Rendering two-loop Feynman diagrams finite

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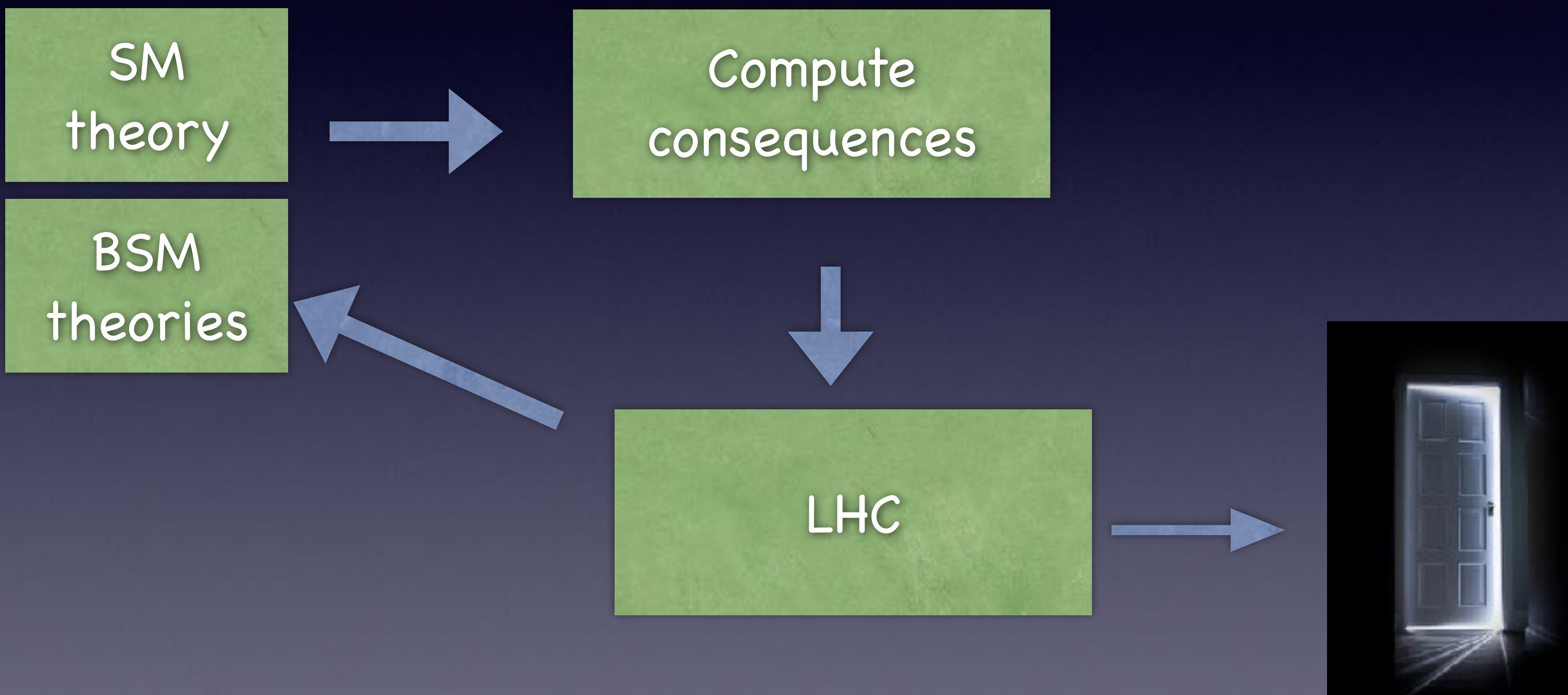
Outline

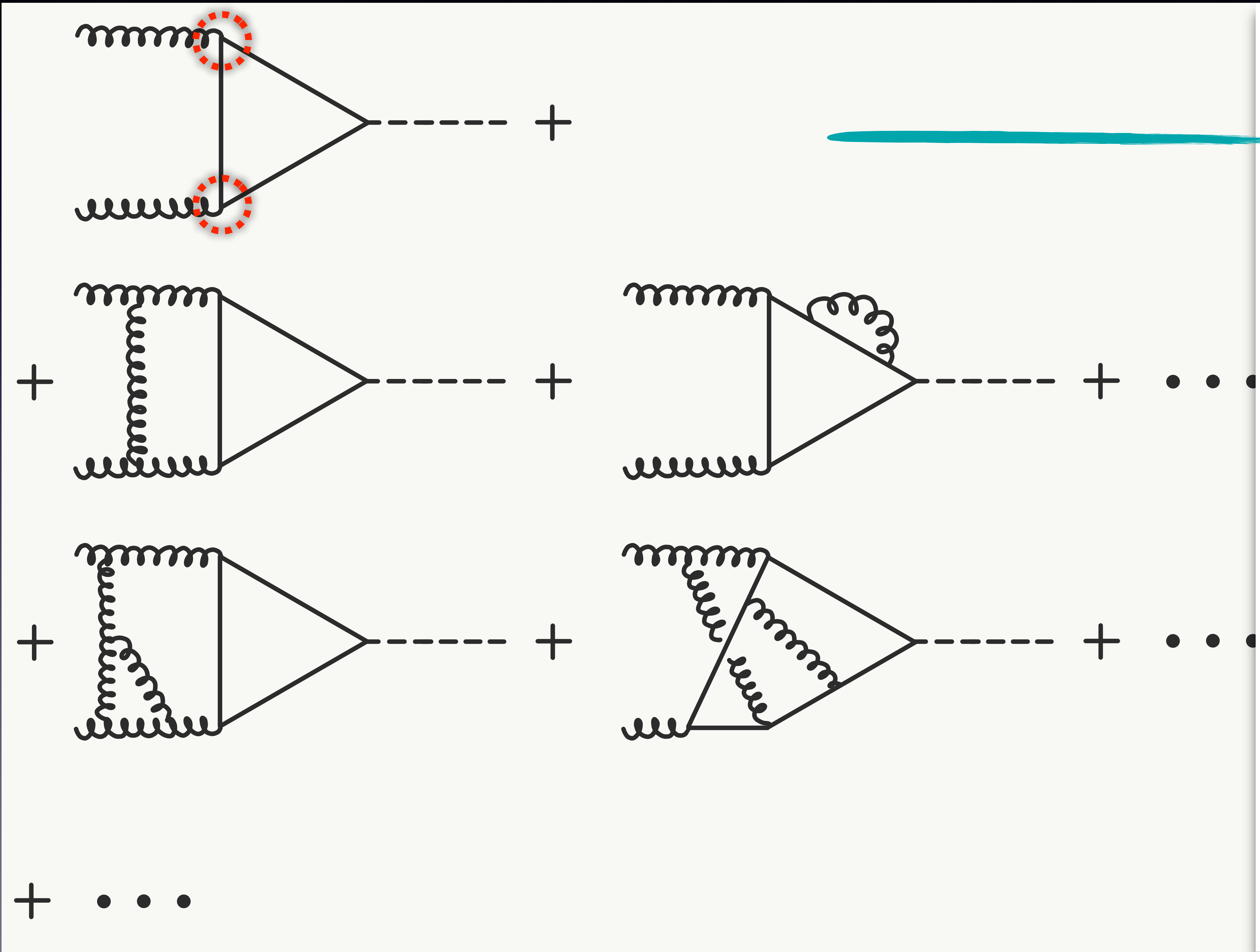
- Motivation
- Infrared divergences of Feynman diagrams
- Nested subtractions
- Examples
- Summary

