Exact correlators from conformal Ward identities in momentum space and perturbative realizations

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I will illustrate recent studies of the conformal anomaly action and its role in the description of a possible conformal extension of the Standard Model.

based on recent work with MM Maglio and A. Costantini, E. Mottola
and on previous work with L. Delle Rose, M. Serino and C. Marzo

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The Standard Models is not a scale invariant theory

It turns out to be scale invariant if we switch off the Higgs vev

The question: how to generate the Higgs mass.

We are not allowed to introduce a dimensionful constant in order to define the electroweak scale because that would violate scale invariance explicitly (no scale invariant Lagrangian)

For a gauge theory SSB leaves the Lagrangian gauge invariant, only the vacuum is not gauge symmetric.

Scale invariance is gained at the cost of introducing a dilaton in the theory but we face the issue of a flat direction in the Higgs-dilaton potential
\[ \mathcal{L} = \frac{1}{2} (\partial \phi)^2 - V_2(\phi) = \frac{1}{2} (\partial \phi)^2 + \frac{\mu^2}{2} \phi^2 - \lambda \frac{\phi^4}{4} - \frac{\mu^4}{4\lambda}, \]

different choices of the unobservable vacuum energy

\[ V_1(H, H^\dagger) = -\mu^2 H^\dagger H + \lambda (H^\dagger H)^2 = \lambda \left( H^\dagger H - \frac{\mu^2}{2\lambda} \right)^2 - \frac{\mu^4}{4\lambda} \]

\[ V_2(H, H^\dagger) = \lambda \left( H^\dagger H - \frac{\mu^2}{2\lambda} \right)^2 \]

V2 is stable in the conformal extension

let's choose V2
\[ \mathcal{L} = \frac{1}{2} (\partial \phi)^2 - V_2(\phi) = \frac{1}{2} (\partial \phi)^2 + \frac{\mu^2}{2} \phi^2 - \frac{\lambda}{4} \frac{\phi^4}{4} - \frac{\mu^4}{4\lambda}, \]

canonical EMT
\[
T^\mu_{\nu}(\phi) = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} \eta^{\mu\nu} \left[ (\partial \phi)^2 + \mu^2 \phi^2 - \frac{\lambda}{2} \frac{\phi^4}{2} - \frac{\mu^4}{2\lambda} \right],
\]
\[
T^\mu_{\nu}(\phi) = -(\partial^\mu \phi)^2 - 2 \mu^2 \phi^2 + \lambda \phi^4 + \frac{\mu^4}{\lambda}.
\]

It is well known that the EMT of a scalar field can be improved in such a way as to make its trace proportional only to the scale breaking parameter, i.e. the mass \( \mu \). This can be done by adding an extra contribution \( T^\mu_{\nu}(\phi, \chi) \) which is symmetric and conserved
\[
T^\mu_{\nu}(\phi, \chi) = \chi \left( \eta^{\mu\nu} \Box \phi^2 - \partial^\mu \partial^\nu \phi^2 \right), \quad \text{term of improvement}
\]
where the \( \chi \) parameter is conveniently chosen. The combination of the canonical plus the improvement EMT,
\[
T^\mu_{\nu} = T^\mu_{\nu} + T^\mu_{\nu}(\phi, \chi)
\]
has the off-shell trace
\[
T^\mu_{\mu}(\phi, \chi) = (\partial \phi)^2 (6\chi - 1) - 2 \mu^2 \phi^2 + \lambda \phi^4 + \frac{\mu^4}{\lambda} + 6\chi \phi \Box \phi.
\]
using the equations of motion
\[
\Box \phi = \mu^2 \phi - \lambda \phi^3.
\]
the grace is proportional to the scaling parameter
\[
T^\mu_{\mu}(\phi, 1/6) = -\mu^2 \phi^2 + \frac{\mu^4}{\lambda}.
\]
\[ \mu \rightarrow \frac{\mu}{\Lambda} \Sigma, \]

promoting scale invariance (field enlarging transformation)

Sigma(x) at this stage is a **compensator**, which needs be rendered dynamical by the addition of a kinetic term

\[ \mathcal{L} = \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} (\partial \Sigma)^2 + \frac{\mu^2}{2 \Lambda^2} \Sigma^2 \phi^2 - \frac{\lambda}{4} \phi^4 - \frac{\mu^4}{4 \lambda \Lambda^4} \Sigma^4 \]

in the form \( V_2 \) of the potential

The new Lagrangian is dilatation invariant

\[ T^\mu_\mu(\phi, \Sigma, \chi, \chi') = (6 \chi - 1) (\partial \phi)^2 + (6 \chi' - 1) (\partial \Sigma)^2 + 6 \chi \phi \boxdot \phi + 6 \chi' \Sigma \Box \Sigma - 2 \frac{\mu^2}{\Lambda^2} \Sigma^2 \phi^2 + \lambda \phi^4 + \frac{1}{\lambda \Lambda^4} \Sigma^4, \]

