Switching it up: *Parameterizing the QCD Equation of State*

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in collaboration with
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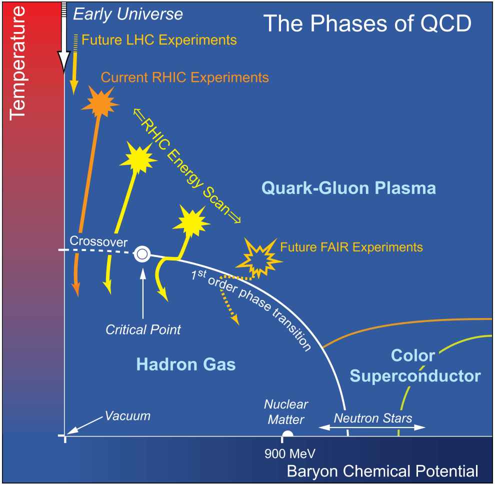
T
μ



Fairly well understood:

- ✓ Hadronic phase
- ✓ Partonic phase
- ✓ T-axis from lattice

Not so well understood:

- → “Coexistence curve”
- → Critical end-point (CEP)?

Hadron Resonance Gas (HRG)

Three different models:

- Point-particle (pt) model

$$P_{HRG}(T,\mu) = \sum_{\alpha \in \text{hadrons}} (2s_\alpha + 1) \int \frac{d^3p}{(2\pi)^3} \frac{p^2/(3E_\alpha)}{e^{\beta[E_\alpha(p)-\mu_\alpha]} \pm 1} \quad (1)$$

  Sum over all hadrons with $m_\alpha < 2.6$ GeV

- Excluded-volume (exI) model
  - extension to pt model
  - assigns finite volume to hadrons
  - volume proportional to hadron energy with coefficient ϵ_0
- Excluded-volume (exII) model
  - variant of exI model
  - also assigns finite volume to hadrons
  - species-dependent volume proportional to hadron mass with universal coefficient ϵ_0

$\rightarrow$ Free parameters: none in pt model, ϵ_0 in exI and exII models

Quark-Gluon Plasma (QGP)

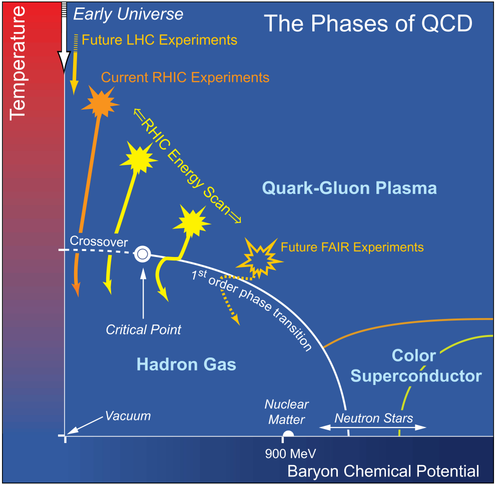
- Write[1, 2, 3]

$$P_{QGP}(T,\mu) = \frac{8\pi^2}{45} T^4 \sum_{n=0}^{6} f_n \left(\frac{\alpha_s}{\pi}\right)^{n/2} \quad (2)$$

- f_n: functions of $\hat{\mu} = \mu/(6\pi T)$, α_s, and $\hat{M} = M/(2\pi T)$
- M: renormalization scale
- Running of α_s determined to three-loop accuracy
- C_S: chosen to eliminate Landau pole in α_s
- C_M: used to adjust renormalization scale

$\rightarrow$ Free parameters: C_S, C_M

[1] A. Vuorinen, PRD **67**, 074032 (2003)
[2] A. Vuorinen, PRD **68**, 054017 (2003)
[3] N. Haque, M. G. Mustafa and M. Strickland, PRD **87**, 105007 (2013)



Checklist:

- ✓ HRG phase
- ✓ QGP phase

Question: How do we put them together?