LHC precision physics



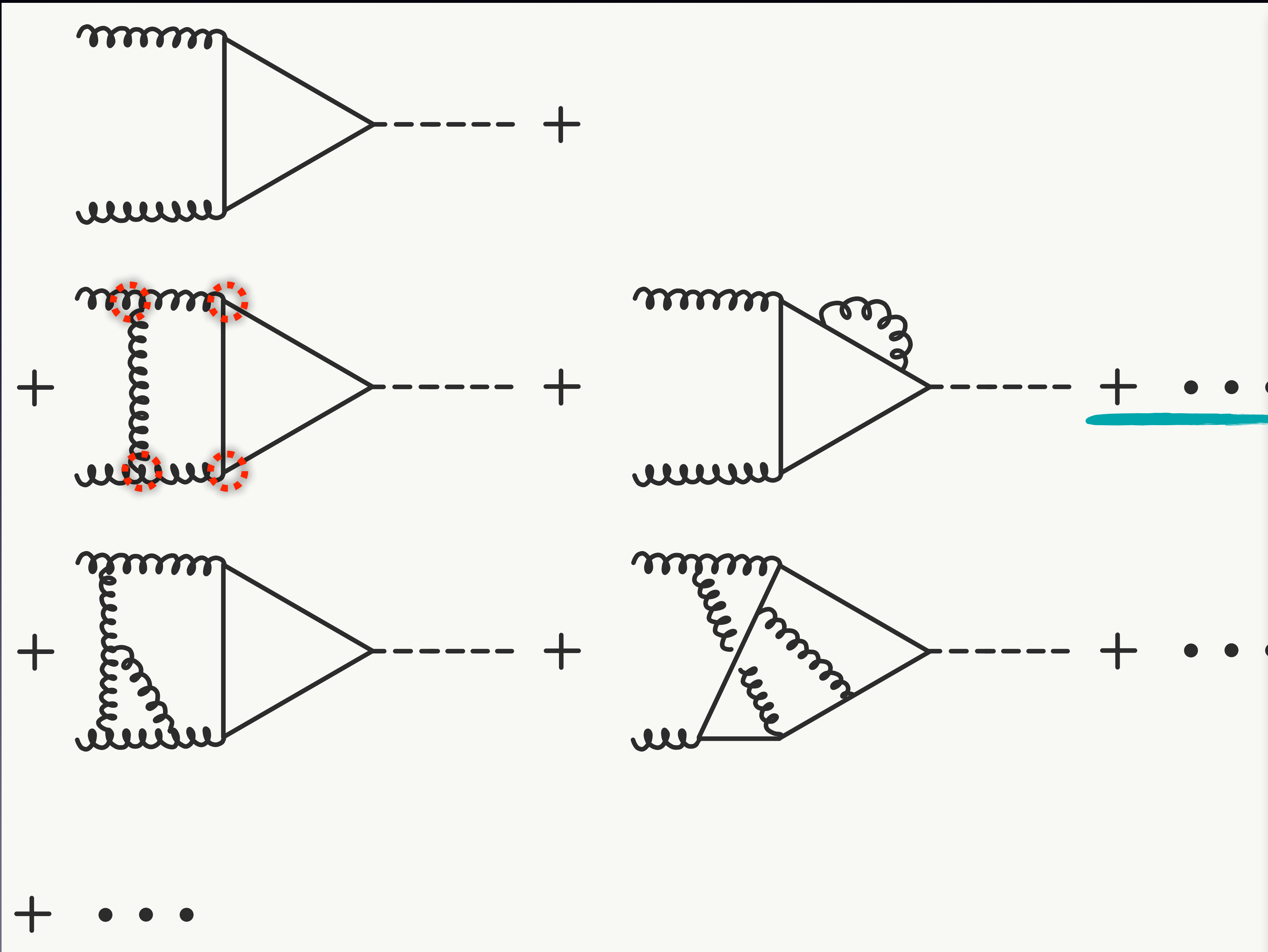
A “Feynman diagram” of finding New Laws

Perturbative expansion



$\mathcal{O}\left(g_s^2\right)$

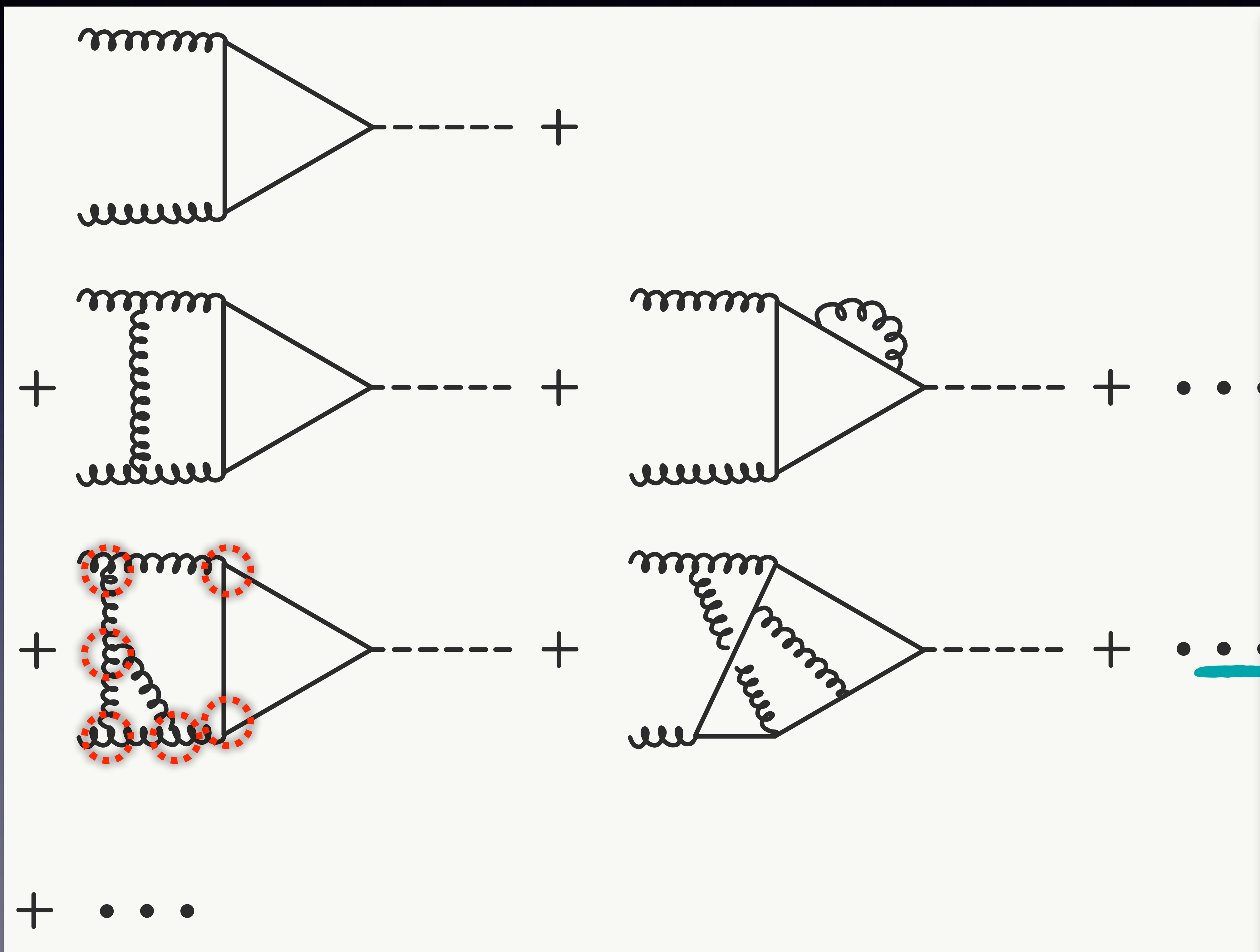
Perturbative expansion



$$\mathcal{O}\left(g_s^2\right)$$

$$\boxed{\mathcal{O}\left(g_s^4\right)}$$

Perturbative expansion

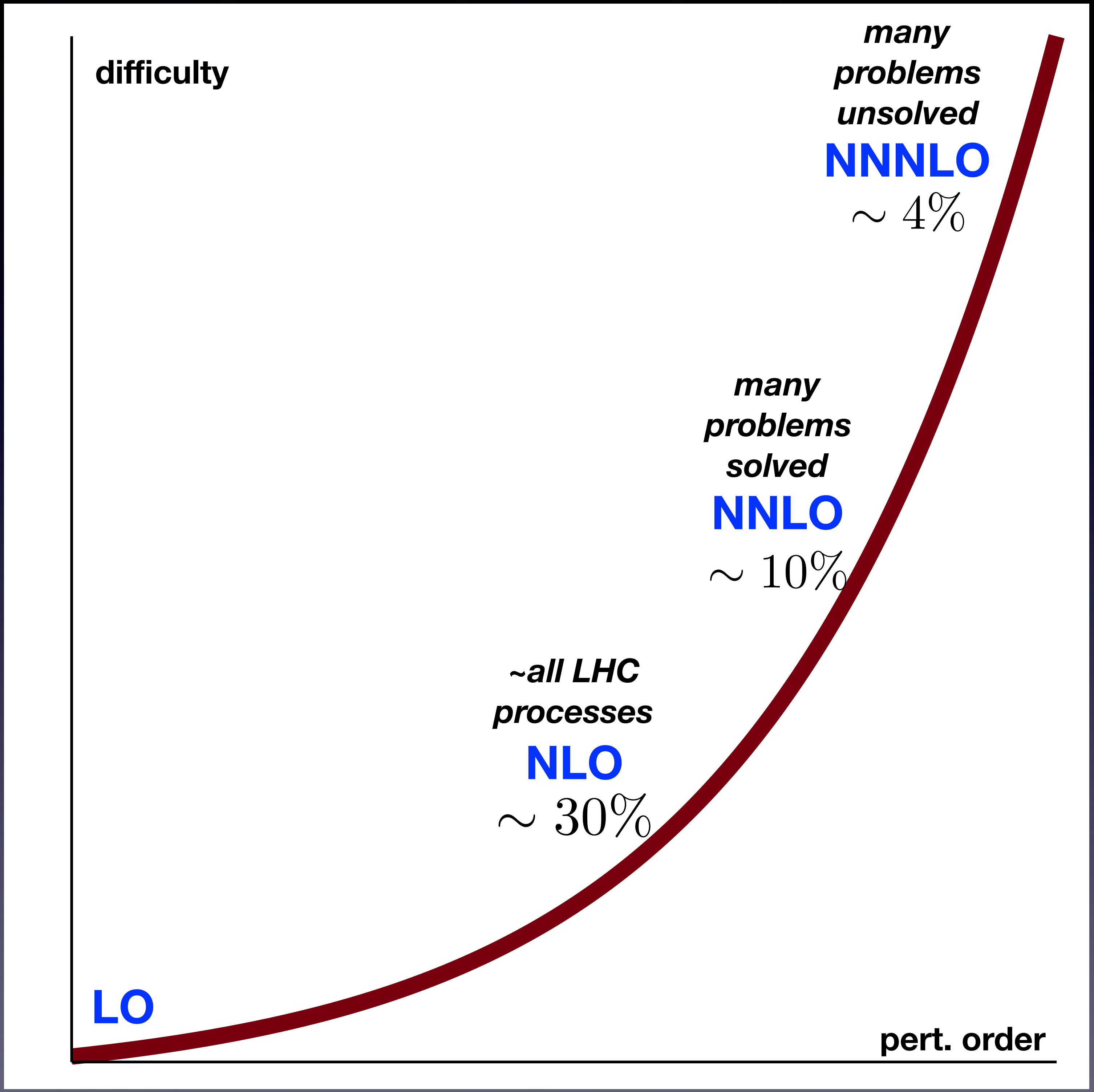


$\mathcal{O}\left(g_s^2\right)$

$\mathcal{O}\left(g_s^4\right)$

$\mathcal{O}\left(g_s^6\right)$

What has been achieved for the LHC?



$$\sigma = \sigma_0 \alpha_s^n + \sigma_1 \alpha_s^{n+1} + \sigma_2 \alpha_s^{n+2} + \ldots$$

Les Houches wish-list

- There is a long agenda for precision physics at the LHC
- Essential theoretical work which is needed to exploit to its maximum the multi-billion investment for the experiments

Process	State of the Art	Desired
$t\bar{t}$	$\sigma_{\rm tot}$(stable tops) @ NNLO QCD dσ(top decays) @ NLO QCD dσ(stable tops) @ NLO EW	dσ(top decays) @ NNLO QCD + NLO EW
$t\bar{t}+j(j)$	dσ(NWA top decays) @ NLO QCD	dσ(NWA top decays) @ NNLO QCD + NLO EW
$t\bar{t}+Z$	dσ(stable tops) @ NLO QCD	dσ(top decays) @ NLO QCD + NLO EW
single-top	dσ(NWA top decays) @ NLO QCD	dσ(NWA top decays) @ NNLO QCD + NLO EW
dijet	dσ @ NNLO QCD (g only) dσ @ NLO EW (weak)	dσ @ NNLO QCD + NLO EW
3j	dσ @ NLO QCD	dσ @ NNLO QCD + NLO EW
$\gamma+j$	dσ @ NLO QCD dσ @ NLO EW	dσ @ NNLO QCD + NLO EW