which vanishes upon using the equations of motion for the \( \Sigma \) and \( \phi \) fields,

\[ \Box \phi = \frac{\mu^2}{\Lambda^2} \Sigma^2 \phi - \lambda \phi^3, \]

\[ \Box \Sigma = \frac{\mu^2}{\Lambda^2} \Sigma \phi^2 - \frac{1}{\lambda \Lambda^4} \Sigma^3, \]

and setting the \( \chi, \chi' \) parameters at the special value \( \chi = \chi' = 1/6 \).
V2 allows to perform the spontaneous breaking of the scale symmetry around a stable minimum, by setting

$$\Sigma = \Lambda + \rho, \quad \phi = v + h.$$  

Lambda is the conformal scale

rho is the dilaton

$$V_1(H, H^\dagger, \Sigma) = -\frac{\mu^2 \Sigma^2}{\Lambda^2} H^\dagger H + \lambda (H^\dagger H)^2,$$

$$V_2(H, H^\dagger, \Sigma) = \lambda \left( H^\dagger H - \frac{\mu^2 \Sigma^2}{2\lambda \Lambda^2} \right)^2,$$

if we expand around the vev of Sigma, leaving the Higgs as it is (i.e. above the ew scale)

$$\mathcal{L} = \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} (\partial \rho)^2 + \frac{\mu^2}{2} \phi^2 - \lambda \frac{\phi^4}{4} - \frac{\mu^4}{4\Lambda} - \frac{\rho}{\Lambda} \left( -\mu^2 \phi^2 + \frac{\mu^4}{\lambda} \right) + \ldots,$$

one can write down a dilaton interaction at order 1/\Lambda

using the eqs of motion

$$\mathcal{L}_\rho = (\partial \rho)^2 - \frac{\rho}{\Lambda} T^\mu_\mu (\phi, 1/6) + \ldots,$$
we could have expanded around two vevs (i.e. below the ew scale) by sequentially invoking a spontaneous breaking of the scale symmetry followed by a breaking of the ew scale (v, Lambda)

\[
\begin{pmatrix}
\rho_0 \\
\h_0
\end{pmatrix} =
\begin{pmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\rho \\
\h
\end{pmatrix}
\]

\[
\cos \alpha = \frac{1}{\sqrt{1 + v^2/\Lambda^2}} \quad \text{and} \quad \sin \alpha = \frac{1}{\sqrt{1 + \Lambda^2/v^2}}.
\]

\[
m_{\h_0}^2 = 2\lambda v^2 \left(1 + \frac{v^2}{\Lambda^2}\right) \quad \text{with} \quad v^2 = \frac{\mu^2}{\lambda},
\]

and with \(m_h^2 = 2\lambda v^2\) being the mass of the Standard Model Higgs. The Higgs mass, in this case, is corrected by the new scale of the spontaneous breaking of the dilatation symmetry (\(\Lambda\)), which remains a free parameter.

clearly, we need to give a mass to rho(x) is we want to justify the vev of Sigma (Lambda)

\[
\mathcal{L}_{\text{break}} = \frac{1}{2} m^2 \rho^2 + \frac{1}{3!} m^2 \rho^3 + \cdots,
\]
The result of such simple considerations are that we cannot get away from the scale symmetric phase without an extra potential which explicitly breaks scale invariance and that we should attribute to some unspecified dynamical mechanism.

SSB's are related to vacuum degeneracies from which we pick up one specific state "the vacuum".

vacuum degeneracies of SIGMA and H are connected

1. somehow we should have an extra potential which is vacuum degenerate only respect to Sigma,
2. assign a vev to Sigma by a vacuum selection (Lambda) (but the Lagrangian should still be scale symmetric)
3. Now end up with a vacuum degeneracy in H, and proceed with the usual electroweak SSB

The problem: How do we guarantee a vacuum degeneracy in Sigma without breaking the scale symmetry of the Lagrangian?
at quantum level the dilaton will couple to the anomaly

one can integrate out matter (in this case the SM) in order to define an effective action
and the interactions of the dilaton.
A stress energy tensor which contains the term of improvement and reproduces all the features discussed above
is obtained via a metric embedding

\[ S = S_G + S_{SM} + S_I = -\frac{1}{\kappa^2} \int d^4x \sqrt{-g} \, R + \int d^4x \sqrt{-g} \mathcal{L}_{SM} + \chi \int d^4x \sqrt{-g} \, R \, H^\dagger H , \]

we can add a dilaton by hand after taking the flat spacetime limit
Wess-Zumino actions