Answer: A switching function

Switching function: philosophy

$$P_{\rm tot}(T,\mu) = S(T,\mu)\, P_{QGP}(T,\mu) + (1 - S(T,\mu))P_{HRG}(T,\mu)$$

Switching function S:

- $S \in [0,1]$: $S = 0$ ($S = 1$) "turns on" P_{HRG} (P_{QGP}) and "turns off" the other

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  - Let $S \in C^n$ be continuously differentiable at most n times in at least one of its arguments
  - Then $P_{\mathrm{tot}} \in C^n$ is also continuously differentiable at most n times in the same argument
    $\Rightarrow P_{\mathrm{tot}}$ has an nth-order phase transition!
  - Must have $S \in C^\infty$ in order not to artificial phase transition

Switching function: strategy

$$P_{\mathrm{tot}}(T,\mu) = S(T,\mu)\,P_{QGP}(T,\mu) + (1 - S(T,\mu))P_{HRG}(T,\mu)$$

The big idea:

- Define EoSs in distinct phases
- Define S to implement desired phase structures
- Combine to obtain full QCD EoS

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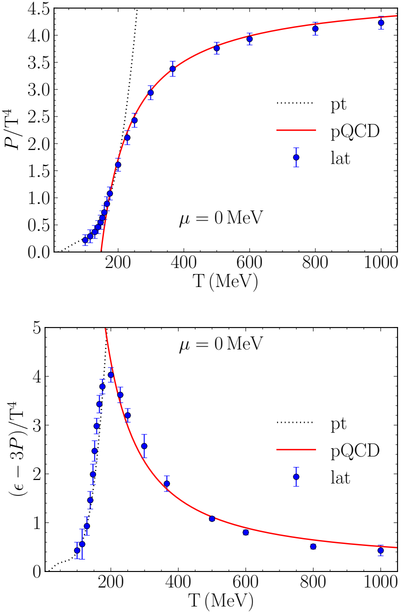
The game plan:

1. Fix lattice observables to describe
2. Construct S
   - **2a** for pure crossover everywhere
   - **2b** for crossover + CEP + 1$^{\text{st}}$-order phase transition
3. Choose free parameters to optimize agreement with lattice data

Step 1: Lattice observables



- Lattice data[a]
  - Normalized pressure
  - Trace anomaly
- Hadronic: valid up to $T \lesssim$ 200 MeV
- Partonic: valid above $T \gtrsim$ 180 MeV
- Try to do matching near $T \sim$ 180-200 MeV

[a] S. Borsanyi, G. Endrodi, Z. Fodor, A. Jakovac, S. D. Katz, S. Krieg, C. Ratti and K. K. Szabo, JHEP **1011**, 077 (2010).

Step 2a: Crossover Switching Function

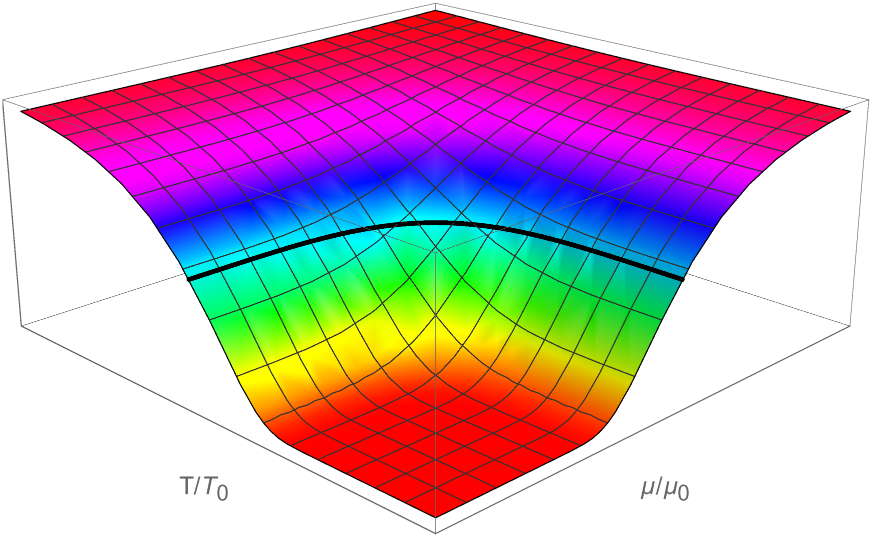
$$S\left(T,\mu\right)\equiv\exp\left[-\theta\left(T,\mu\right)\right] \tag{3}$$

$$\theta\left(T,\mu\right)\equiv\left[\left(\frac{T}{T_0}\right)^r+\left(\frac{\mu}{\mu_0}\right)^r\right]^{-1} \tag{4}$$

