The NS sector of SUGRA from graded Poisson algebra

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Table of Contents

Table of Contents

Motivations

- generalized geometry: invariance under diffeomorphisms, B-transforms, more generally T-dualities $\Longrightarrow$ bundle $TM \oplus T^*M$ with bracket on sections and bilinear form;
- graded Poisson algebra of $C^\infty(T^*[2]M \oplus T[1]M)$, degree -2 $\{\cdot,\cdot\}$, $T^*[2]T[1]M$ is symplectic with Θ deg-3 Hamiltonian

$\Longrightarrow$ generalized geometry (Courant algebroid) is derived structure of graded and symplectic $T^*[2]T[1]M$

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[Roytenberg, 2002]

- gravity multiplet (g, H, ϕ) not complete:
  when graded algebra side incorporates ϕ and $\mathbf{G} := g + B$, it provides a (g, H, ϕ)-dependent connection $\widetilde{\nabla}$ on E

10-dim SUGRA NS sector:

$$\mathcal{S} = \int_M \mathrm{Vol}_{\mathrm{d}=10} \left(R - \frac{1}{12} H^2 + 4 (\nabla\phi)^2 \right) e^{-2\phi}$$

is scalar from Ricci tensor of this $\widetilde{\nabla}$.

Table of Contents

Derived Courant structure

$T^*[2]T[1]M$ with standard symplectic ω, $(x^i, \underbrace{\chi_a, \theta^b}_{=:\xi_a}, p_j)$ coordinates

Hamiltonian vector field (shifted) Θ of deg 3

$\{\{e_1, \Theta\}, e_2\} = [e_1, e_2]$, $\quad e_1, e_2$ linear functions in ξ

$\{\{e_1, \Theta\}, f\} = \rho(e_1)f$, $\quad f \in C^\infty(M)$,

well-defined derived object if:

$\{\Theta, \Theta\} = 0$

i.e. a bracket $[\cdot, \cdot]$ on sections $\Gamma(TM \oplus T^*M)$ and an anchor map $\rho : TM \oplus T^*M \to TM$ for a Courant algebroid remain defined, provided that

$$\Theta = \theta^i p_i + \frac{1}{3!} C_{abc} \tilde{\xi}^a \tilde{\xi}^b \tilde{\xi}^c$$

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$(TM \oplus T^*M, \rho, [\cdot,\cdot], \langle\cdot,\cdot\rangle)$ Courant algebroid if:

- $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$ Jacobi id;
- $[e_1, fe_2] = f[e_1, e_2] + \rho(e_1)f\, e_2$ Leibniz rule;
- $\rho(e)\langle e_1, e_2\rangle = \langle [e, e_1], e_2\rangle + \langle e_1, [e, e_2]\rangle$ compatibility of $[\cdot,\cdot]$ and $\langle\cdot,\cdot\rangle$;
- $\rho(e_1)\langle e, e\rangle = 2\langle [e, e], e_1\rangle$ control of symmetric part of $[\cdot,\cdot]$.

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$U = X + \eta$, $V = Y + \zeta$, $\quad X, Y \in \Gamma(TM)$, $\eta, \zeta \in \Gamma(T^*M)$:

$$[U, V]_{\text{Dorfman}} = [X, Y] + \mathcal{L}_X \zeta - \iota_Y d\eta \quad \text{is bracket for Courant algebroid.}$$

New Poisson brackets

$$T^*[2]T[1]M \ni (x^i, \underbrace{\chi_a, \theta^b}_{\xi_a}, p_j) \text{ Darboux chart,} \qquad \omega = dx^i\, dp_i + d\chi_a\, d\theta^a$$

$$\{x, p\}_{(-)} = \delta, \qquad \{\chi, \theta\}_{(+)} = \delta\,;$$

but more general Poisson brackets are allowed:

$$\{\xi_a, \xi_b\}' = G_{ab}, \quad \{p_i, \xi_a\}' = \Gamma^b{}_{ai}\xi_b, \quad \{p_i, p_j\}' = R_{ij}\,.$$

