A Connection for Born Geometry

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Dualities and Generalized Geometries

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Physics & Geometry

Geometrical Structures		Physical Concepts
symplectic geometry Riemannian geometry geometry of principle bundles	⟺	classical & quantum mechanics general relativity gauge theories

Physics & Geometry

Geometrical Structures

symplectic geometry
Riemannian geometry
geometry of principle bundles

⟺

Physical Concepts

classical & quantum mechanics
general relativity
gauge theories

What is the geometry for Quantum Gravity?

Context

- Consider possible geometry for "quantum gravity"
- Motivated by principles from GR and QM
- String Theory naturally lives in such geometry
- Study geometric objects such as connections, curvature, fluxes, …

Outline

1. Introduction: Context & Motivation
2. What is Born Geometry?
3. Para-Hermitian Geometry & D-Structures
4. Born Connection
5. Outlook

What is Born Geometry?

Born Geometry

[Freidel, Minic, Leigh '14]

- Doubled Space: 2d-dimensional manifold $\mathcal{P}$

Born Geometry

[Freidel, Minic, Leigh '14]

- Doubled Space: 2d-dimensional manifold $\mathcal{P}$ - motivated by
  - ➡ Doubled target space (DFT) $X = \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}$

Born Geometry

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- Doubled Space: 2d-dimensional manifold $\mathcal{P}$ - motivated by
  - ➡ Doubled target space (DFT) $X = \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}$ $\quad \tilde{x}' = p$
  - ➡ Phase space of string

$$\mathrm{Ham}(x,p) = \dot{x}\cdot p - \mathrm{Lag} = \frac{1}{2} Z^T \mathcal{H} Z \qquad Z = \begin{pmatrix} x' \\ p \end{pmatrix}$$

Born Geometry

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Three compatible structures on $\mathcal{P}$

- Generalized metric $\mathcal{H}$
- Neutral metric η
- (Almost) Symplectic form ω

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Three compatible structures on $\mathcal{P}$

- Generalized metric $\mathcal{H}$
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- Para-Hermitian manifold + generalized metric

String Theory

String Theory naturally lives in a Born geometry.

- Doubled Formalism
  - ➡ World sheet: Doubled String Sigma Model (Tseytlin)
  - ➡ Target space: Double Field Theory (DFT)

$$X = \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}$$

with dual coordinates defined via $\tilde{x}' = p$

Doubled String Sigma Model

includes topological term

$$S_{\text{doubled}} = \frac{1}{2}\int \mathrm{d}\tau\mathrm{d}\sigma \left[(\eta_{MN} + \omega_{MN})\partial_\tau X^M \partial_\sigma X^N - \mathcal{H}_{MN}\partial_\sigma X^M \partial_\sigma X^N\right]$$

[Tseytlin '90; Giveon, Rocek '91; Hull '06]

String sigma model action containing all three structures

- Global $O(d,d)$ symmetry manifest
- Not Lorentz invariant — impose as constraint
- Interpret target space as DFT / Born geometry

Double Field Theory

[Siegel '93; Hull & Zwiebach '09]

- Doubled Target Space

$$S_{\rm DFT} = \int \mathrm{d}^{2d}X e^{-2\Phi}\, \mathcal{R}(\mathcal{H},\Phi)$$

Generalized Ricci scalar

- Unification of background fields in target space

$$\mathcal{H} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}$$

- Unification of local symmetries (diffeos & gauge trafos)

➡ Closure of symmetry algebra gives constraint

Connections

Riemann vs Born

Differential geometry Riemannian manifold	“Generalized” geometry Born manifold

Levi-Civita Connection

- Recall: Connection ∇ in Riemannian geometry (M, g)
  - ➡ Metric compatible: $\nabla_X g = 0$
  - ➡ Torsion-free: $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$

Lie bracket

Levi-Civita Connection

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  - ➡ Torsion-free: $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$

Lie bracket

- Such a connection always exists and is unique
  - ➡ Koszul formula:

$$g(\nabla_X Y, Z) = \frac{1}{2}\Big\{X[g(Y,Z)] + Y[g(Z,X)] - Z[g(X,Y)] + g([X,Y],Z) - g([Y,Z],X) - g([X,Z],Y)\Big\}$$

Levi-Civita connection

Riemann vs Born

Differential geometry
Riemannian manifold

“Generalized” geometry
Born manifold

- Riemannian manifold M
- Metric g

- Metric compatibility
- Torsion T
- Levi-Civita connection ∇^g

The Unique Connection Problem

- In DFT one only considers the structures $(\eta, \mathcal{H})$ on $\mathcal{P}$
- Compatibility with $(\eta, \mathcal{H})$ and vanishing (generalized) torsion $\mathcal{T}$ is not enough to fix a unique connection!
- Certain components remain undetermined