An economical way to couple the dilaton to the Standard Model, and to address issues such as renormalization, anomalies and so on, is to couple it to gravity

\[ S = S_G + S_{SM} + S_I = -\frac{1}{\kappa^2} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \mathcal{L}_{SM} + \chi \int d^4x \sqrt{-g} R \, H^\dagger H, \]

\[ \mathcal{H} = \begin{pmatrix} -i\phi^+ \\ \frac{1}{\sqrt{2}}(v + H + i\phi) \end{pmatrix}, \quad T_{\mu\nu}^I = -\frac{1}{3} \left[ \partial_{\mu} \partial_{\nu} - \eta_{\mu\nu} \Box \right] \mathcal{H}^\dagger \mathcal{H} = -\frac{1}{3} \left[ \partial_{\mu} \partial_{\nu} - \eta_{\mu\nu} \Box \right] \left( \frac{H^2}{2} + \frac{\phi^2}{2} + \phi^+ \phi^- + vH \right), \]

The complete expression of the energy-momentum tensor can be found in (2.1)

\[ \langle T_{\mu}^{\mu} \rangle = \mathcal{A} (z), \]

\[ C^2 = C_{\lambda\mu\nu\rho}C^{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{R^2}{3}, \]

\[ E = ^*R_{\lambda\mu\nu\rho}^*R^{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \]

Delle Rose, Serino, C.C.


TVV
dilaton interactions

\[ \Gamma_{VV'}^{\alpha \beta}(k, p, q) = (2 \pi)^4 \delta^4(k - p - q) \frac{i}{\Lambda} \left( \mathcal{A}^{\alpha \beta}(p, q) + \Sigma^{\alpha \beta}(p, q) + \Delta^{\alpha \beta}(p, q) \right), \]

\[
\mathcal{A}^{\alpha \beta}(p, q) = \int d^4x \, d^4y \, e^{ip \cdot x + iq \cdot y} \frac{\delta^2 \mathcal{A}(0)}{\delta A^\alpha(x) \delta A^\beta(y)}
\]

\[
\Sigma^{\alpha \beta}(p, q) + \Delta^{\alpha \beta}(p, q) = \int d^4x \, d^4y \, e^{ip \cdot x + iq \cdot y} \left( T^\mu_\alpha(0) V^\alpha(x) V^\beta(y) \right).
\]

The computation of the SM corrections to TVV vertices shows that renormalization does not require additional counterterms if the coupling of the Higgs is conformal.

Delle Rose, Serino, C.C.
Conformal couplings

Figure 12. The decay branching fractions of the dilaton (a) to gluon pair and (b)-(c) gauge boson pairs for different $\xi$ parameters.

Mass combinatorics can give us much earlier hint for such resonance peak. The results of our analysis can be easily extrapolated for higher conformal breaking scale $\xi$ and higher mass values of the dilaton. In section 6 we also discuss the effect of the mixing between dilaton and Higgs boson. Our results can be easily interpreted for any non-zero $\xi$ case by rescaling the decay branching fraction of the dilaton. Finally we expect LHC run II will unveil more data that will give more insight into this extension of SM.

References


\[
\frac{\Gamma_{\rho \rightarrow gg}}{\Gamma_{h \rightarrow gg}} = \frac{v^2 \sum_i x_i [1 + (1 - x_i) f(x_i)]^2}{\Lambda^2 [\sum_i x_i [1 + (1 - x_i) f(x_i)]]^2}.
\]

\[
x_i = \frac{4m_i^2}{m_\rho^2}.
\]

Any value of Lambda \(> 5\) TeV is not ruled out by the current data.

P. Bandyopadhyay, A. Costantini, L. Delle Rose, C.C.

**SUMMARY/ PERSPECTIVES**

The coupling to the anomaly is introduced "by hand".

What kind of interactions should we include?
not all the dilaton interactions are functionally independent since there are some conformal trace relations which constrain those of order higher than 4

(Delle Rose, Marzo, Serino, C.C)
\[
\Gamma_{WZ}[g,\tau] = \int d^4x \sqrt{g} \left\{ \beta_\alpha \left[ \frac{\tau}{\Lambda} \left( F - \frac{2}{3} \Box R \right) + \frac{2}{\Lambda^2} \left( \frac{R}{3} \partial^\alpha \tau \partial_\alpha \tau + (\Box \tau)^2 \right) \right] - \frac{4}{\Lambda^3} \partial^\alpha \tau \partial_\alpha \tau \Box \tau + \frac{2}{\Lambda^4} \left( \partial^\alpha \tau \partial_\alpha \tau \right)^2 \right\} + \beta_b \left[ \frac{\tau}{G} - \frac{4}{\Lambda^2} \left( R^{\alpha\beta} - \frac{R}{2} g^{\alpha\beta} \right) \partial_\alpha \tau \partial_\beta \tau - \frac{4}{\Lambda^3} \partial^\alpha \tau \partial_\alpha \tau \Box \tau + \frac{2}{\Lambda^4} \left( \partial^\alpha \tau \partial_\alpha \tau \right)^2 \right].
\]

This is in agreement with the quartic nature of the dilaton Lagrangian or by the Noether method.