$\rightarrow$ Free parameters: T_0, μ_0, $r \in \mathbb{Z}^+$

$\Rightarrow$ **Total** free parameters: T_0, μ_0, r, ϵ_0, C_S, C_M

Step 2a: Crossover Switching Function



Step 2b: Switching Function with Crossover and CEP

Requirements:

- S has a discontinuity along a curve in the complex plane
- Discontinuity terminates at a single point
- $S \in C^{\infty}$ everywhere else
- $S \in [0, 1]$ for any point $z \in \mathbb{C}$

Natural candidate: $S(x, y) \sim \arg(x + iy)$!

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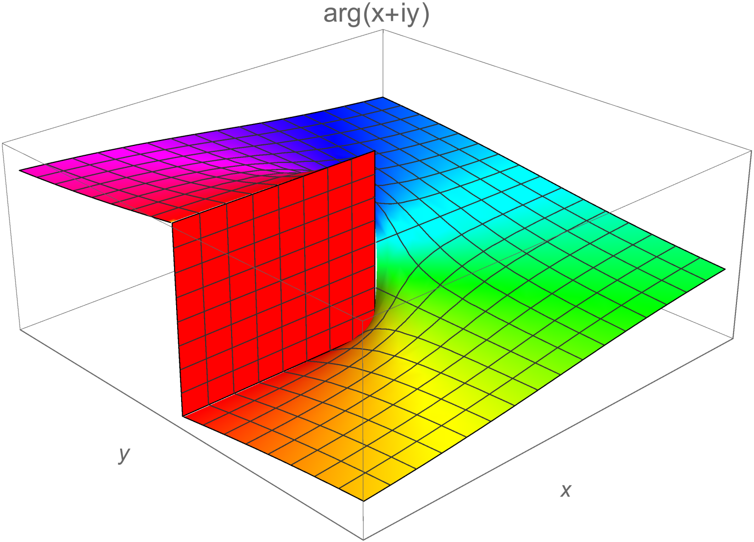

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Natural candidate: $S(x, y) \sim \arg(x + iy)$!

- ✓ Discontinuous branch cut across negative real axis
- ✓ Branch cut terminates at the origin
- ✓ Infinitely differentiable everywhere else

Upshot: choose an appropriate conformal mapping which takes $(x, y) \to (T, \mu)$ in a reasonable way

Step 2b: Switching Function with Crossover and CEP

End result:

$$
\begin{aligned}
S\left(T,\mu,\psi_c,r\right) &= \frac{1}{2}+\frac{1}{\pi}\arg\left(\eta_1\left(m,t,\psi_c\right)+i\,\eta_2\left(m,t,r\right)\right)\\
\eta_1\left(\mu,T,\psi_c\right) &\equiv \frac{1}{2}\left[1+\tanh\left(\frac{a\left(b-\left|\frac{\psi}{\psi_c}\right|\right)}{\left|\frac{\psi}{\psi_c}\right|\left(1-\left|\frac{\psi}{\psi_c}\right|\right)}\right)\right],\\
\eta_2(\mu,T,r) &\equiv \tan\left[\frac{\pi}{2^\theta}-\frac{\pi}{2}\right],\\
\theta\left(\mu,T,r\right) &= \left(\frac{T^2+\mu^2}{R^2\left(\psi\right)}\right)^{-r/2}, \quad \psi=\arctan\left(\mu/T\right),\\
T_{\mathrm{co}}\left(\mu\right) &\equiv T_0\left[1-k_2\left(\frac{\mu}{\mu_c}\right)^2-k_4\left(\frac{\mu}{\mu_c}\right)^4-\cdots\right].
\end{aligned}
$$