Table 2: Wishlist part 2 – Jets and Heavy Quarks

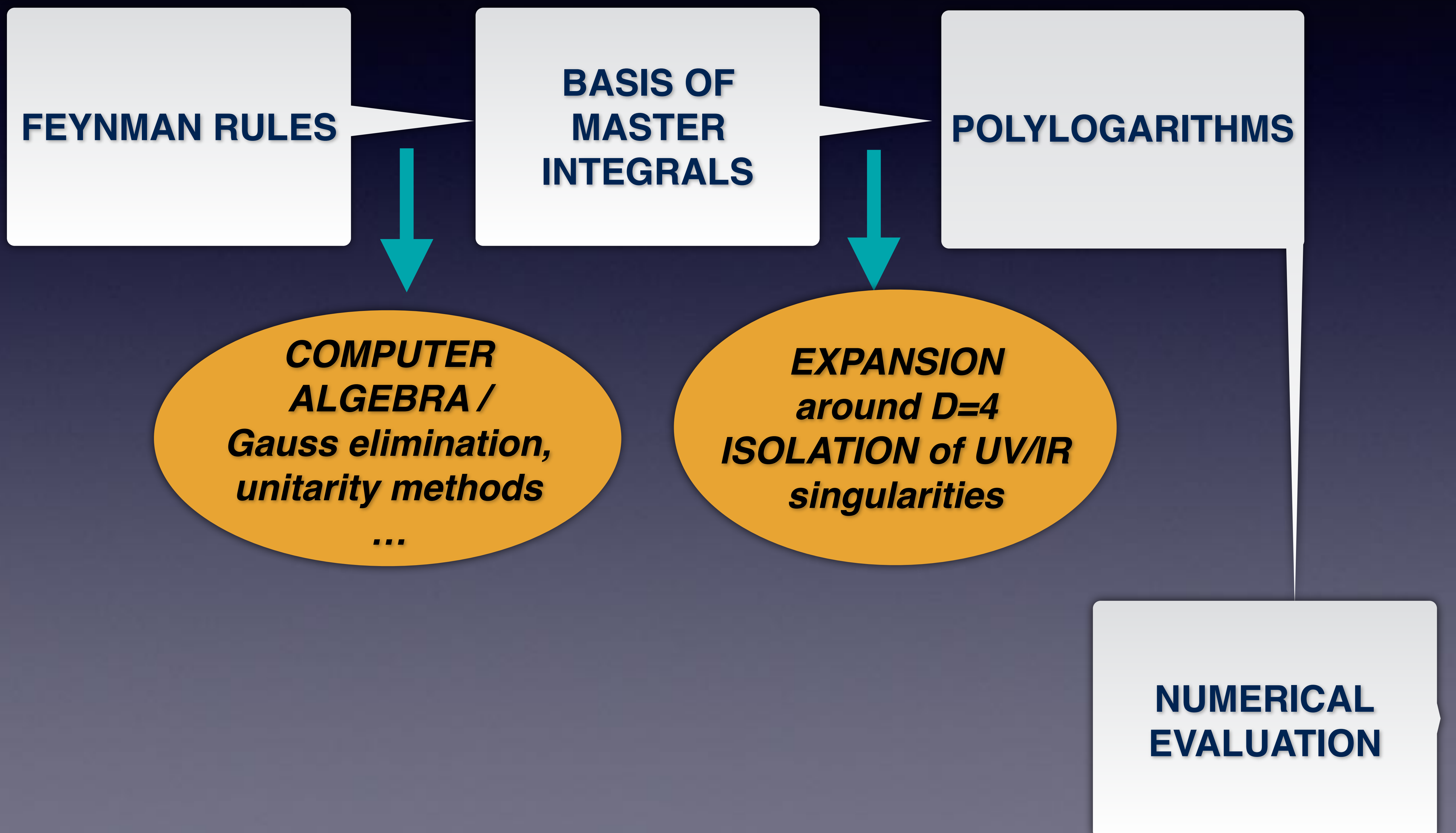
Process	State of the Art	Desired
H	dσ @ NNLO QCD (expansion in $1/m_t$) full m_t/m_b dependence @ NLO QCD and @ NLO EW NNLO+PS, in the $m_t \to \infty$ limit	dσ @ NNNLO QCD (infinite-m_t limit) full m_t/m_b dependence @ NNLO QCD and @ NNLO QCD+EW NNLO+PS with finite top quark mass effects
H + j	dσ @ NNLO QCD (g only) and finite-quark-mass effects @ LO QCD and LO EW	dσ @ NNLO QCD (infinite-m_t limit) and finite-quark-mass effects @ NLO QCD and NLO EW
H + 2j	$\sigma_{\rm tot}$(VBF) @ NNLO(DIS) QCD dσ(VBF) @ NLO EW dσ(gg) @ NLO QCD (infinite-m_t limit) and finite-quark-mass effects @ LO QCD	dσ(VBF) @ NNLO QCD + NLO EW dσ(gg) @ NNLO QCD (infinite-m_t limit) and finite-quark-mass effects @ NLO QCD and NLO EW
H + V	dσ @ NNLO QCD dσ @ NLO EW $\sigma_{\rm tot}$(gg) @ NLO QCD (infinite-m_t limit)	with $H \to b\bar{b}$ @ same accuracy dσ(gg) @ NLO QCD with full m_t/m_b dependence
tH and $\bar{t}$H	dσ(stable top) @ LO QCD	dσ(top decays) @ NLO QCD and NLO EW
$t\bar{t}H$	dσ(stable tops) @ NLO QCD	dσ(top decays) @ NLO QCD and NLO EW
$gg \to HH$	dσ @ NLO QCD (leading m_t dependence) dσ @ NNLO QCD (infinite-m_t limit)	dσ @ NLO QCD with full m_t/m_b dependence

Table 1: Wishlist part 1 – Higgs ($V = W, Z$)

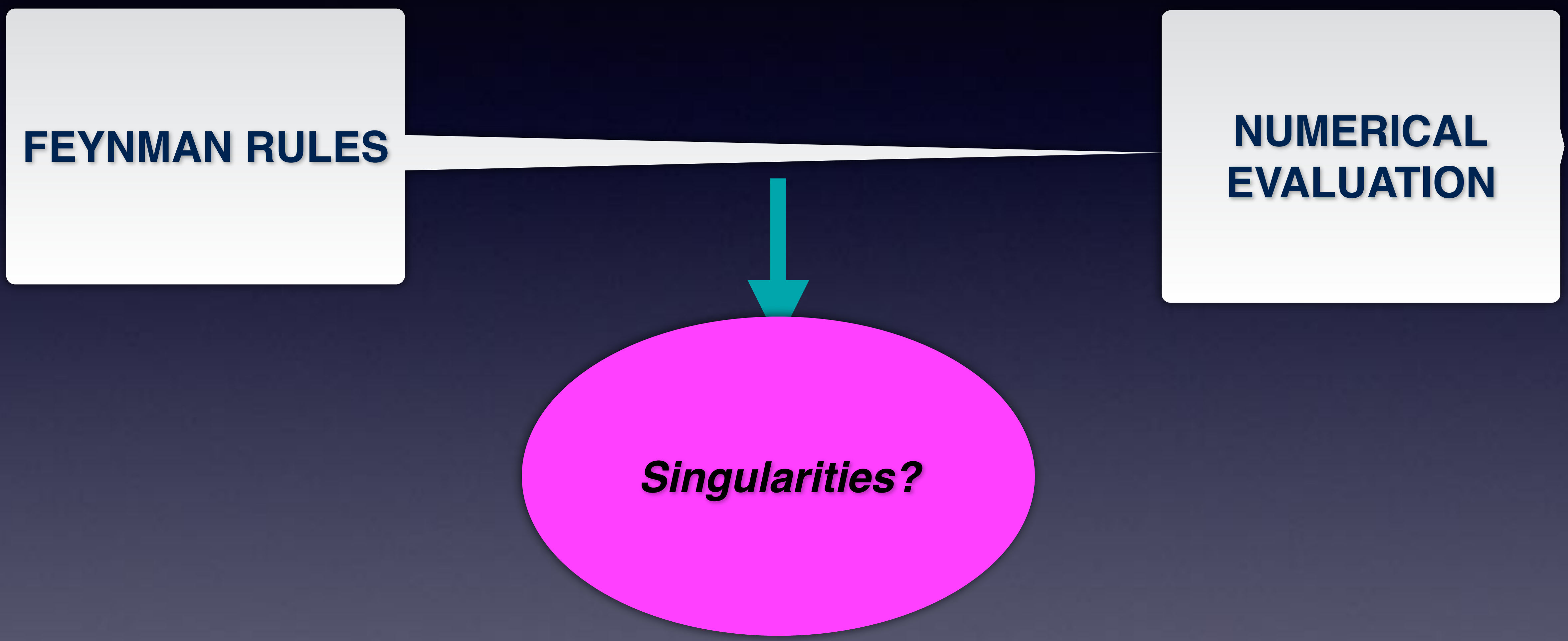
Process	State of the Art	Desired
V	dσ(lept. V decay) @ NNLO QCD dσ(lept. V decay) @ NLO EW	dσ(lept. V decay) @ NNNLO QCD and @ NNLO QCD+EW NNLO+PS
V + j(j)	dσ(lept. V decay) @ NLO QCD dσ(lept. V decay) @ NLO EW	dσ(lept. V decay) @ NNLO QCD + NLO EW
VV'	dσ(V decays) @ NLO QCD dσ(on-shell V decays) @ NLO EW	dσ(decaying off-shell V) @ NNLO QCD + NLO EW
$gg \to VV$	dσ(V decays) @ LO QCD	dσ(V decays) @ NLO QCD
$V\gamma$	dσ(V decay) @ NLO QCD dσ(PA, V decay) @ NLO EW	dσ(V decay) @ NNLO QCD + NLO EW
$Vb\bar{b}$	dσ(lept. V decay) @ NLO QCD massive b	dσ(lept. V decay) @ NNLO QCD + NLO EW, massless b
$VV'\gamma$	dσ(V decays) @ NLO QCD	dσ(V decays) @ NLO QCD + NLO EW
$VV'V''$	dσ(V decays) @ NLO QCD	dσ(V decays) @ NLO QCD + NLO EW
$VV'+j$	dσ(V decays) @ NLO QCD	dσ(V decays) @ NLO QCD + NLO EW
$VV'+jj$	dσ(V decays) @ NLO QCD	dσ(V decays) @ NLO QCD + NLO EW
$\gamma\gamma$	dσ @ NNLO QCD + NLO EW	q_T resummation at NNLL matched to NNLO