\[
\delta \Gamma_{WZ}[g,\tau] = \int d^4x \sqrt{g} \left[ \beta_\alpha \left( F - \frac{2}{3} \Box R \right) + \beta_b G \right].
\]

Marzo, Delle Rose, Serino, C.C.

\[\Gamma^{(1)}_{WZ}[g,\tau] = \int d^4x \sqrt{g} \left[ \frac{\tau}{\Lambda} \left( F - \frac{2}{3} \Box R \right) + \beta_b G \right].\]

\[\Gamma^{(2)}_{WZ}[g,\tau] = \Gamma^{(1)}_{WZ}[g,\tau] + \frac{1}{\Lambda^2} \int d^4x \sqrt{g} \left\{ \beta_\alpha \left( \frac{2}{3} R \partial^\alpha \tau \partial_\alpha \tau + 2 (\Box \tau)^2 \right) - 4 \beta_b \left( R^{\alpha\beta} - \frac{g^{\alpha\beta}}{2} R \right) \partial_\alpha \tau \partial_\beta \tau \right\}.\]

WZ action

\[g_{\mu\nu}'(x) = e^{2\sigma(x)} g_{\mu\nu}(x),\]

\[V'_{\alpha\rho}(x) = e^{\sigma(x)} V_{\alpha\rho}(x),\]

\[\Phi'(x) = e^{d\sigma(x)} \Phi(x),\]

quartic action

the quartic nature of the WZ action

Dilaton interactions and constraints from \(\Gamma_{WZ}\)

Marzo, Delle Rose, Serino, C.C

higher order interactions exactly fixed by the quartic expansion
Weyl tensor squared

\[ g_{\mu\nu}(z) \langle T^{\mu\nu}(z) \rangle = \sum_{I=F,S,G} n_{I} \left[ \beta_{a}(I) C^{2}(z) + \beta_{b}(I) E(z) \right] + \frac{\kappa}{4} n_{G} F^{a}_{\mu\nu} F_{\mu\nu}^{a}(z) \]
\[ \equiv \mathcal{A}(z, g), \]

\[ C^{2} = R_{abcd} R^{abcd} - \frac{4}{d-2} R_{ab} R^{ab} + \frac{2}{(d-2)(d-1)} R^{2}, \]
\[ E = R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} + R^{2} \]

Euler-Poincare density

\[ \mathcal{A}[g] = \sum_{i=1}^{3} c_{i} \left( I_{i} + \nabla_{\mu} J_{i}^{\mu} \right) + a E_{6}, \]

\[ E_{6} = K_{1} - 12 K_{2} + 3 K_{3} + 16 K_{4} - 24 K_{5} - 24 K_{6} + 4 K_{7} + 8 K_{8} \]

d=4

d=6

EP density in d=6

F. Bastianelli et al,
L. Delle Rose, Carlo Marzo, M. Serino, C.C.
\[ \Gamma_{WZ}[\delta, \tau] = - \int d^6x \sqrt{g} \left\{ - \frac{c_3}{\Lambda^2} \Box \tau \Box^2 \tau + \frac{1}{\Lambda^3} \left[ \left( - \frac{7}{16} c_1 + \frac{11}{4} c_2 + 2 c_3 \right) (\Box \tau)^3 
right. \\
+ \left( \frac{3}{2} c_1 - 6 c_2 - 8 c_3 \right) (\partial \tau)^2 \Box \tau \right] + \frac{1}{\Lambda^4} \left[ \left( - \frac{3}{2} c_1 + 6 c_2 + 16 c_3 + 24 a \right) (\partial \tau)^2 (\partial \tau)^2 
right. \\
- \left( \frac{3}{8} c_1 + \frac{9}{2} c_2 + 4 c_3 + 24 a \right) (\partial \tau)^2 (\Box \tau)^2 + \left( \frac{3}{4} c_1 - 3 c_2 - 4 c_3 \right) \partial^\mu (\partial \tau)^2 \partial_\mu (\partial \tau)^2 
right. \\
\left. \left. \left. \frac{1}{\Lambda^5} \left( \frac{3}{2} c_1 + 6 c_2 + 36 a \right) (\partial \tau)^4 \Box \tau - \frac{1}{\Lambda^6} \left( c_1 + 4 c_2 + 24 a \right) (\partial \tau)^6 \right\} . \right. \] 

(91)

higher order dilaton interactions determined by the first 6 traces
Which action?
There is not a unique viewpoint, since anomaly actions are not unique, if we just want to reproduce an anomaly functional.

Can we describe the breaking of the conformal dynamics without introducing an asymptotic dilaton field?