$\rightarrow$ Fixed parameters: $R(\psi)$, T_0, k_2, a, b

$\rightarrow$ Free parameters: k_4, $r\in\mathbb{Z}^+$

$\Rightarrow$ **Total** free parameters: k_4, r, ϵ_0, C_S, C_M

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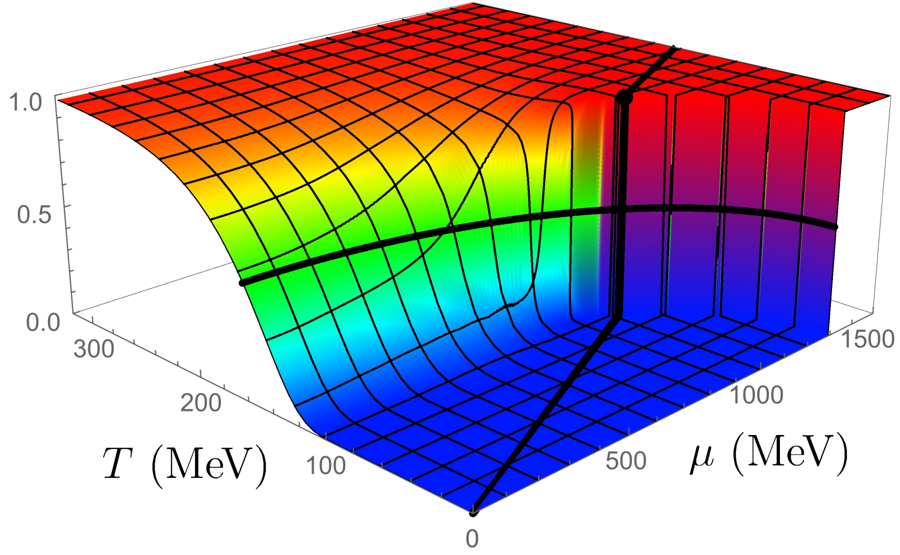

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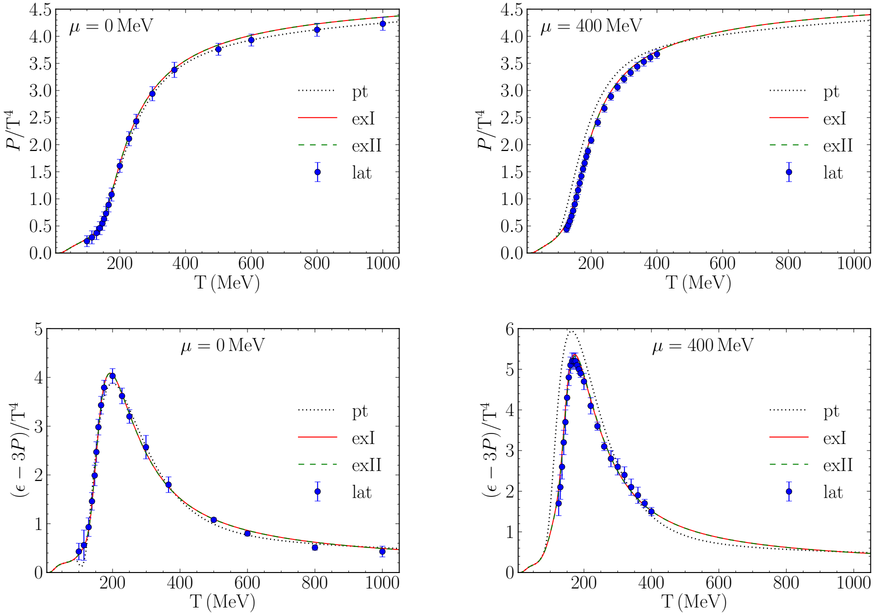
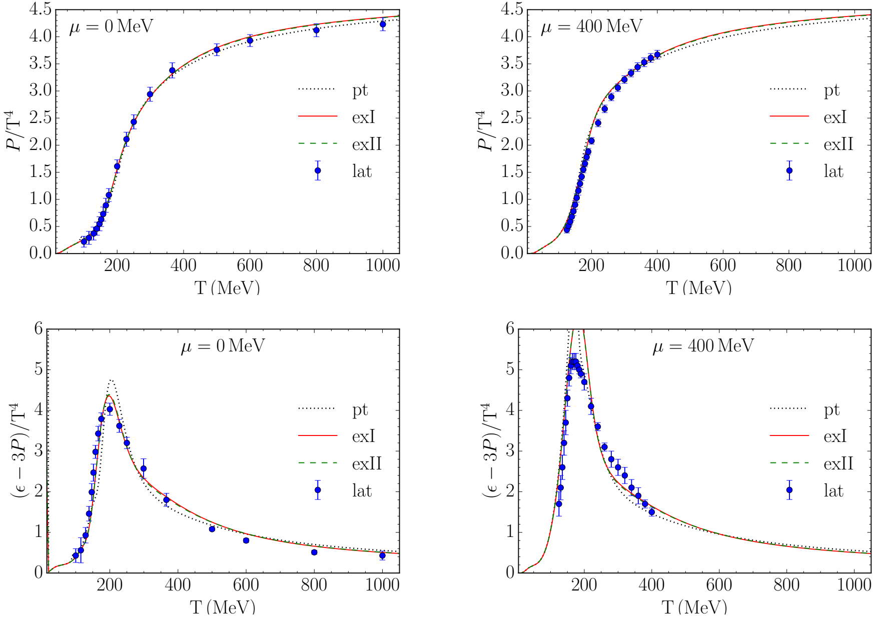
Results for a Pure Crossover



