}_\lambda$$

where $\mathcal{S}_\lambda = -\frac{\lambda}{6g^2}\mathrm{Trtr}(\mathcal{X}_\mu\mathcal{X}^\mu)$

- Using the explicit form of the algebra trace:

$$\mathrm{Tr}C^{\mu\nu\rho}\left(e_{\mu a}T^a_{\nu\rho} - 4\omega_{\mu a}R^a_{\nu\rho} - (X_\mu + A_\mu)F_{\nu\rho} + \tilde{A}_\mu\tilde{F}_{\nu\rho}\right)$$

## *Variation of the action*

- Two ways of variation lead to the (same) equations of motion:
  - Variation with respect to the covariant coordinate, $\mathcal{X}_\mu$
  - Variation with respect to the gauge fields

- The equations of motion are:

$$\mathcal{R}_{\mu\nu} = 0$$

$$T_{\mu\nu}{}^{a} = 0\,, \quad R_{\mu\nu}{}^{a} = 0\,, \quad F_{\mu\nu} = 0\,, \quad \tilde{F}_{\mu\nu} = 0$$

*The Euclidean case*

- Group of symmetries: $SO(4) \cong Spin(4) = SU(2) \times SU(2)$
- Anticommutators *do not close* → Extension to $U(2) \times U(2)$
- Each $U(2)$: four 4x4 matrices as generators:

$$J_a^L = \begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix}, \; J_a^R = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_a \end{pmatrix}, \; J_0^L = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \; J_0^R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

- Identification of the correct nc dreibein and spin connection fields:

$$P_a = \frac{1}{2}(J_a^L - J_a^R)\,,\; M_a = \frac{1}{2}(J_a^L + J_a^R)\,,\; \mathbb{1} = J_0^L + J_0^R\,,\; \gamma_5 = J_0^L - J_0^R$$

- Calculations give the CRs and aCRs

$$\begin{aligned} &[P_a, P_b] = i\epsilon_{abc}M_c\,, \quad [P_a, M_b] = i\epsilon_{abc}P_c\,, \quad [M_a, M_b] = i\epsilon_{abc}M_c\,, \\ &\{P_a, P_b\} = \tfrac{1}{2}\delta_{ab}\mathbb{1}\,, \quad \{P_a, M_b\} = \tfrac{1}{2}\delta_{ab}\gamma_5\,, \quad \{M_a, M_b\} = \tfrac{1}{2}\delta_{ab}\mathbb{1}\,. \\ &[\gamma_5, P_a] = [\gamma_5, M_a] = 0\,, \quad \{\gamma_5, P_a\} = 2M_a\,, \quad \{\gamma_5, M_a\} = 2P_a \end{aligned}$$

- Gauging proceeds in the same way as before

## *Summary*

- 3-d gravity described as C-S gauge theory
- Translation to nc regime (gauge theories through cov. coord.)
- 3-d nc spacetimes built from $\mathrm{SU}(2)$ and $\mathrm{SU}(1,1)$
- Gauge their symmetry groups
- Transformations of fields - Curvatures - Action - E.o.M.

## *Future plans*

- Further analysis of the Lorentzian case space structure (algebra of functions, differential calculus, etc.)
- Move to the 4-d case of gravity as noncommutative gauge theory
- Embed gauge group and space structure into a larger symmetry

*Heckman-Verlinde '14, Madore-Burić '15*

*Thank you for your attention!*