One example: the nonlocal Riegert action.
nonlocal anomaly action

\[ S_{\text{anom}}[g, A] = \frac{1}{8} \int d^4x \sqrt{-g} \int d^4x' \sqrt{-g'} \left( E - \frac{2}{3} \Box R \right)_x G_A(x, x') \left[ 2b C^2 + b' \left( E - \frac{2}{3} \Box R \right) + 2c F_{\mu \nu} F^{\mu \nu} \right]_{x'} \]

\[ \Delta_4 \equiv \nabla_\mu \left( \nabla^\mu \nabla^\nu + 2R^{\mu \nu} - \frac{2}{3} R g^{\mu \nu} \right) \nabla_\nu = \Box^2 + 2R^{\mu \nu} \nabla_\mu \nabla_\nu + \frac{1}{3} (\nabla^\mu R) \nabla_\mu - \frac{2}{3} \Box R. \]

\[ S = S_G + S_{\text{SM}} + S_T = -\frac{1}{\kappa^2} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \mathcal{L}_{\text{SM}} + \frac{1}{6} \int d^4x \sqrt{-g} R \mathcal{H}^\dagger \mathcal{H}, \]

TJJ for instance, this action shows that the anomaly is mediated by the emergence of an effective massless pole

E. Mottola,
M. Giannotti

mediation by an anomaly pole

Armillis, Delle rose, C.C.

This action generates the anomaly contribution to $T T$, $T T T$, $T T T T$ etc (with open indices) but also the corresponding traces (dilaton interactions), without introducing an asymptotic dilaton


\[ S = S_G + S_{SM} + S_I = -\frac{1}{\kappa^2} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \mathcal{L}_{SM} + \frac{1}{6} \int d^4x \sqrt{-g} R \mathcal{H}^\dagger \mathcal{H}, \]

\[ T_{\mu\nu}(x) = \frac{2}{\sqrt{-g(x)}} \frac{\delta[S_{SM} + S_I]}{\delta g^{\mu\nu}(x)}, \]

\[ \mathcal{L}_{grav}(x) = -\frac{\kappa}{2} T^{\mu\nu}(x) h_{\mu\nu}(x). \]

\[ T_{\mu\nu}^{Min} = T^{f.s.}_{\mu\nu} + T^{ferm.}_{\mu\nu} + T^{Higgs}_{\mu\nu} + T^{Yukawa}_{\mu\nu} + T^{g.fix.}_{\mu\nu} + T^{ghost}_{\mu\nu}. \]

and many more
Supersymmetry. The case of the Ferrara Zumino supercurrent

Superconformal Sum Rules and the Spectral Density Flow of the Composite Dilaton (ADD) Multiplet in \( N = 1 \) Theories

Costantini, Delle Rose, Serino, C.C.,

N=1 SYM theory shares a similar behaviour: it is clearly universal

The supercurrent combines a stress energy tensor, a chiral and a susy anomaly

\[
R^\mu = \bar{\chi} \sigma^\mu \chi + \frac{1}{3} \left( -\bar{\chi} \sigma^\mu \chi + 2i \phi_i^\dagger D_{ij}^\mu \phi_j - 2i (D_{ij}^\mu \phi_j)^\dagger \phi_i \right),
\]

\[
S_A^\mu = i (\sigma^{\mu \rho} \sigma^\alpha \bar{\chi})_A F_{\rho \sigma}^A - \sqrt{2} (\sigma^{\mu \rho} \chi_i)_A (D_{ij}^\mu \phi_j)^\dagger - i \sqrt{2} (\sigma^{\mu \rho} \chi_i) W_i^j (\phi^j)
\]

\[
T^{\mu \nu} = - F^{\alpha \mu \nu} F_{\alpha \gamma} + \frac{i}{4} \left[ \bar{\chi} \sigma^{\mu \nu} (\delta^\alpha \beta - g \epsilon^{\beta \gamma \epsilon}) \chi + \bar{\chi} \sigma^{\mu \nu} (\delta^\alpha \beta + g \epsilon^{\beta \gamma \epsilon}) \chi + (\mu \leftrightarrow \nu) \right]
\]

\[
\partial_\mu R^\mu = \frac{g^2}{16 \pi^2} \left( T(A) - \frac{1}{3} T(R) \right) F^{\alpha \mu \nu} F_{\alpha \mu \nu},
\]

\[
\bar{\sigma}_A S_A^\mu = -i \frac{3 g^2}{8 \pi^2} \left( T(A) - \frac{1}{3} T(R) \right) (\bar{\chi}^A \sigma^{\mu \nu}) A F^{\alpha \mu \nu},
\]

\[
\eta_{\mu \nu} T^{\mu \nu} = - \frac{3 g^2}{32 \pi^2} \left( T(A) - \frac{1}{3} T(R) \right) F^{\alpha \mu \nu} F_{\alpha \mu \nu},
\]
R current and vector currents

S current and vector currents

Similar behaviour

TVV
In each sector, only 1 form factor is responsible for the anomaly.