Table 3: Wishlist part 3 – Electroweak Gauge Bosons ($V = W, Z$)

How to compute a multi - loop amplitude



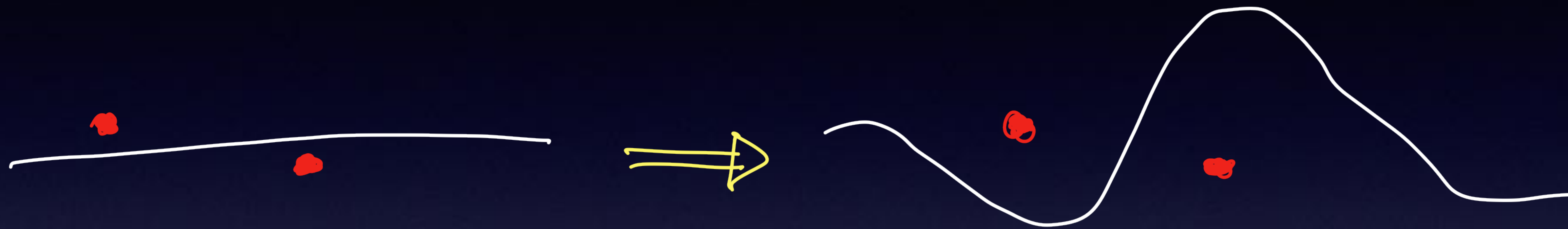
A PURELY NUMERICAL APPROACH?



Bypasses the problem of algebraically demanding reductions to master integrals

Bypasses the problem of computing the master integrals

Singularities of Feynman diagrams with loops



Some singularities can be avoided with a contour deformation

Endpoint singularities:

Pinched singularities:

Cannot be avoided with a deformation of the contour

Subtraction of non-deformable singularities

$$\int d^d\ell \; G(\ell) = \underbrace{\int d^d\ell \; G_{sing}(\ell)}_{\text{Analytic Integration}}$$

$$+ \underbrace{\int d^d\ell \left[G(\ell) - G_{sing}(\ell) \right]}_{\text{Numerical Integration}}$$

- Identify all inescapable singular limits and remove them from the integrand

- Integrate the singular contributions analytically.

- Integrate the smooth bounded remainder numerically.

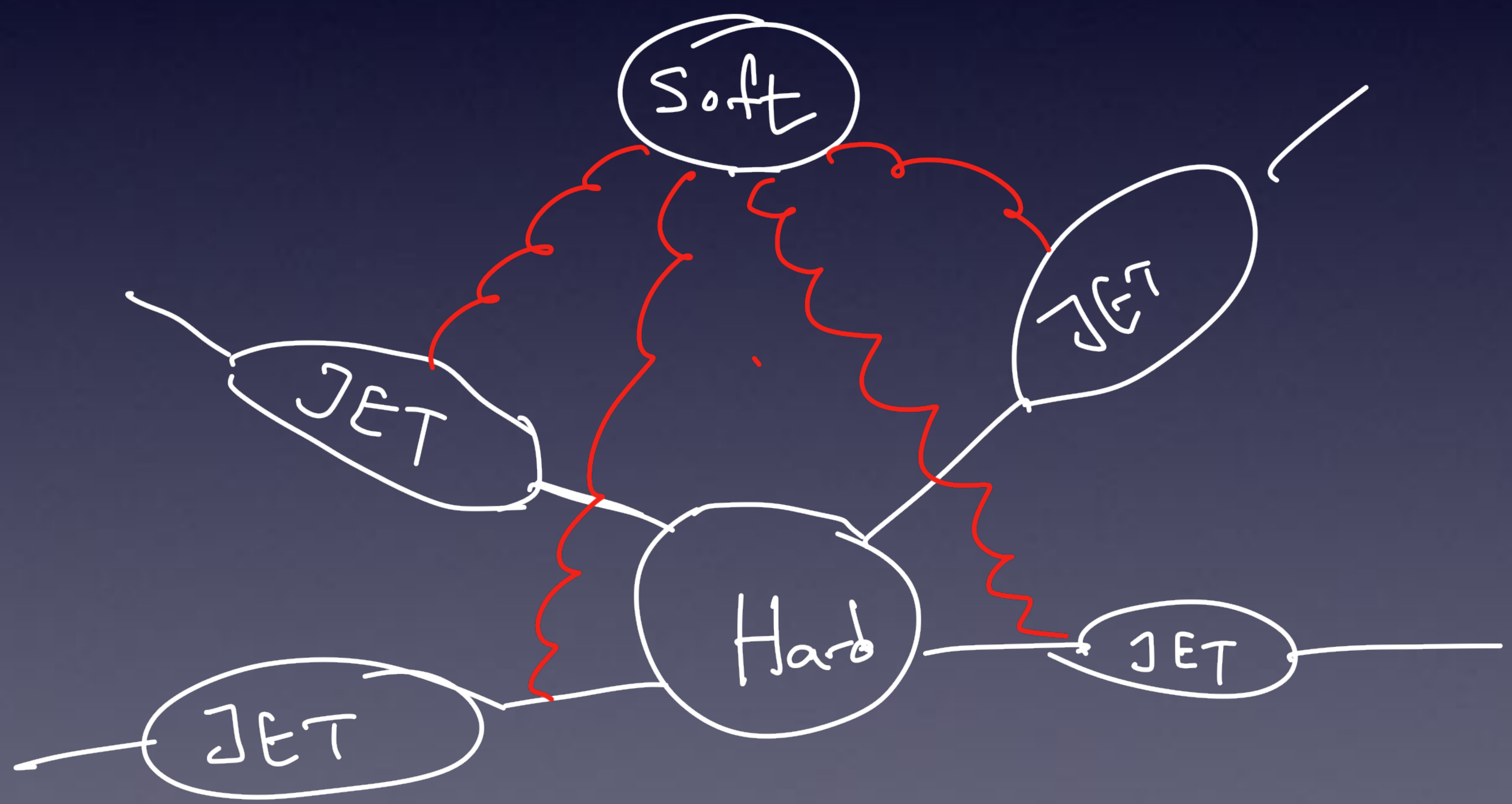
Physical picture of “inescapable” singularities

- Singularities that cannot be avoided with a contour deformation are:

  - **Ultraviolet**
  - **Soft**
  - **Collinear**

- Can be found systematically

- But they overlap!



Subtraction of singularities

Feynman parameter space

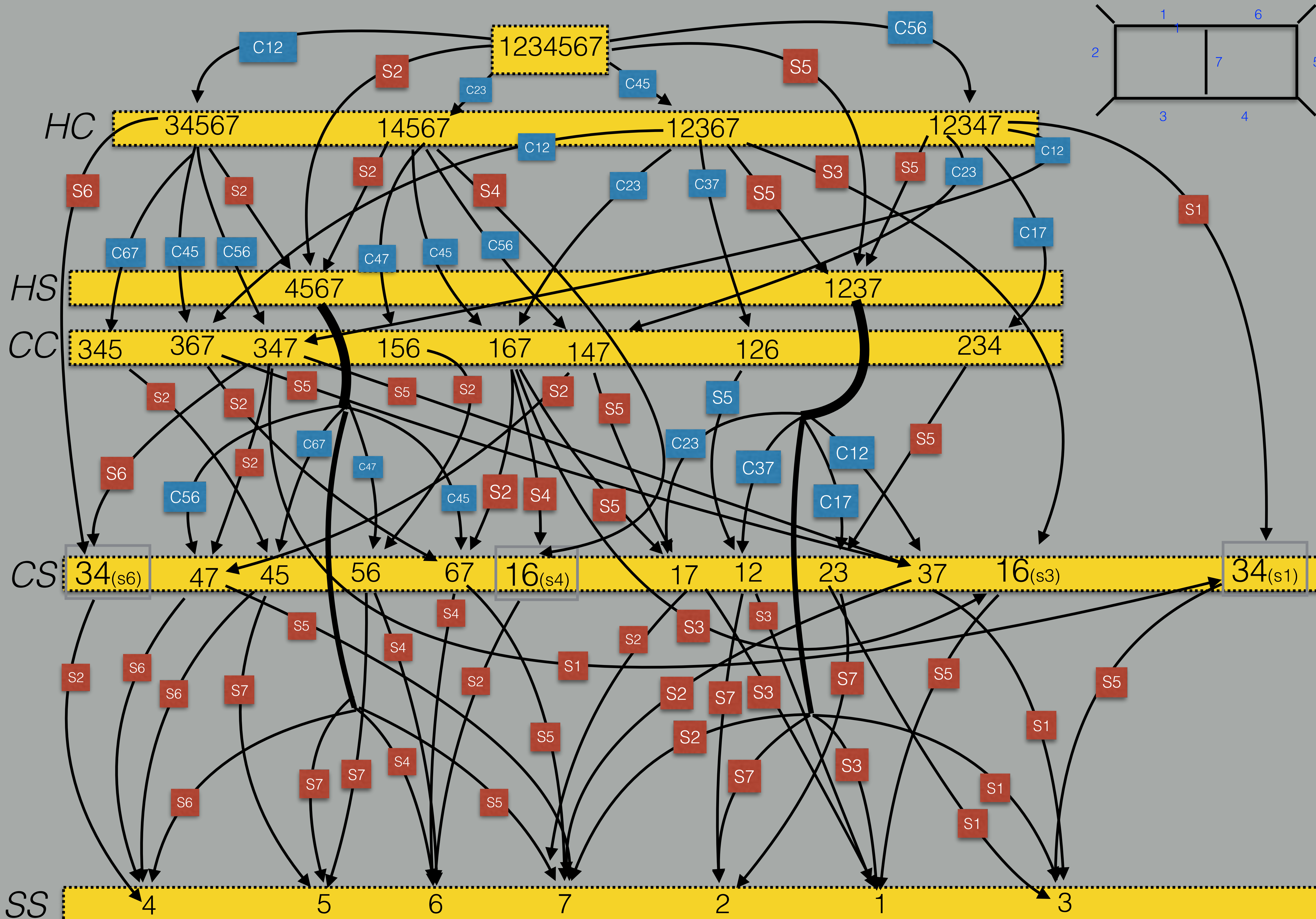
- IR/UV counterterms can be found algorithmically for arbitrary loops
- A sector-decomposition algorithm can disentangle overlapping singularities
  (*Binoth, Heinrich; …)*
- Contour deformations can be produced algorithmically for arbitrary loops
  (Nagy, Soper; …)