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Choose

- $G_{ab} = \lambda^2 \begin{pmatrix} 2g(x) & \delta \\ \delta & \mathbf{0} \end{pmatrix}, \quad \lambda := e^{-\frac{1}{3}\phi(x)}$,
- $R_{ij} = 0$;

then the connection coefficients are provided by:

$$\begin{aligned}\{p_i, \xi_a\}' &= \lambda^{-1}\partial_i\lambda\,\xi_a + \partial_i(g(x) + B(x))_{ac}\,\tilde{\xi}^c \\ &= \Gamma^c{}_{ai}\xi_c \qquad \text{(Weitzenboeck connection } \leftrightarrow =: \nabla\text{)}\end{aligned}$$

Courant structure, general Poisson algebra

$(TM \oplus T^*M = E, \rho', [\cdot,\cdot]'_{\text{Dorf}}, \langle\cdot,\cdot\rangle')$ from the Poisson algebra of $T^*[2]T[1]M$ with metric G_{ab} and Weitzenboeck connection ∇:

- $\{e_a, e_b\}' = G_{ab} = \langle e_a, e_b\rangle'$.
- $\{\{e_1, \Theta'\}', e_2\}' = \lambda[e_1, e_2]_{\text{Dorf}} + \underbrace{(\rho(e_1)\lambda)\, e_2 - (\rho(e_2)\lambda)\, e_1} + \lambda\, \underbrace{\mathcal{L}_{\rho(e_1)}\mathbf{G}(e_2)}$

  $-\lambda\mathbf{G}([e_1, e_2]) - \lambda\, \underbrace{\iota_{\rho(e_2)} d\, \mathbf{G}(e_1)} + d\lambda\, (\langle e_1, e_2\rangle + 2g(e_1, e_2)) = [e_a, e_b]'_{\text{Dorf}}$.
- $\{\{e, \Theta'\}', f\}' = \lambda\rho(e)\, f = \rho'(e)\, f$

$\mathbf{G}(x) := (g(x) + B(x)) : \Gamma(TM) \to \Gamma(T^*M)$
$\Theta' = \lambda^{-1}\theta p$

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Fully contracted bracket with the fluxes C_{abc}:

$$\{\{\{e_1, \frac{1}{3!}\lambda^{-3}\, C_{abc}\tilde{\xi}^a\tilde{\xi}^b\tilde{\xi}^c\}', e_2\}', e_3\}' = C(e_1, e_2, e_3) .$$

Takes into account the geometric and non-geometric fluxes:

$$H_{ijk} \xrightarrow{T_k} f_{ij}{}^k \xrightarrow{T_j} Q_i{}^{jk} \xrightarrow{T_i} R^{ijk} .$$

Table of Contents

Koszul formula

If

$$[U,V]_{\mathrm{L}} \stackrel{\text{def}}{=} \rho(U)V - \rho(V)U\,,$$

for the Courant algebroid $(E', \rho', [\cdot,\cdot]', \langle\cdot,\cdot\rangle')$, the previous bracket can be written as:

$$\begin{aligned}
[U,V]'_{\mathrm{D}} &= \langle \nabla_U V, \cdot\rangle' - \langle \nabla_V U, \cdot\rangle' + \langle \nabla_\cdot U, V\rangle' \\
\langle [U,V]'_{\mathrm{D}}, W\rangle' - \langle [U,V]'_{\mathrm{L}}, W\rangle' &= \langle \nabla_U V, W\rangle' - \langle \nabla_V U, W\rangle' - \langle [U,V]'_{\mathrm{L}}, W\rangle' + \langle \nabla_W U, V\rangle' \\
&= \langle \underbrace{\mathrm{T}^{\nabla}(U,V)}_{\text{gen. torsion}}, W\rangle' + \langle \nabla_W U, V\rangle' =: \langle \widetilde{\nabla}_W U, V\rangle'
\end{aligned}$$

In components:

$$\widetilde{\nabla}_k \xi_i = \begin{pmatrix} \left(\Gamma^{\mathrm{L.C.}\,j}{}_{ik} + \frac{1}{2} H^j{}_{ik} - \frac{1}{3}\left(\partial_i\phi\,\delta^j{}_k - \partial^j\phi\, g_{ik} + \partial_k\phi\,\delta^j{}_i\right)\right)\partial_j & -\frac{1}{3}\left(\partial_i\phi\, g_{kj} + \partial_k\phi\, g_{ij}\right) dx^j \\ -\frac{1}{3}\left(\partial_k\phi\, g^{ij} - \partial^j\phi\delta^i{}_k\right)\partial_j & 0 \end{pmatrix}.$$