Results for a Crossover and Critical Point



Conclusions

What have we done?

- Constructed switching functions for rapid crossover between QGP and HRG phases
- Both with[4] and without[5] CEP and first-order phase transition
- Obtained phenomenological EoS for various T_c

What's left?

- Improve description of QGP and HRG phases in small-T, large-μ regime
- Include critical exponents at CEP (currently mean-field only)[6]
- Extend to include other phase structures[7,8]

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[4] M. Albright, J. Kapusta and C. Young, Phys. Rev. C **90**, 024915 (2014)
[5] CP, T. Welle, and J. Kapusta, forthcoming
[6] J. I. Kapusta, Phys. Rev. C **81**, 055201 (2010)
[7] M. G. Alford, K. Rajagopal and F. Wilczek, Phys. Lett. B **422**, 247 (1998)
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Backup slides

Switching function: philosophy

Why do we need a switching function?

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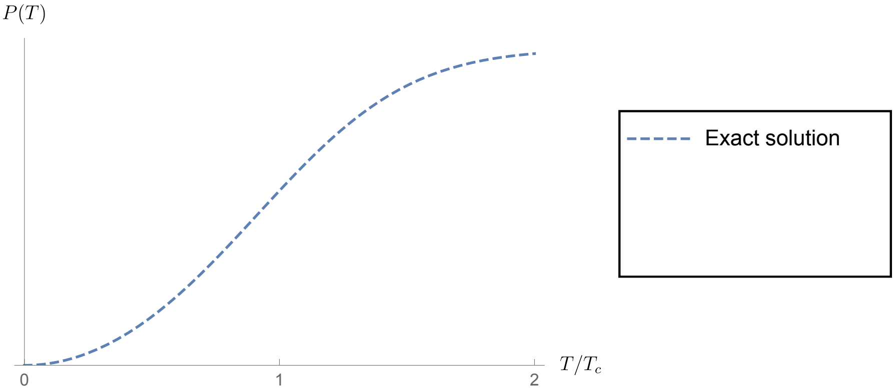
Toy example:



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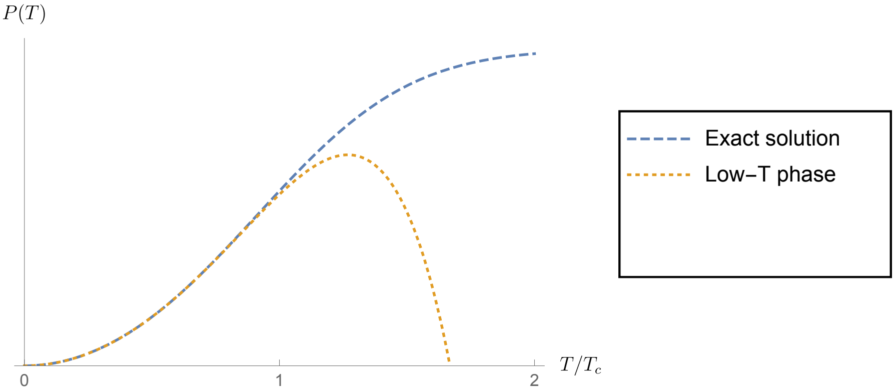