Momentum space

- IR/UV counterterms have been found only at one-loop
  (Nagy, Soper)
- Contour deformations are known at one-loop and beyond for processes with massless propagators. *(Nagy, Soper; Becker, Weinzierl), But not efficient!*
- A promising field of research with space for new ideas

REMOVING THE SINGULARITIES OF TWO-LOOP AMPLITUDES

- Loop integrals become divergent when internal particles in any of the two loops become collinear to external particles, or they are soft.
- However, the web of singularities at two-loops is complicated.
- Singularities are many and highly entangled



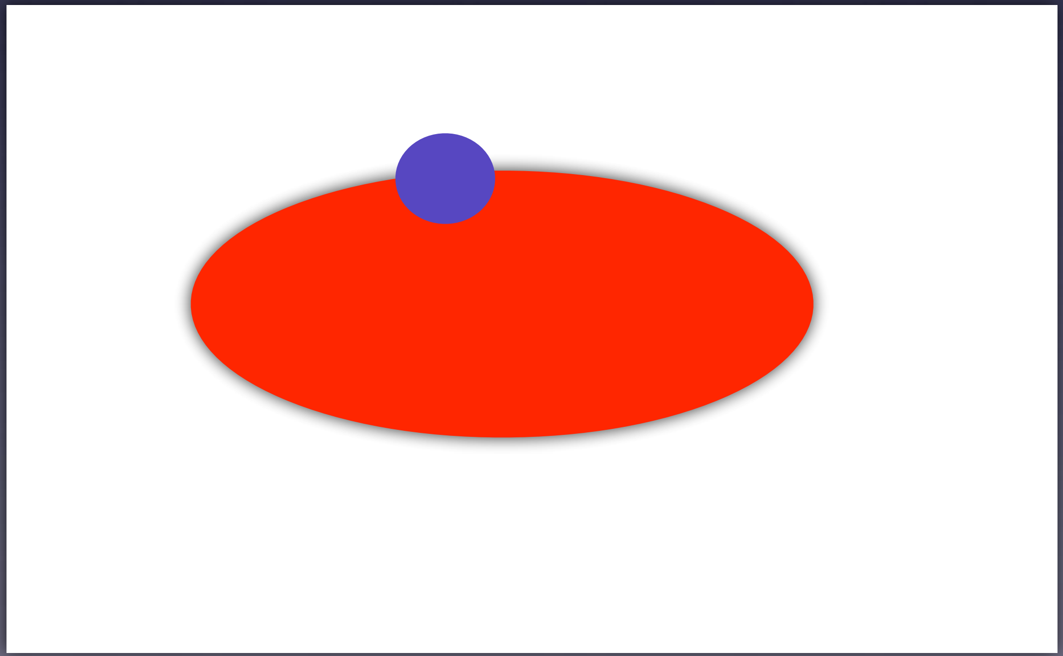
1234567
HC
34567
14567
12367
12347
HS
4567
1237
CC
345
367
347
156
167
147
126
234
CS
34(s6)
47
45
56
67
16(s4)
17
12
23
37
16(s3)
34(s1)
SS
4
5
6
7
2
1
3

Nested subtractions

Ozan Erdogan, George Sterman

- Order the singular regions by their “volume”
- Subtract an approximation of the integrand in the smallest volume
- Then, proceed to the next volume and repeat until there are no more singularities to remove.

$$R^{(n)}\,\gamma^{(n)} = \gamma^{(n)} + \sum_{N\in\mathcal{N}[\gamma^{(n)}]} \prod_{\rho\in N} (-t_\rho)\,\gamma^{(n)}\,,$$



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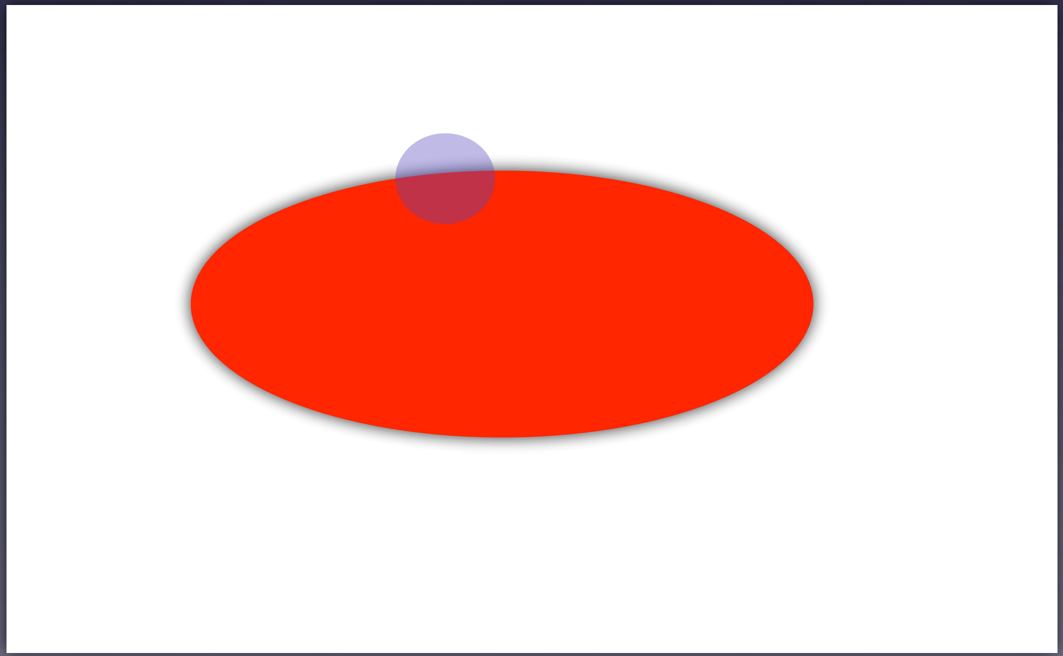
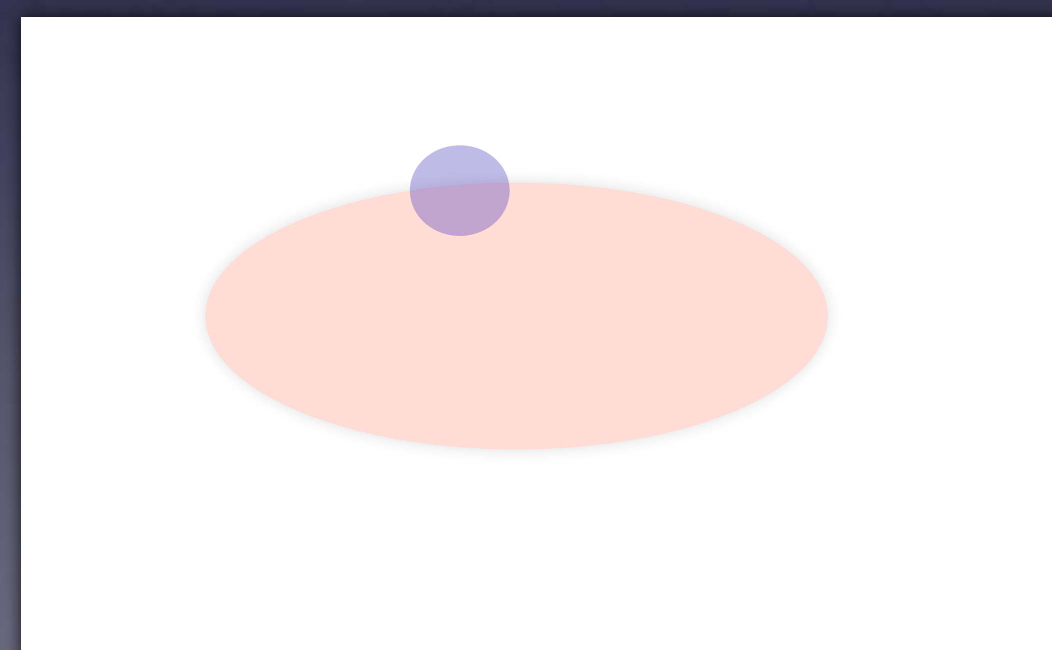


Nested subtractions

Ozan Erdogan, George Sterman

- Order the singular regions by their "volume"
- Subtract an approximation of the integrand in the smallest volume
- Then, proceed to the next volume and repeat until there are no more singularities to remove.
- Method should work at all orders in perturbation theory.
- This structure gives rise to factorisation into Jet, Soft and Hard functions for scattering amplitudes.

