Twisted D-bracket and Deformations of para-Kähler Manifolds

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PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

Papers

- Algebroid Structures on para-Hermitian manifolds (arXiv:1802.08180)

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- A Unique Connection for Born Geometry (arXiv:1806.05992, joint w/ F. Rudolph and L. Freidel)

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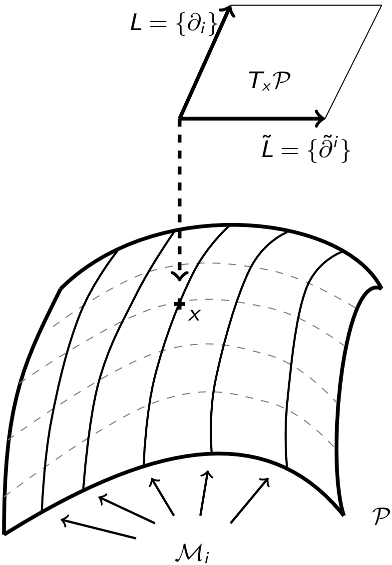
- For our applications we demand that L is integrable $\longrightarrow$ $\mathcal{P}$ is foliated by **Space-time leaves**: $\mathcal{P} = \bigcup_i \mathcal{M}_i$



$L = \{\partial_i\}$
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- Compatibility with K: vanishing generalized Nijenhuis tensor

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- Relationship with the Lie bracket

$$[\![PX, PY]\!] = P([PX, PY]),$$

$$[\![\tilde{P}X, \tilde{P}Y]\!] = \tilde{P}([\tilde{P}X, \tilde{P}Y]),$$

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Theorem (Freidel, Rudolph, DS)

A unique D-bracket exists on any almost para-Hermitian manifold and is given by the formula

$$\eta([\![X, Y]\!], Z) = \eta(\nabla^c_X Y - \nabla^c_Y X, Z) + \eta(\nabla^c_X Z, Y), \tag{1}$$

where $\nabla^c_X Y = \mathring{\nabla}_X Y + \frac{1}{2} K(\mathring{\nabla}_X K) Y$, $\mathring{\nabla}$ *being the Levi-Civita connection of* η.

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Remarks:

- In the "DFT limit", when $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, (1) recovers the usual expression

$$[\![X,Y]\!]^D = \left(X^I \partial_I Y^J - Y^I \partial_I X^J + \eta_{IL}\eta^{KJ} Y^I \partial_K X^L\right)\partial_J .$$

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- (1) recovers Dorfman brackets on $\mathcal{M}/\tilde{\mathcal{M}}$ by the formula

$$\eta([\![X,Y]\!]_\pm, Z) = \eta(\nabla^c_{P/\tilde{P}X} Y - \nabla^c_{P/\tilde{P}Y} X, Z) + \eta(\nabla^c_{P/\tilde{P}X} Z, Y),$$

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- Eigenbundles of K_B are $L = L + B(L)$ and $\tilde{L}$ (unchanged)

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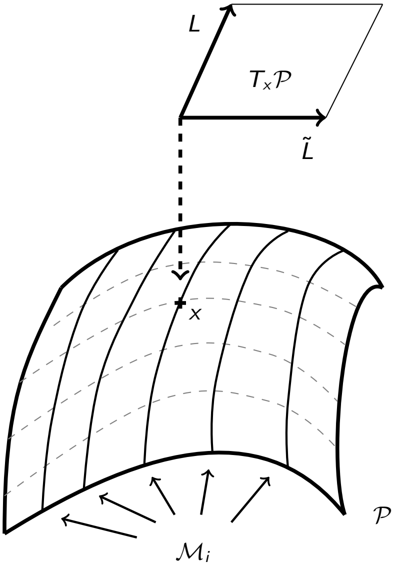

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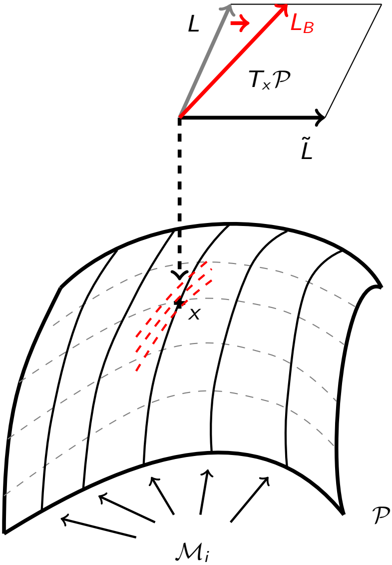

- Because B satisfies $\eta(BX, Y) = -\eta(X, BY)$, it defines a 2-form b and a bi-vector β:

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- If K_B is a B-transformation of K by B, then K is a B-transformation of K_B by $-B$.

B-transform as a deformation

Recall: the D-bracket associated to any para-Hermitian structure (η, K) is compatible with K in the following sense:

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Let K_B be a B-transformation of a para-Hermitian structure (η, K). Then the D-bracket of K_B is compatible with K if and only if the covariantized H-flux

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The vanishing of a covariantized H-flux takes the coordinate-free expression

$$\mathrm{d}_+ b + (\Lambda^3 \eta)[\beta, \beta]_- = 0,$$

and therefore can be understood as a Maurer-Cartan element associated to the **deformation** $K \mapsto K_B$.

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Remarks, discussion and outlook

- "Reverse construction": We can understand the appearance of the fluxes as a result of a choice of non-integrable splitting of $T\mathcal{P}$ (K_B), which can be B-transformed into an integrable para-Kahler structure (K)

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- Full deformation theory of Para-Hermitian geometry