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- Compatibility with $(\eta, \mathcal{H})$ and vanishing (generalized) torsion $\mathcal{T}$ is not enough to fix a unique connection!
- Certain components remain undetermined
- But: Generalized Ricci scalar in action is fully determined
- Issues may arise for higher order corrections
- Unsatisfactory - hints at missing ingredient

Towards the Born Connection

- Can be fixed in Born geometry!
- Missing ingredient is ω
- First need to understand **para-Hermitian** sector (η, ω) of Born geometry

Need generalized differentiable structure

Para-Hermitian Geometry

Para-Complex Manifold

- 2d-dimensional para-complex manifold $(\mathcal{P}, K)$
- Para-complex structure $K^2 = +\mathbb{1}$
- Eigenbundles (of equal rank)

$$K\Big|_L = +1 \qquad K\Big|_{\tilde{L}} = -1 \qquad P, \tilde{P} = \frac{1}{2}(\mathbb{1} \pm K)$$

- Splitting of tangent space $K : T\mathcal{P} = L \oplus \tilde{L}$
- Integrability of L and $\tilde{L}$ is independent

Para-Hermitian Manifold

- Include pseudo-Riemannian metric η ⇨ $(\mathcal{P}, \eta, K)$
- Skew-orthogonality: $K^{\mathsf{T}}\eta K = -\eta$
- Split signature (d, d) since eigenbundles have same rank
- Fundamental form $\omega = \eta K$ (almost symplectic)

$$\omega = \eta K = -K^{\mathsf{T}}\eta = -\omega^{\mathsf{T}}$$

- Eigenbundles L and $\tilde{L}$ are isotropic w.r.t η and ω

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"Generalized" geometry
Born manifold

- para-Hermitian manifold $\mathcal{P}$
- Structure (η, K)

D-Structure

Differentiable Structure

- Two main notions of differentiation on a smooth manifold
  - ➡ Lie derivative $\mathcal{L}_X$ — always exists
  - ➡ Covariant derivative ∇_X — need connection
- For vector fields ⇨ Lie bracket

$$\mathcal{L}_X Y = [X, Y]$$

Differentiable Structure

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  - ➡ Lie derivative $\mathcal{L}_X$ — always exists
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$$\mathcal{L}_X Y = [X, Y] = \nabla_X Y - \nabla_Y X$$

any torsion-free connection

Riemann vs Born

Differential geometry
Riemannian manifold

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- Differentiable structure
  - Lie derivative $\mathcal{L}_X$
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"Generalized" geometry
Born manifold

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- Structure (η, K)

D-structure

- Generalized Lie derivative

$$\mathbb{L}_X Y = [\![X, Y]\!]$$

D-bracket

- Bracket operation $[\![\ ,\ ]\!]$ for vector fields on $T\mathcal{P}$ with
  - ➡ Leibniz property $[\![X, fY]\!] = f[\![X, Y]\!] + X[f]Y$
  - ➡ Compatible with the metric:

$$\mathbb{L}_X \eta = 0$$

not true in general
for Lie derivative

D-structure

- Compatible with K ⇒ generalized integrability
  - ➡ Eigenbundles L and $\tilde{L}$ are Dirac structures (involutive)

$$[\![L, L]\!] \subseteq L \qquad [\![\tilde{L}, \tilde{L}]\!] \subseteq \tilde{L}$$

$$\tilde{P}[\![P(X), P(Y)]\!] = 0 \qquad P[\![\tilde{P}(X), \tilde{P}(Y)]\!] = 0$$

D-structure

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D-structure

$$(\mathcal{P}, \eta, K, [\![\ ,\ ]\!])$$

D-bracket $[\![\ ,\ ]\!]$

- Leibniz property
- Compatible with η
- Compatible with K

[Freidel, FJR, Svoboda '18]

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“Generalized” geometry
Born manifold

- para-Hermitian manifold $\mathcal{P}$
- Structure (η, K)
- D-structure
  - Gen. Lie derivative $\mathbb{L}_X$
  - D-bracket $[\![\ ,\ ]\!]$

D-structure

- Can define a D-bracket for any metric-compatible connection ∇

$$\eta([\![X,Y]\!]^{\nabla},Z)=\eta(\nabla_X Y-\nabla_Y X,Z)+\eta(\nabla_Z X,Y)$$

D-structure

- Can define a D-bracket for any metric-compatible connection ∇

$$\eta([\![X,Y]\!]^\nabla, Z) = \eta(\nabla_X Y - \nabla_Y X, Z) + \eta(\nabla_Z X, Y)$$