Dispersion relation for the anomaly for factor, away from the conformal limit. As \( m \to 0 \), the branch cut turns into a pole.

\[
F(Q^2, m^2) = \frac{1}{\pi} \int_0^\infty ds \frac{\rho(s, m^2)}{s + Q^2},
\]

\[
\lim_{Q^2 \to \infty} Q^2 F(Q^2, m^2) = f.
\]

A nonlocal action is responsible for this behaviour

\[
\lim_{m \to 0} \rho_\chi(s, m^2) = \lim_{m \to 0} \frac{2\pi m^2}{s^2} \log \left( \frac{1 + \sqrt{s(m^2)}}{1 - \sqrt{s(m^2)}} \right) \theta(s - 4m^2) = \pi \delta(s)
\]

\[
\frac{1}{\pi} \int_{4m^2}^\infty ds \rho_\chi(s, m^2) = 1.
\]
This action predicts a certain structure for multiple correlators of stress energy tensors.

**TTT in CFT: Trace Identities and the Conformal Anomaly Effective Action.**
Matteo Maria Maglio, Emil Mottola, C.C.

[arXiv:1703.08860 [hep-th]].
These effective interactions mediated by "anomaly poles" are now used in the theory of topological insulators and in Weyl semimetals.

\[ S_{an} \sim \beta(e) \int d^4x \, d^4y R^{(1)}(x) \left( \frac{1}{\Box} \right)(x, y) FF(y) \]

Mottola, Giannotti Armillis, Delle Rose, C.C.

\[ S_{an} \sim \int d^4x \, d^4y R^{(1)}(x) \left( \frac{1}{\Box} \right)(x, y) \left( \beta_b E^{(2)}(y) + \beta_a C^2(2)(y) \right) \]

Maglio, C.C.
The nonlocal action is part of a general construct which is the "true" anomaly action.

To define such a "true" action, we should change our perspective and not simply solve a variational problem either with or without a dilaton, but compute – as far as we can – directly from CFT's the correlation function of multiple stress-energy tensors.

Use conformal Ward identities.
Studies of the exact Conformal Anomaly Action

The General 3-Graviton Vertex ($TTT$) of Conformal Field Theories in Momentum Space in $d = 4$

Matteo Maria Maglio, C.C.
[arXiv:1802.01501 [hep-th]].

Exact Correlators from Conformal Ward Identities in Momentum Space and the Perturbative $TJJ$ Vertex

COMPUTE $TT$, $TTT$, $TTTT$ vertices exactly!
Anomalous breakings are controlled by a limited number of constants, at least for correlators of lower orders (up to 3 point functions).

Osborn and Petkou

Reconstruction in momentum space for tensor correlators, Bzowski, McFadden, Skenderis 2014-2017
use the transverse traceless sector to build the entire correlator

in the scalar case investigated by Delle Rose, Mottola, Serino, C.C.

$$\sum_{j=1}^{n-1} \left( p_j^\kappa \frac{\partial^2}{\partial p_j^\alpha \partial p_j^\alpha} + 2(\Delta_j - d) \frac{\partial}{\partial p_j^\kappa} - 2p_j^\alpha \frac{\partial^2}{\partial p_j^\kappa \partial p_j^\alpha} \right) \Phi(p_1, \ldots, p_{n-1}, \vec{p}_n) = 0.$$
transverse-traceless + local terms. This introduces a minimal number of form factors in the correlator.

\begin{align}
\langle T_{\mu_1 \nu_1} J_{\mu_2} J_{\mu_3} \rangle &= \langle t_{\mu_1 \nu_1} j_{\mu_2} j_{\mu_3} \rangle + \langle T_{\mu_1 \nu_1} J_{\mu_2} j^{\mu_3}_{loc} \rangle + \langle T_{\mu_1 \nu_1} j^{\mu_2}_{loc} J_{\mu_3} \rangle + \langle t^{\mu_1 \nu_1}_{loc} J_{\mu_2} J_{\mu_3} \rangle \\
&- \langle T_{\mu_1 \nu_1} j_{\mu_2} j^{\mu_3}_{loc} \rangle - \langle t_{\mu_1 \nu_1} j^{\mu_2}_{loc} J^{\mu_3}_{loc} \rangle - \langle t^{\mu_1 \nu_1}_{loc} J^{\mu_2}_{loc} j_{\mu_3} \rangle - \langle t^{\mu_1 \nu_1}_{loc} J^{\mu_2}_{loc} J^{\mu_3}_{loc} \rangle.
\end{align}

The formalism is very heavy.

