Pseudo-Riemannian Structure of The Noncommutative Standard Model

Arkadiusz Bochniak[1]

[1] Institute of Physics of the Jagiellonian University

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Reference

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What is a geometry ?

Connes' reconstruction theorem

The whole metric and spin structure of a compact, orientable, Riemannian, spin^c manifold can be encoded in the $*$-algebra $C^\infty(M)$ of smooth functions, Hilbert space $L^2(S)$ of square-integrable spinors and the Dirac operator $\not{D}_M = i\gamma^\mu (\partial_\mu + \omega_\mu)$ together with the γ_5 grading and the charge conjugation operator.

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Spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma, J)$

$\mathcal{A}$ is a $*$-algebra represented on Hilbert space $\mathcal{H}$, $\gamma = \gamma^\dagger$, $\gamma^2 = 1$ is a $\mathbb{Z}/2\mathbb{Z}$-grading commuting with $\mathcal{A}$, J is an antilinear isometry s.th. $[Ja^*J^{-1}, b] = 0$ for all $a, b \in \mathcal{A}$.
$\mathcal{D}$ is essentially self-adjoint operator with compact resolvent and s.th. $[\mathcal{D}, a]$ is bounded for all $a \in \mathrm{Dom}(\mathcal{D})$ and $\mathcal{D}\gamma = -\gamma\mathcal{D}$.

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Moreover $\mathcal{D}J = \epsilon J\mathcal{D}$, $J^2 = \epsilon'\mathrm{id}$ and $J\gamma = \epsilon''\gamma J$ with $\epsilon, \epsilon', \epsilon'' = \pm 1$ defining KO-dimension.

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Moreover $\mathcal{D}J = \epsilon J\mathcal{D}$, $J^2 = \epsilon'\mathrm{id}$ and $J\gamma = \epsilon''\gamma J$ with $\epsilon, \epsilon', \epsilon'' = \pm 1$ defining KO-dimension.
There are additional compatibility conditions for $\mathcal{D}$ and for γ.

Almost-commutative geometry for the Standard Model

$$\left(C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})), L^2(S) \otimes H_f, \not{D}_M \otimes 1 + \gamma_5 \otimes D_f, \gamma_5 \otimes \gamma_f, J_M \otimes J_f\right)$$

$$H_f = H_L \oplus H_R \oplus H_L^c \oplus H_R^c$$

$$D_f \in M_{96}(\mathbb{C})$$

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Expansion of the Euclidean spectral action reproduces the effective action for the SM and allows for the expression of bosonic parameters by fermionic one.

Questions

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3. What does it imply for the SM ?

Finite pseudo-Riemannian spectral triple of signature (p, q)

$$(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma, J, \beta)$$

1. $\mathcal{A}$ is a $*$-algebra represented on an Hilbert space $\mathcal{H}$
2. For $p+q$ even $\gamma^* = \gamma,\ \gamma^2 = 1$ is a $\mathbb{Z}/2\mathbb{Z}$-grading commuting with $\mathcal{A}$
3. J is antilinear isometry with $[Ja^*J^{-1}, b] = 0$
4. $\beta = \beta^\dagger, \beta^2 = 1$ commuting with $\mathcal{A}$
5. $\mathcal{D}^\dagger = (-1)^p \beta \mathcal{D} \beta$
6. $[\mathcal{D}, a]$ is bounded
7. $\mathcal{D}\gamma = -\gamma\mathcal{D}$
8. $\mathcal{D}J = \epsilon J\mathcal{D},\ J^2 = \epsilon' \mathrm{id},\ J\gamma = \epsilon'' \gamma J$

$p-q$ mod 8	0	1	2	3	4	5	6	7
ϵ	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$
ϵ'	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
ϵ''	$+$		$-$		$+$		$-$	

9. $\beta\gamma = (-1)^p \gamma\beta$, $\beta J = (-1)^{\frac{p(p-1)}{2}} \epsilon^p J\beta$
10. $\left[JaJ^{-1}, [\mathcal{D}, b]\right] = 0$
11. orientability : there exist $\mathcal{A} \ni a^i, a_0^i, ..., a_n^i$, $i = 1, ..., k$ s.th.

$$\sum_{i=1}^{k} Ja^i J^{-1} a_0^i [\mathcal{D}, a_1^i]...[\mathcal{D}, a_n^k] = \begin{cases} \gamma, & n \text{ even} \\ 1, & n \text{ odd} \end{cases}$$

12. time-orientation : there exist $\mathcal{A} \ni b^i, b_0^i, ..., b_p^i$, $i = 1, ..., k'$ s.th.

$$\beta = \sum_{i=1}^{k'} Jb^i J^{-1} b_0^i [\mathcal{D}, b_1^i]...[\mathcal{D}, b_p^k].$$