- Among all D-brackets, there is a **canonical** one which projects onto the Lie bracket when restricted to L and $\tilde{L}$

$$[\![P(X), P(Y)]\!] = P([P(X), P(Y)])$$

$$[\![\tilde{P}(X), \tilde{P}(Y)]\!] = \tilde{P}([\tilde{P}(X), \tilde{P}(Y)])$$

D-structure

[Freidel, FJR, Svoboda '18]

- The canonical D-bracket is unique:

Proposition

On a para-Hermitian manifold $(\mathcal{P}, \eta, K)$ there exists a unique canonical D-bracket $[\![\ ,\ ]\!]^c$ with the above properties (Leibniz, compatibility with η and K) and which projects onto the Lie bracket $[\ ,\ ]$ when restricted to L and $\tilde{L}$.

- Note:
The bracket is unique but the connection is not, i.e. there are different connections giving the canonical bracket.

Generalized Torsion

- Ordinary torsion $T^{\nabla}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$
  - ➡ Difference between brackets $[X,Y]^{\nabla} := \nabla_X Y - \nabla_Y X$

Generalized Torsion

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  - ➡ Difference between brackets $[X,Y]^{\nabla} := \nabla_X Y - \nabla_Y X$

- Generalized torsion

$$\mathcal{T}^{\nabla}(X,Y) = [\![X,Y]\!]^{\nabla} - [\![X,Y]\!]^{c}$$

  - ➡ Defined in terms of the unique canonical D-bracket

Riemann vs Born

Differential geometry Riemannian manifold	"Generalized" geometry Born manifold

- Riemannian manifold M
- Metric g
- Differentiable structure
  - Lie derivative $\mathcal{L}_X$
  - Lie bracket $[\ ,\ ]$
- Metric compatibility
- Torsion T
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- para-Hermitian manifold $\mathcal{P}$
- Structure (η, K)
- D-structure
  - Gen. Lie derivative $\mathbb{L}_X$
  - D-bracket $[\![\ ,\ ]\!]$
- Gen. torsion $\mathcal{T}$

Born Geometry

- Start from para-Hermitian manifold $(\mathcal{P}, \eta, \omega)$
- Include another metric $\mathcal{H}$ ⇨ $(\mathcal{P}, \eta, \omega, \mathcal{H})$

➡ Chiral structure $J = \eta^{-1}\mathcal{H}$ $\quad J^2 = +\mathbb{1}$

Proposition

Born Geometry is equivalent to a para-Hermitian structure (η, K) along with a choice of Riemannian metric g on L .

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"Generalized" geometry
Born manifold

- Born manifold $\mathcal{P}$
- Structures $(\eta, \omega, \mathcal{H})$
- D-structure
  - Gen. Lie derivative $\mathbb{L}_X$
  - D-bracket $[\![\ ,\ ]\!]$
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Born Connection

Born Connection

[Freidel, FJR, Svoboda '18]

- Connection compatible with Born geometry $(\mathcal{P}, \eta, \omega, \mathcal{H})$

$$\nabla^B \eta = \nabla^B \omega = \nabla^B \mathcal{H} = 0$$

- Vanishing generalized torsion

$$\mathcal{T}^{\nabla^B} = 0$$

The Born connection exists and is unique!

(Like Levi-Civita connection for Riemannian geometry.)

Born Connection

Theorem (Freidel, FJR, Svoboda)

There exists a unique connection ∇^{B} compatible with the Born structure $(\eta, \omega, \mathcal{H})$ and with vanishing generalized torsion $\mathcal{T}$. It is given by

$$\nabla^{\mathrm{B}}_X Y = [\![X_-, Y_+]\!]^c_+ + [\![X_+, Y_-]\!]^c_- \\ + (K[\![X_+, KY_+]\!]^c)_+ + (K[\![X_-, KY_-]\!]^c)_-$$

where $X_\pm = \frac{1}{2}(\mathbb{1} \pm J)X$ are the projected components associated with the splitting of $T\mathcal{P}$ given by the chiral structure J.

Born Connection

$$\nabla^{\mathrm{B}}_X Y = [\![X_-, Y_+]\!]^c_+ + [\![X_+, Y_-]\!]^c_- + (K[\![X_+, KY_+]\!]^c)_+ + (K[\![X_-, KY_-]\!]^c)_-$$

- Lagrangian subspaces $K : T\mathcal{P} = L \oplus \tilde{L}$
- Chiral subspaces $J : T\mathcal{P} = C_+ \oplus C_-$

$$P_\pm = \frac{1}{2}(\mathbb{1} \pm J) \qquad X_\pm = P_\pm(X)$$

Born Connection

$$\nabla^{\mathrm{B}}_X Y = [\![X_-, Y_+]\!]^c_+ + [\![X_+, Y_-]\!]^c_- + (K[\![X_+, KY_+]\!]^c)_+ + (K[\![X_-, KY_-]\!]^c)_-$$