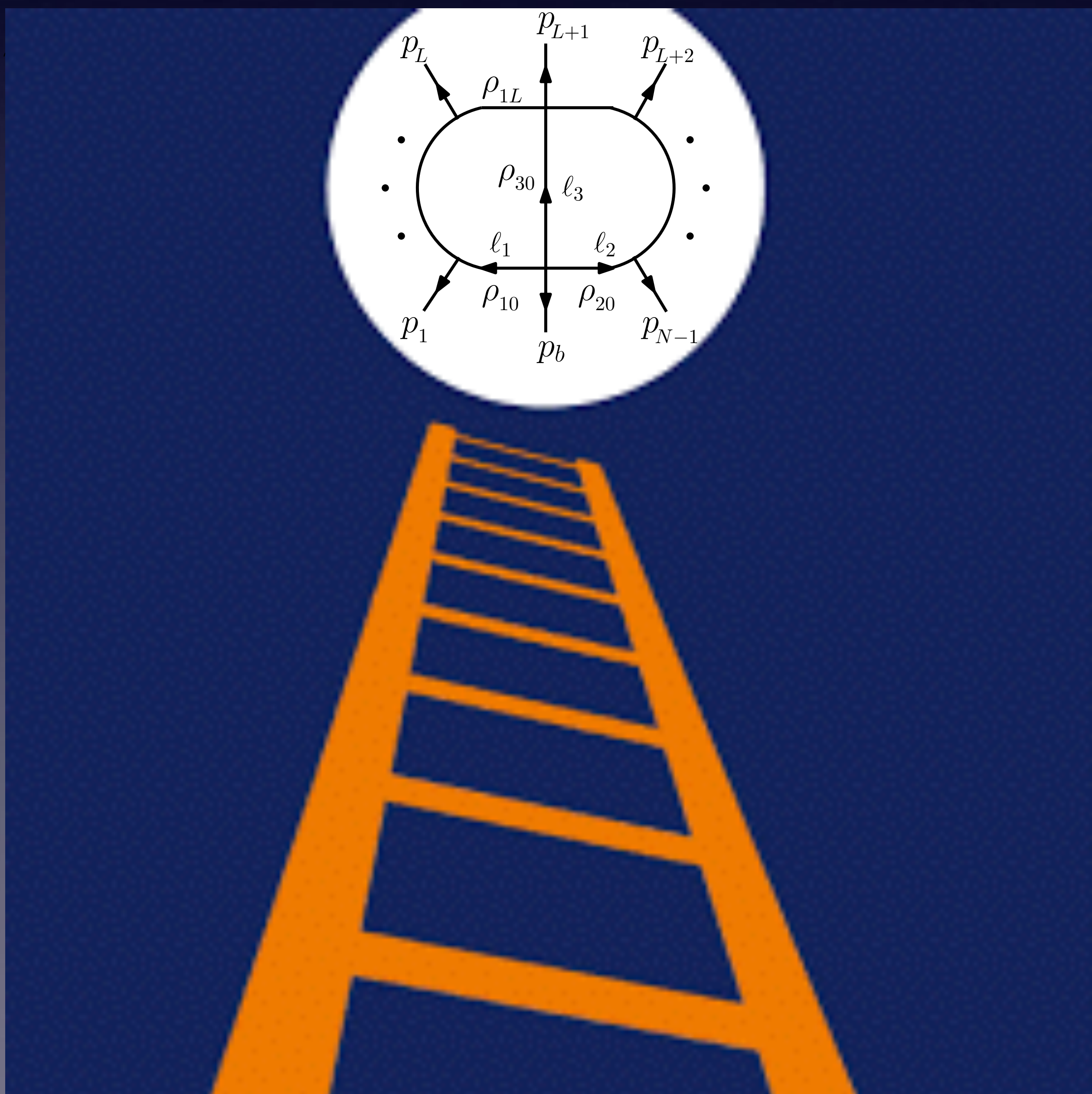
$$R^{(n)}\,\gamma^{(n)} = \gamma^{(n)} + \sum_{N\in\mathcal{N}[\gamma^{(n)}]} \prod_{\rho\in N} (-t_\rho)\,\gamma^{(n)}\,,$$



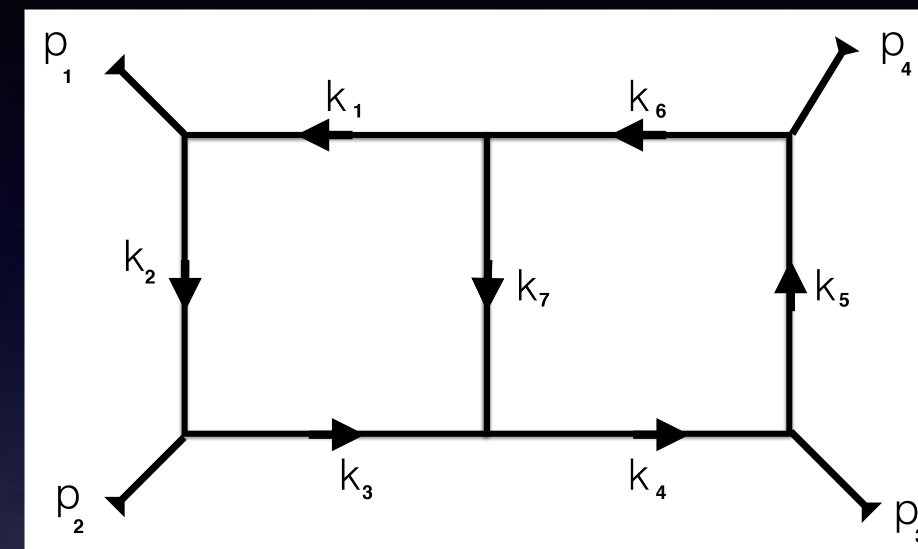
Nested subtractions at 2-loops

CA, George Sterman

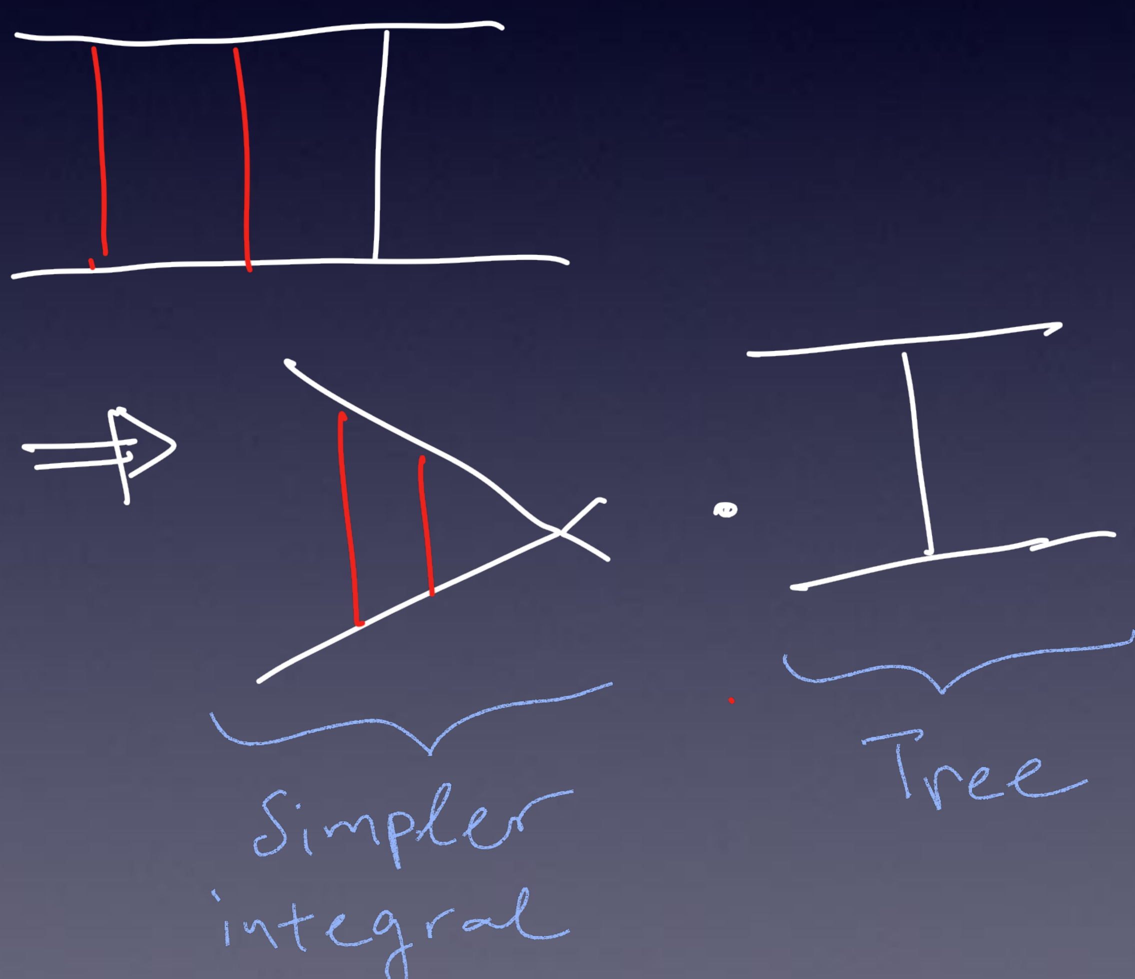
- Order of subtractions:
  - double-soft
  - soft-collinear
  - double-collinear
  - single-soft
  - single-collinear
- Approximations in singular regions do not need to be strict limits!
- Good approximations should not introduce ultraviolet divergences
- Good approximations should be easy to integrate exactly.