The part for vector fields provides a generalized Koszul formula for $\mathbf{G}$ and ϕ:

$$\begin{aligned}
\lambda^3 2g(\widetilde{\nabla}_Z X, Y) = &\lambda^3\left[X\cdot \mathbf{G}(Y,Z) + Z\cdot \mathbf{G}(X,Y) - Y\cdot\mathbf{G}(X,Z)\right] \\
&+ \lambda^3\left[\mathbf{G}(X,[Y,Z]) - \mathbf{G}(Y,[X,Z]) - \mathbf{G}([X,Y],Z)\right] \\
&+ \lambda^2\left[X(\lambda)2g(Y,Z) - Y(\lambda)2g(X,Z) + Z(\lambda)2g(X,Y)\right].
\end{aligned}$$

SUGRA

The connection $\widetilde{\nabla}$ involves the metric g, the Neveu-Schwarz field H and the dilaton ϕ; once projected onto the maximal isotropic subbundle TM, we can compute the Riemann tensor $\mathcal{R}$:

$$\mathcal{R}_{kl}X = \left[\widetilde{\nabla}_k, \widetilde{\nabla}_l\right] X - \widetilde{\nabla}_{[\partial_k, \partial_l]}X \,.$$

The Ricci tensor is:

$$\begin{aligned} \mathrm{R}_{il} &= \mathcal{R}^{j}{}_{ikl}\,\delta^k_j \\ &= \mathrm{R}^{\text{L.C.}}_{il} + \frac{1}{2}\nabla^{\text{L.C.}}_k H_{li}{}^{k} - \frac{1}{4}H_{im}{}^{k}H_{kl}{}^{m} - \frac{d-4}{6}\nabla^{\text{L.C.}}_k \phi H_{li}{}^{k} \\ &+ \frac{(d-2)}{9}\left(\nabla^{\text{L.C.}}_i \phi \nabla^{\text{L.C.}}_l \phi - g_{il}\left(\nabla^{\text{L.C.}}\phi\right)^2\right) - \frac{1}{3}\left[(2-d)\nabla^{\text{L.C.}}_i \nabla^{\text{L.C.}}_l \phi - g_{il}(\nabla^{\text{L.C.}})^2\phi\right] . \end{aligned}$$

Proposal for a scalar built from R:

$$\mathrm{Tr}\left(e^{\frac{2}{3}\phi}g^{-1}\left(g+B\right)e^{\frac{2}{3}\phi}g^{-1}\,\mathrm{R}\,\sqrt{\det g\, e^{-\frac{d}{3}\phi}}\right) \quad \text{in 10 dim, gives SUGRA NS sector :}$$

$$\mathcal{S} = \int_M \mathrm{Vol}_{\mathrm{d}=10}\left(R^{\text{L.C.}} - \frac{1}{12}H^2 + 4\left(\nabla^{\text{L.C.}}\phi\right)^2\right)e^{-2\phi}$$

Summary and comments

In brief:

1. In the most general Poisson algebra of functions of $(T^*[2]T[1]M, \omega, \Theta)$ we set a metric G_{ab} for ξ_a (vector fields and forms) and the curvature to be null $\Longrightarrow$ a torsionful connection ∇ remains defined;
2. We looked at the change in the structures of $(E, \rho', [\cdot,\cdot]', \langle\cdot,\cdot\rangle')$ and found that ∇ combines into a new connection $\widetilde{\nabla}$ (through definition of $[\cdot,\cdot]_{\mathrm{L}}$);
3. We projected onto TM and computed the curvature, the Ricci tensor R and a scalar through a particular trace of R that combined $\mathbf{G}$ and g^{-1}.

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Few comments:

- seen also as a homomorphism ϑ: $\xi \to \lambda(\xi + j \circ \mathbf{G} \circ \rho(\xi))$;
- even if in the end we restrict to TM, it is important to start with graded structure: no "magic" of derived bracket otherwise;
- results not new for Courant algebroid side: e.g. [Coimbra, Strickland-Constable, Waldram 2012], [Garcia-Fernandez 2014], [Jurco, Vysoky 2015], [Deser, Saemann 2016].

Table of Contents

Conclusions and Outlook

Graded symplectic geometry can be given a torsionful connection. That induces a connection for the Courant algebroid; its Ricci tensor can be contracted in a way that yields the SUGRA Lagrangian in 10 *d.*

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Thanks for the attention!