\begin{align}
\langle t_{\mu_1 \nu_1}(p_1) j_{\mu_2}(p_2) j_{\mu_3}(p_3) \rangle &= \Pi_{1}^{\mu_1 \nu_1} \Pi_{2}^{\mu_2 \nu_2} \Pi_{3}^{\mu_3 \nu_3} \left( A_1 p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_1^{\alpha_3} + A_2 \delta^{\alpha_2 \alpha_3} p_2^{\alpha_1} p_1^{\alpha_3} + A_3 \delta^{\alpha_1 \alpha_2} p_2^{\beta_1} p_1^{\alpha_3} \\
&+ A_3(p_2 \leftrightarrow p_3) \delta^{\alpha_1 \alpha_3} p_2^{\beta_1} p_3^{\alpha_2} + A_4 \delta^{\alpha_1 \alpha_3} \delta^{\alpha_2 \beta_1} \right). \tag{6.30}
\end{align}

\begin{align}
\left[ 2d + N_n - \sum_{j=1}^{3} \Delta_j + \sum_{j=1}^{2} p_j^{\alpha} \frac{\partial}{\partial p_j^{\alpha}} \right] A_n(p_1, p_2, p_3) &= 0,
\end{align}

\begin{align}
\sum_{j=1}^{n-1} \left( p_j^{\kappa} \frac{\partial^2}{\partial p_j^{\kappa} \partial p_j^{\alpha}} + 2(\Delta_j - d) \frac{\partial}{\partial p_j^{\kappa}} - 2 p_j^{\alpha} \frac{\partial^2}{\partial p_j^{\kappa} \partial p_j^{\alpha}} \right) \Phi(p_1, \ldots p_{n-1}, \bar{p}_n) &= 0.
\end{align}

\begin{align}
0 &= C_{11} = K_{13} A_1 \\
0 &= C_{12} = K_{13} A_2 + 2A_1 \\
0 &= C_{13} = K_{13} A_3 - 4A_1 \\
0 &= C_{14} = K_{13} A_3(p_2 \leftrightarrow p_3) \\
0 &= C_{15} = K_{13} A_4 - 2A_3(p_2 \leftrightarrow p_3)
\end{align}

\begin{align}
0 &= C_{21} = K_{23} A_1 \\
0 &= C_{22} = K_{23} A_2 \\
0 &= C_{23} = K_{23} A_3 - 4A_1 \\
0 &= C_{24} = K_{23} A_3(p_2 \leftrightarrow p_3) + 4A_1 \\
0 &= C_{25} = K_{23} A_4 + 2A_3 - 2A_3(p_2 \leftrightarrow p_3)
\end{align}

dilatations

tensor correlators

special conformal equations
For 3-point functions, in D=3 and D=4 CFT's are exactly matched by free field theories with a specific numbers of scalars, fermions and spin 1
Lorentz Ward identities (Maglio, C.C.)

\[ 0 = \sum_{j=1}^{2} \left[ p^\mu_j \frac{\partial}{\partial p^\mu_j} - p^\mu_j \frac{\partial}{\partial p^\mu_j} \right] \langle T^{\mu_1 \nu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3) \rangle \]
\[ + 2 \left( \delta^\nu_{\alpha_1} \delta^{\mu}(\mu_1) - \delta^\nu_{\alpha_1} \delta^{\mu}(\mu_1) \right) \langle T^{\nu_1}(\alpha_1) (p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3) \rangle \]
\[ + \left( \delta^\nu_{\alpha_2} \delta^{\mu}(\mu_1) - \delta^\nu_{\alpha_2} \delta^{\mu}(\mu_1) \right) \langle T^{\nu_1}(\alpha_2) (p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3) \rangle \]
\[ + \left( \delta^\nu_{\alpha_3} \delta^{\mu}(\mu_3) - \delta^\nu_{\alpha_3} \delta^{\mu}(\mu_3) \right) \langle T^{\nu_1}(\alpha_3) (p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3) \rangle, \]

\[ x = \frac{p_1^2}{p_3^2}, \quad y = \frac{p_2^2}{p_3^2}, \]

Appell

Campes de Feriet

Solutions expressed in terms of 3K integrals (Bzowski, McFadden, Skenderis) or built directly using the F4 (universality of the Fuchsian points of such systems of equations) (Maglio, C.C.)
$T^T T T$

3-graviton vertex

\[ K_{13} A_1 = 0 \]
\[ K_{13} A_2 = 8 A_1 \]
\[ K_{13} A_2(p_1 \leftrightarrow p_3) = -8 A_1 \]
\[ K_{13} A_2(p_2 \leftrightarrow p_3) = 0 \]
\[ K_{13} A_3 = 2 A_2 \]
\[ K_{13} A_3(p_1 \leftrightarrow p_3) = -2 A_2(p_1 \leftrightarrow p_3) \]
\[ K_{13} A_3(p_2 \leftrightarrow p_3) = 0 \]
\[ K_{13} A_4 = -4 A_2(p_2 \leftrightarrow p_3) \]
\[ K_{13} A_4(p_1 \leftrightarrow p_3) = 4 A_2(p_2 \leftrightarrow p_3) \]
\[ K_{13} A_4(p_2 \leftrightarrow p_3) = 4 A_2(p_1 \leftrightarrow p_3) - 4 A_2 \]
\[ K_{13} A_5 = 2 [ A_4 - A_4(p_1 \leftrightarrow p_3) ] \]