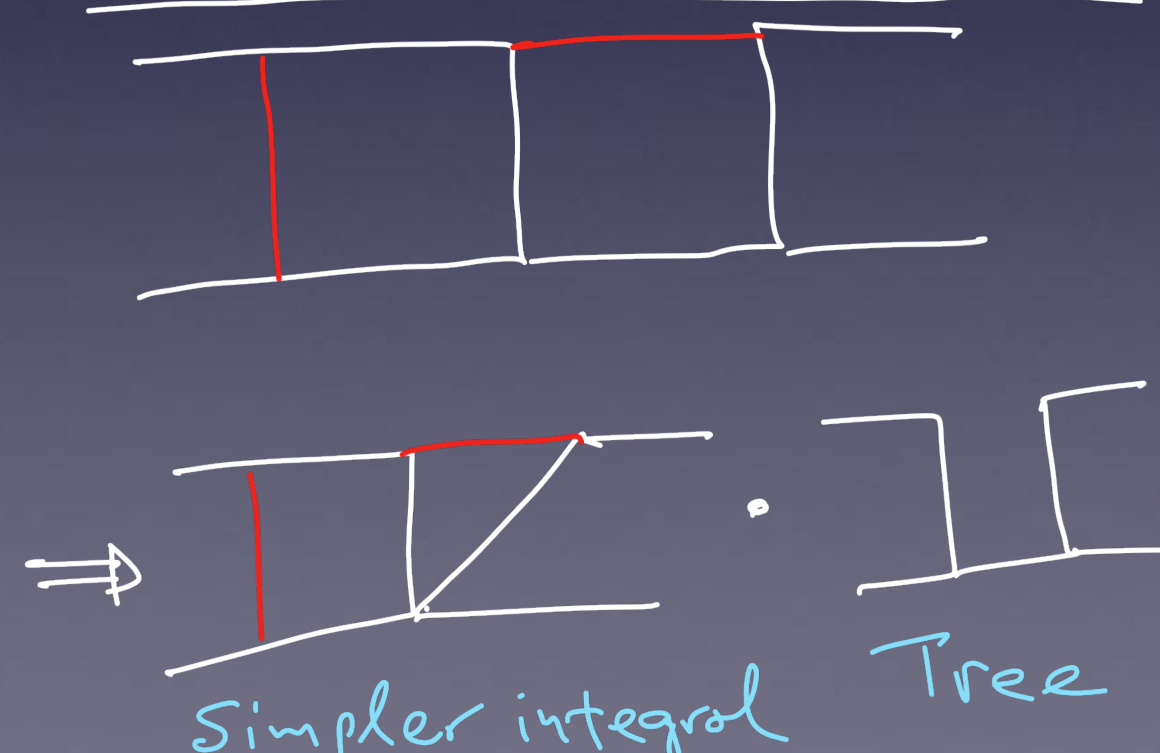
An example: 2-loop planar box



• Double soft: $k_2 \sim k_7 \to 0$

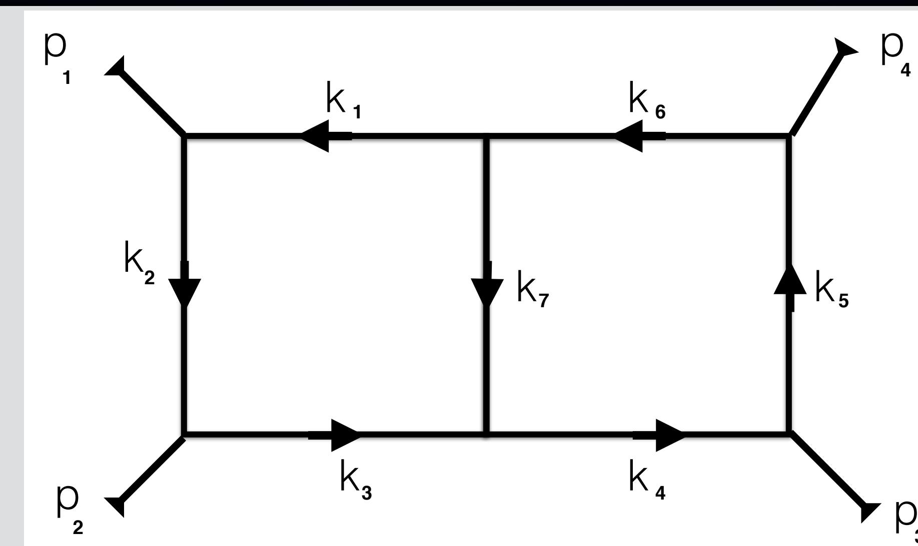


• Double soft: $k_2 \sim k_6 \to 0$



Example: planar double-box

CA, George Sterman



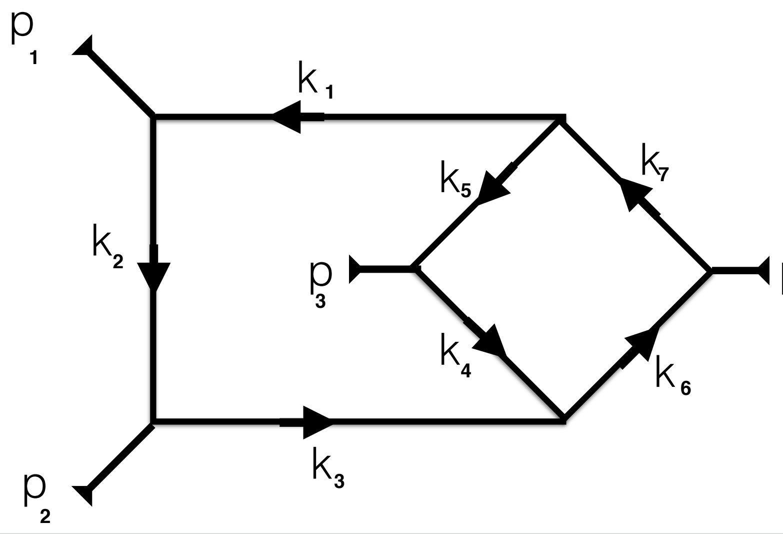
$$F_{Pbox} = \frac{F^{(2)}_{Pbox}}{P_1P_2P_3P_4P_5P_6P_7} + F^{(1s)}_{Pbox} + F^{(1c)}_{Pbox},$$

$$F^{(2)}_{Pbox} = 1 \overbrace{- \frac{P_{257}}{t} - \frac{P_{1346}}{s} + \frac{P_1P_6 + P_3P_4}{s^2}}^{\text{double soft}} + \underbrace{\frac{P_{13}P_5 + P_{46}P_2}{st}}_{\text{two collinear pairs}} \underbrace{+ \frac{s+t}{s^2t}(P_1P_4 + P_3P_6)}_{\text{triple-collinear}}$$

$$F^{(1c)}_{Pbox} = -\frac{\frac{\mu}{\mu^2-P_1}}{P_1P_2s(1-x_1)}\left\{\left[\frac{F^{(2)}_{Pbox}}{P_4P_5P_6P_7}\right]_{k_1=-x_1p_1} - \left[\frac{F^{(2)}_{Pbox}}{P_4P_5P_6P_7}\right]_{k_2=0}\right\}$$
$$\underbrace{\begin{aligned}&-\frac{\frac{\mu^2}{\mu^2-P_3}}{P_3P_2s(1-x_2)}\left\{\left[\frac{F^{(2)}_{Pbox}}{P_4P_5P_6P_7}\right]_{k_3=-x_2p_2} - \left[\frac{F^{(2)}_{Pbox}}{P_4P_5P_6P_7}\right]_{k_2=0}\right\}\\ &-\frac{\frac{\mu^2}{\mu^2-P_6}}{P_6P_5s(1-x_4)}\left\{\left[\frac{F^{(2)}_{Pbox}}{P_1P_2P_3P_7}\right]_{k_6=x_4p_4} - \left[\frac{F^{(2)}_{Pbox}}{P_1P_2P_3P_7}\right]_{k_5=0}\right\}\\ &-\frac{\frac{\mu^2}{\mu^2-P_4}}{P_4P_5s(1-x_3)}\left\{\left[\frac{F^{(2)}_{Pbox}}{P_1P_2P_3P_7}\right]_{k_4=-x_3p_3} - \left[\frac{F^{(2)}_{Pbox}}{P_1P_2P_3P_7}\right]_{k_5=0}\right\}\end{aligned}}_{\text{single collinear}}$$

$$F^{(1s)}_{Pbox} = \underbrace{-\frac{1}{P_1P_2P_3}\left[\frac{F^{(2)}_{Pbox}}{P_4P_5P_6P_7}\right]_{k_2=0} - \frac{1}{P_4P_5P_6}\left[\frac{F^{(2)}_{Pbox}}{P_1P_2P_3P_7}\right]_{k_5=0}}_{\text{single-soft}}$$

Example:cross-box



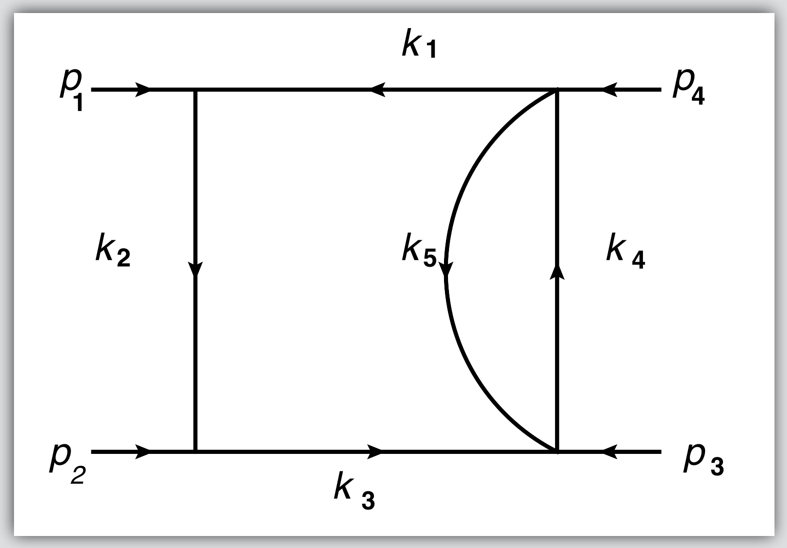
$$F_{Xbox} = \frac{F^{(2)}_{Xbox}}{P_1P_2P_3P_4P_5P_6P_7} + F^{(1s)}_{Xbox} + F^{(1c)}_{Xbox},$$

$$\begin{aligned} F^{(2)}_{Xbox} = {} & \left(1 - \frac{P_{13}}{s}\right)^2 + \frac{P_2}{tu}\left(P_2 + s - P_{13}\right) \\ & - \left(1 - \frac{P_1}{s}\right)\left(\frac{P_5}{t} + \frac{P_7}{u}\right) - \left(1 - \frac{P_3}{s}\right)\left(\frac{P_4}{u} + \frac{P_6}{t}\right) + \frac{P_2P_{4567}}{tu} \\ & - \frac{P_3}{s}\left(\frac{P_7}{t} + \frac{P_5}{u}\right) - \frac{P_1}{s}\left(\frac{P_6}{u} + \frac{P_4}{t}\right) + \frac{(t-u)^2}{s^2}\frac{P_1P_3}{tu}. \end{aligned}$$

$$\begin{aligned} F^{(1c)}_{Xbox} = {} & -\frac{\frac{\mu^2}{\mu^2 - P_1}}{P_1P_2s(1-x_1)}\left\{\left[\frac{F^{(2)}_{Xbox}}{P_4P_5P_6P_7}\right]_{k_1=-x_1p_1} - \left[\frac{F^{(2)}_{Xbox}}{P_4P_5P_6P_7}\right]_{k_2=0}\right\} \\ & -\frac{\frac{\mu^2}{\mu^2 - P_1}}{P_2P_3s(1-x_3)}\left\{\left[\frac{F^{(2)}_{Xbox}}{P_4P_5P_6P_7}\right]_{k_3=-x_2p_2} - \left[\frac{F^{(2)}_{Xbox}}{P_4P_5P_6P_7}\right]_{k_2=0}\right\} \\ & -\frac{\frac{\mu^2}{\mu^2 - P_4}}{P_4P_5}\left[\frac{F^{(2)}_{Xbox}}{P_1P_2P_3P_6P_7}\right]_{k_5=-x_3p_3} - \frac{\frac{\mu^2}{\mu^2 - P_6}}{P_6P_7}\left[\frac{F^{(2)}_{Xbox}}{P_1P_2P_3P_4P_5}\right]_{k_5=-x_4p_4} \end{aligned}$$

$$F^{(1s)}_{Xbox} = -\frac{1}{P_1P_2P_3}\left[\frac{F^{(2)}_{Xbox}}{P_4P_5P_6P_7}\right]_{k_2=0}$$