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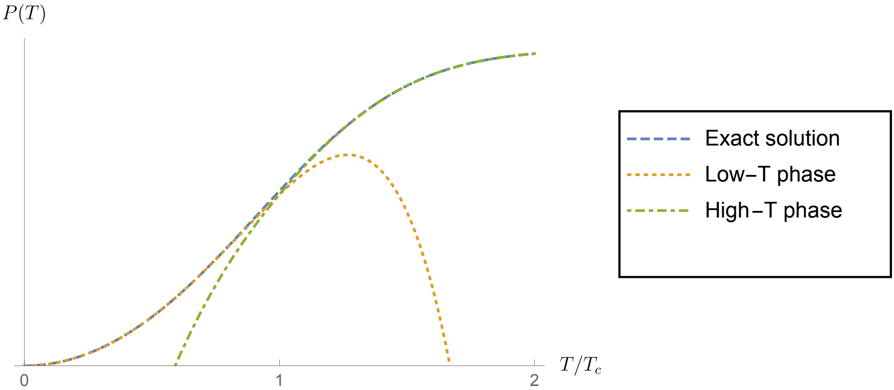

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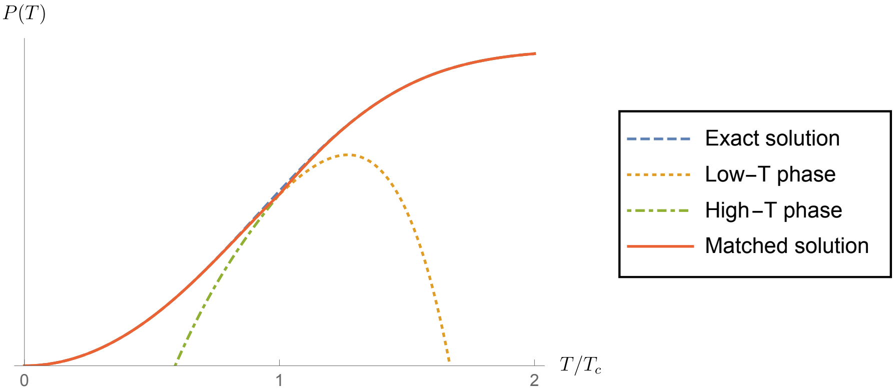

$P(T)$
T/T_c
0
1
2
Exact solution
Low−T phase
High−T phase
Matched solution

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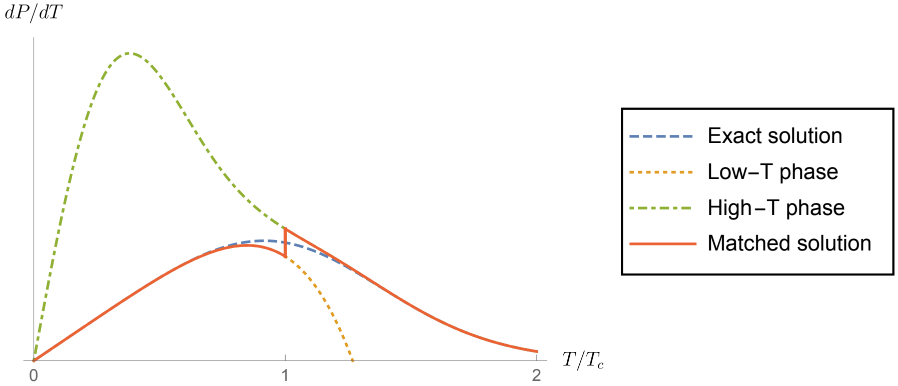

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