$A_5 = p_3^2 \sum_{ab} x^a y^b \left\{ C_1 f_{5,1}(a, b, d) F_4(\alpha + 1, \beta, \gamma - 1, \gamma' - 1, x, y) + h_1(C_1, C_2) f_{5,2}(a, b, d) F_4(\alpha + 1, \beta, \gamma, \gamma' - 1, x, y) + h_2(C_1, C_2) f_{5,3}(a, b) F_4(\alpha + 1, \beta, \gamma, \gamma', x, y) + h_3(C_1, C_2) f_{5,4}(a, b, d) F_4(\alpha + 1, \beta, \gamma - 1, \gamma', x, y) + C_2 \left[ f_{5,5}(a, b, d) F_4(\alpha, \beta, \gamma - 1, \gamma', x, y) + f_{5,6}(a, b, d) F_4(\alpha, \beta, \gamma, \gamma' - 1, x, y) + f_{5,7}(a, b, d) F_4(\alpha, \beta, \gamma - 1, \gamma' - 1, x, y) \right] + h_4(C_2, C_5) f_{5,8}(a, b, d) F_4(\alpha, \beta, \gamma, \gamma', x, y) \right\} \quad (6.81)$
The renormalization program in D=4 is very involved, especially in either formalism (3K or Fuchsian).

But is can be bypassed

Use different free field theory sectors and show that the nonperturbative and the perturbative solutions match (Maglio, C.C.)
Simplifications

Free field theory in d=4 can be used to simplify the solutions

Maglio, C.C.

All the form factors take a lengthy but simple form, with renormalized anomalous CWI's in terms of the free field content.

\[
\begin{align*}
K_{13}A_{3}^{\text{Ren}} & = 2A_{2}^{\text{Ren}} - \frac{2\pi^{2}}{45} (7n_{F} - 26n_{G} + 2n_{S}) \\
K_{23}A_{3}^{\text{Ren}} & = 2A_{2}^{\text{Ren}} - \frac{2\pi^{2}}{45} (7n_{F} - 26n_{G} + 2n_{S}) \\
K_{13}A_{4}^{\text{Ren}} & = -4A_{2}^{\text{Ren}}(p_{2} \leftrightarrow p_{3}) + \frac{4\pi^{2}}{45} (7n_{F} - 26n_{G} + 2n_{S}) \\
K_{23}A_{4}^{\text{Ren}} & = -4A_{2}^{\text{Ren}}(p_{1} \leftrightarrow p_{3}) + \frac{4\pi^{2}}{45} (7n_{F} - 26n_{G} + 2n_{S}) \\
K_{13}A_{5}^{\text{Ren}} & = 2 [A_{4}^{\text{Ren}} - A_{4}^{\text{Ren}}(p_{1} \leftrightarrow p_{3})] - \frac{4\pi^{2}}{9} (s_{1} - s_{2}) (5n_{F} + 2n_{G} + n_{s}) \\
K_{23}A_{6}^{\text{Ren}} & = 2 [A_{4}^{\text{Ren}} - A_{4}^{\text{Ren}}(p_{2} \leftrightarrow p_{3})] - \frac{4\pi^{2}}{9} (s_{1} - s_{2}) (5n_{F} + 2n_{G} + n_{s})
\end{align*}
\]
Result:

One derives the exact renormalized TTT as a vertex (before any trace) in a simple form, expressed uniquely in terms of B0 and C0 integrals.

It is given by the anomaly terms + one traceless contribution

\[ \langle T_{\mu_1 \nu_1} T_{\mu_2 \nu_2} T_{\mu_3 \nu_3} \rangle_{\text{Ren}} = \langle t_{\mu_1 \nu_1} t_{\mu_2 \nu_2} t_{\mu_3 \nu_3} \rangle_{\text{Ren}} + \langle T_{\mu_1 \nu_1} T_{\mu_2 \nu_2} T_{\mu_3 \nu_3} \rangle_{\text{Ren}} \text{tr} + \langle T_{\mu_1 \nu_1} T_{\mu_2 \nu_2} T_{\mu_3 \nu_3} \rangle_{\text{anomaly}} \]

The anomaly (trace) terms are in agreement with those predicted by Riegert's action. The trace-free parts are associated to the renormalized form factors in terms of the scalar 2 and 3 point functions B0 and C0.

The match has been checked in d=3 and d=5, where there is perfect agreement.
Conclusions

There are two significant versions of the anomaly action

1. WZ for with an asymptotic dilaton field. This is introduced by hand, enlarging the number of degrees of freedom

2. The exact, computable form (at least up to 3-point functions) obtained by solving the conformal constraints.

They both contain information about the breaking of a conformal symmetry.

They may describe two phases of the same theory (UV/IR)

in d=4 free field theory saturates the exact solution by adding independent sectors and performing a matching.

The approach can be extended to TTTT. This will give a new perspective on the a-theorem and the irreversibility of the RG flow, which has been discussed only using an external compensator.