Example:bubble-box



$$\mathrm{B_{box}}|_{\mathrm{fin}} = \int \frac{d^dk_2}{i\pi^{\frac{d}{2}}}\frac{d^dk_5}{i\pi^{\frac{d}{2}}}\left\{\frac{1}{A_1A_2A_3A_4A_5} - \frac{1}{A_1A_2A_3}\left[\frac{1}{A_4A_5}\right]_{k_2=0}\right.$$

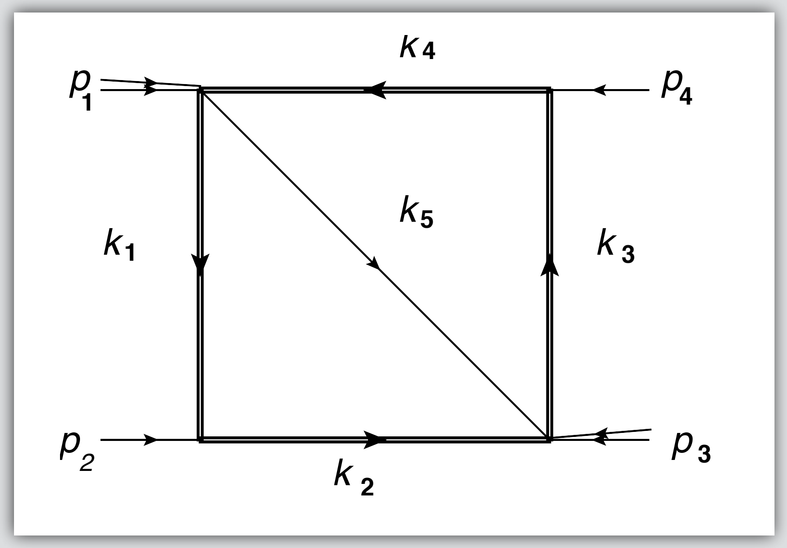
$$-\frac{\frac{-\mu^2}{A_1-\mu^2}}{A_1A_2s(1-x_1)}\left[\left.\frac{1}{A_4A_5}\right|_{k_1=-x_1p_1} - \left.\frac{1}{A_4A_5}\right|_{k_1=-p_1}\right]$$

$$\left.= -\frac{\frac{-\mu^2}{A_3-\mu^2}}{A_2A_3s(1-x_2)}\left[\left.\frac{1}{A_4A_5}\right|_{k_3=x_2p_2} - \left.\frac{1}{A_4A_5}\right|_{k_3=p_2}\right]\right\}$$

Physical regulators

- The subtraction counterterms are local.
- They can be invented with dimensional regularisation in mind, but they can also be adapted to other regularisation schemes for the IR divergences.
- Small quark masses act as physical regulators.
- In such case, the infrared counterterms integrate to yield the logarithmically enhanced terms of the integral.

Large logs from small masses easily determined.



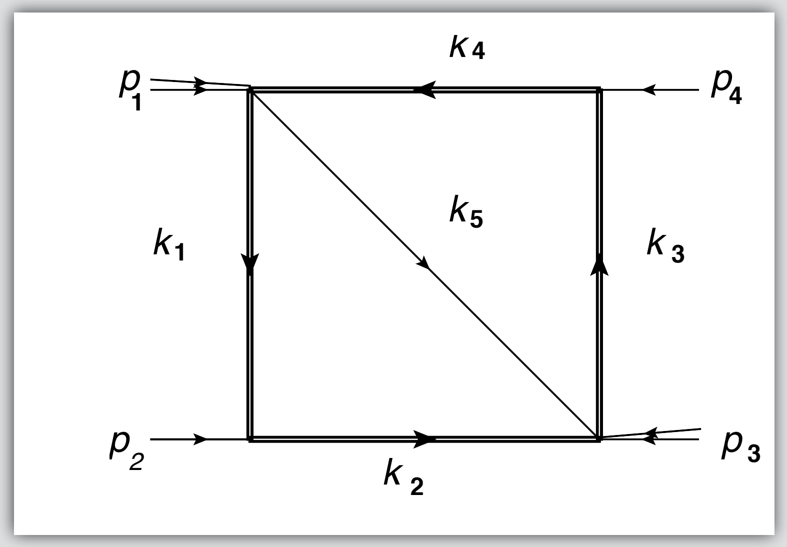
$$
\begin{aligned}
\mathrm{D}_{\mathrm{box}} = & \left[\frac{1}{(A_1-m^2)\,(A_2-m^2)} - \frac{1}{(A_1-\mu^2)\,(A_2-\mu^2)}\right]\left[\frac{1}{A_3A_4A_5}\right]_{k_1=-x_2p_2} \\
& + \left[\frac{1}{(A_3-m^2)\,(A_4-m^2)} - \frac{1}{(A_3-\mu^2)\,(A_4-\mu^2)}\right]\left[\frac{1}{A_1A_2A_5}\right]_{k_4=x_4p_4} \\
& - \left[\frac{1}{(A_1-m^2)\,(A_2-m^2)\,(A_3-m^2)\,(A_4-m^2)}\right. \\
& \qquad \times \left.\left[-\frac{1}{(A_1-\mu^2)\,(A_2-\mu^2)\,(A_3-\mu^2)\,(A_4-\mu^2)}\right]\left[\frac{1}{A_5}\right]\right]_{\substack{k_4=x_4p_4,\\ k_1=-x_2p_2}} \\
& \qquad + \mathrm{D}_{\mathrm{box}}|_{\mathrm{fin}} + \mathcal{O}(m^2)
\end{aligned}
$$

Large logs from small masses easily determined.

$$
\begin{aligned}
(m_1^2+m_3^2-s-t)\ \mathrm{D}_{\mathrm{box}} = {} & 1/3\ \ln\left(-\frac{(m_1{}^2-s)\,(m_1{}^2-t)}{m_1{}^2m_3{}^2-st}\right)\ln^3\left(-\frac{m_1{}^2}{m^2}\right) \\
& +\mathrm{Li}_2\left(\frac{(m_1{}^2+m_3{}^2-s-t)\,m_1{}^2}{m_1{}^2m_3{}^2-st}\right)\ln^2\left(-\frac{m_1{}^2}{m^2}\right) - 2\,\mathrm{Li}_3\left(\frac{(m_1{}^2+m_3{}^2-s-t)\,m_1{}^2}{m_1{}^2m_3{}^2-st}\right)\ln\left(-\frac{m_1{}^2}{m^2}\right) \\
& -1/3\ \ln^3\left(-\frac{s}{m^2}\right)\ln\left(\frac{(m_3{}^2-s)\,(m_1{}^2-s)}{m_1{}^2m_3{}^2-st}\right) - 1/3\ \ln^3\left(-\frac{t}{m^2}\right)\ln\left(\frac{(m_3{}^2-t)\,(m_1{}^2-t)}{m_1{}^2m_3{}^2-st}\right) \\
& +1/3\ \ln^3\left(-\frac{m_3{}^2}{m^2}\right)\ln\left(-\frac{(m_3{}^2-t)\,(m_3{}^2-s)}{m_1{}^2m_3{}^2-st}\right) - \mathrm{Li}_2\left(\frac{s\,(m_1{}^2+m_3{}^2-s-t)}{m_1{}^2m_3{}^2-st}\right)\ln^2\left(-\frac{s}{m^2}\right) \\
& +2\,\mathrm{Li}_3\left(\frac{s\,(m_1{}^2+m_3{}^2-s-t)}{m_1{}^2m_3{}^2-st}\right)\ln\left(-\frac{s}{m^2}\right) - \mathrm{Li}_2\left(\frac{t\,(m_1{}^2+m_3{}^2-s-t)}{m_1{}^2m_3{}^2-st}\right)\ln^2\left(-\frac{t}{m^2}\right) \\
& +2\,\mathrm{Li}_3\left(\frac{t\,(m_1{}^2+m_3{}^2-s-t)}{m_1{}^2m_3{}^2-st}\right)\ln\left(-\frac{t}{m^2}\right) + \mathrm{Li}_2\left(\frac{m_3{}^2\,(m_1{}^2+m_3{}^2-s-t)}{m_1{}^2m_3{}^2-st}\right)\ln^2\left(-\frac{m_3{}^2}{m^2}\right) \\
& -2\,\mathrm{Li}_3\left(\frac{m_3{}^2\,(m_1{}^2+m_3{}^2-s-t)}{m_1{}^2m_3{}^2-st}\right)\ln\left(-\frac{m_3{}^2}{m^2}\right) + 2\,\mathrm{Li}_4\left(\frac{(m_1{}^2+m_3{}^2-s-t)\,m_1{}^2}{m_1{}^2m_3{}^2-st}\right) \\
& -2\,\mathrm{Li}_4\left(\frac{s\,(m_1{}^2+m_3{}^2-s-t)}{m_1{}^2m_3{}^2-st}\right) - 2\,\mathrm{Li}_4\left(\frac{t\,(m_1{}^2+m_3{}^2-s-t)}{m_1{}^2m_3{}^2-st}\right) \\
& +2\,\mathrm{Li}_4\left(\frac{m_3{}^2\,(m_1{}^2+m_3{}^2-s-t)}{m_1{}^2m_3{}^2-st}\right)
\end{aligned}
$$

Large log

$$+\,O(m^2)$$



Summary/Prospects

- I presented a method for the removal of singularities in multi-loop integrals.
- We are testing the method on complicated two-loop examples.
- It paves the way for a direct numerical evaluation of two-loop amplitudes in momentum space.
- It can also be used for extracting the asymptotic behaviour of Feynman diagrams in a small mass limit or other kinematic limits.
- Work in progress…