- Born geometry analogue of Koszul formula for Levi-Civita connection
- No coincidence: (L, g) is a Riemannian vector bundle recalling $P(X) = x$

Projected Born Connection

- Born geometry analogue of Koszul formula for Levi-Civita connection
- No coincidence: (L, g) is a Riemannian vector bundle recalling $P(X) = x$

Theorem

The Born connection ∇^{B} restricts on L to the Levi-Civita connection ∇^g of g

$$\nabla^{\mathrm{B}}_{P(X)} P(Y) = \nabla^g_x y$$

Riemann vs Born

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Riemannian manifold

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"Generalized" geometry
Born manifold

- Born manifold $\mathcal{P}$
- Structures $(\eta, \omega, \mathcal{H})$
- D-structure
  - Gen. Lie derivative $\mathbb{L}_X$
  - D-bracket $[\![\ ,\ ]\!]$
- Structure compatibility
- Gen. torsion $\mathcal{T}$
- Born connection ∇^{B}

Outlook

Outlook

- Make relation to DFT precise
  - ➡ Include Dilaton: need weighted bundle (tensor density)
  - ➡ Include Fluxes: obstruction to closure, integrability
- Generalized Fluxes in Born geometry
  - ➡ Twisted D-bracket: $[\![\ ,\ ]\!] = [\![\ ,\ ]\!]_0 + \mathcal{F}$
- Analogue of Riemann Normal Coordinates
- …

Summary

- Born Geometry: possible geometry for Quantum Gravity
- para-Hermitian manifold + dynamical metric $(\mathcal{P}, \eta, \omega, \mathcal{H})$
- Generalized differentiable structure: D-bracket $[\![\ ,\ ]\!]$
- Born Connection ∇^{B}: unique, torsion-free, compatible
- Reduces to Levi-Civita connection ∇^g on L

Extensions

Riemann vs Born

Differential geometry Riemannian manifold	"Generalized" geometry Born manifold
Riemannian manifold M	Born manifold $\mathcal{P}$
Metric g	Structures $(\eta, \omega, \mathcal{H})$
Differentiable structure	D-structure
Lie derivative $\mathcal{L}_X$	Gen. Lie derivative $\mathbb{L}_X$
Lie bracket $[\ ,\ ]$	D-bracket $[\![\ ,\ ]\!]$
Metric compatibility	Structure compatibility
Torsion T	Gen. torsion $\mathcal{T}$
Levi-Civita connection ∇^g	Born connection ∇^{B}

Born Geometry

Proposition

There always exists a frame $E \in GL(2d)$ in which the triple $(\eta, \omega, \mathcal{H})$ takes the form

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}.$$

- The frame E is only determined up to an $O(d)$ trafo
- h is a constant metric on L
- Riemannian metric $g = e^{\mathsf{T}} h e$

Structure Groups

$$Sp(2d) \cap O(d,d) \cap O(2d) = O(d)$$

Structure Groups

neutral metric η

Lorentz group

$$Sp(2d) \cap O(d,d) \cap O(2d) = O(d)$$

sympletic form ω

dynamical metric $\mathcal{H}$

Generalized Kinematics & Dynamics for DFT

Generalization

- Generalization of geometry suitable for strings in a doubled space
- Kinematical structure encoded in (η, ω)
  - ➡ Neutral metric $\eta \in O(d,d)$
  - ➡ Symplectic form $\omega \in Sp(2d)$
- Dynamical d.o.f. in generalized metric $\mathcal{H}(g,B)$

Generalizations for DFT

- DFT is limit of Born geometry
- Right setup to allow for general η and ω
- Accommodate:
  - ➡ Geometric and non-geometric fluxes
  - ➡ Non-commutativity, non-associativity, non-geometry, ...

T-Duality

T-Duality

- What is T-duality?
  - ➡ Sigma model with two sets of fields x^μ and $\tilde{x}_\mu$
  - ➡ Integrate out one or the other

$$\tilde{x}' = p = \dot{x} \qquad\qquad x' = w = \dot{\tilde{x}}$$

  - ➡ Get T-duality related actions
- Canonical transformation on phase space
  - ➡ No need for compact dimensions or isometries

T-Duality

[Tseytlin '90, Duff '90, Siegel '93]

- Worldsheet symmetry (exchanges σ and τ)

$$\mathrm{d}x^\mu \to *\mathrm{d}x^\mu$$

- Target space symmetry

chiral structure

$$X \to J(X)$$

- Doubled string actions with manifest symmetry