Clifford algebra : $\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}1$

- $\gamma = i^{\frac{p-q}{2}}\gamma_1 \ldots \gamma_{p+q}$
- there exists unitary B s.th. $B\gamma_i = \epsilon\gamma_i^* B$ and $BB^* = \epsilon'$. Define $J\psi := B\psi^*$.
- $\mathcal{D} = -\sum_j \eta_{jj}\gamma_j\partial_j$
- $B\gamma = \epsilon''\gamma B$
- $\beta = i^{\frac{1}{2}p(p-1)}\gamma_1 \ldots \gamma_p$
- $\beta\mathcal{D}\beta = (-1)^p\mathcal{D}^\dagger$

Finite triples

$$\mathcal{A} = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}), \qquad \mathcal{H} = \bigoplus_{i,j} \mathcal{H}_{ij}$$

$$\mathcal{H}_{ij} = P_i \mathcal{H} P_j \cong \mathbb{C}^{n_i} \otimes \mathbb{C}^{r_{ij}} \otimes \mathbb{C}^{n_j}$$

- $\gamma_{ij} = \gamma|_{\mathcal{H}_{ij}} = 1_{n_i} \otimes \Gamma_{ij} \otimes 1_{n_j}$
- $q_{ij} := r_{ij}\gamma_{ij}$ is symmetric for KO-dimension 0 and 4 and antisymmetric for KO-dimension 2 and 6
- $\mathcal{D}_{ij,kl} := P_i J P_j J^{-1} \mathcal{D} P_k J P_l J^{-1}$
- there exists $\xi = \sum_{i \neq j} P_i d P_j$ s.th. $\mathcal{D} = \xi + J\xi J^{-1} + \delta$
- $\mathcal{D}_{ji,lk} = \epsilon J \mathcal{D}_{ji,lk} J^{-1}$
- for odd p and some $\gamma_{ij} = \pm 1,\ r_{ij} > 0$ there is no pseudo-Riemannian structure

$$\mathcal{D}_+ = \frac{1}{2}(\mathcal{D} + \mathcal{D}^\dagger), \quad \mathcal{D}_- = \frac{i}{2}(\mathcal{D} - \mathcal{D}^\dagger)$$

We get two Riemannian spectral triples $(\mathcal{A}, \pi, \mathcal{H}, \mathcal{D}_\pm, J, \gamma)$, that differ by KO-dimensions, with additional selfadjoint grading β s.th.

$$\beta\mathcal{D}_\pm = \pm(-1)^p \mathcal{D}_\pm \beta,$$

$$\beta\gamma = (-1)^p \gamma\beta, \quad \beta J = (-1)^{\frac{1}{2}p(p-1)} \epsilon^p J\beta.$$

Riemannian from pseudo-Riemannian

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$$\mathcal{D}_E = \mathcal{D}_+ + \mathcal{D}_-$$

$$J_E = J\beta, \quad \text{or} \quad J_E = J\beta\gamma$$

$(\mathcal{A}, \pi, \mathcal{H}, \mathcal{D}_E, J_E, \gamma)$ is a Riemannian spectral triple of signature $(0, -(p+q))$.

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Define two unitary operators

$$\pi(U)\,|n, m, \pm\rangle = |n+1, m, \pm\rangle\,, \qquad \pi(V)\,|n, m, \pm\rangle = \lambda^{-n}\,|n, m+1, \pm\rangle$$

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$$\gamma\,|n, m, \pm\rangle = \pm\,|n, m, \pm\rangle\,, \quad J\,|n, m, \pm\rangle = \mp\lambda^{mn}\,|-n, -m, \pm\rangle$$

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$$\beta\,|n, m, \pm\rangle = \pm i\,|n, m, \mp\rangle\,, \quad \mathcal{D}\,|n, m, \pm\rangle = (n \pm m)\,|n, m, \mp\rangle\,.$$

Example : Noncommutative Torus

It is a pseudo-Riemannian spectral triple of signature $(1,1)$ that is time-orientable, i.e. $\beta = iV^\dagger[\mathcal{D}, V]$.

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These give the data of the usual equivariant Riemannian spectral triple over the noncommutative torus.

Example : Functions over 2-point space

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$$\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \circ *.$$

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Now, we can easily identify a nontrivial additional symmetry ($\mathbb{Z}/2\mathbb{Z}$-grading) β and construct a Dirac operator $\mathcal{D}_+$, which is real, satisfies first-order condition and commutes with β:

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \mathcal{D}_+ = \begin{pmatrix} 0 & d & d^* & 0 \\ d^* & 0 & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

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Both these triples could be seen as Riemannian parts of a pseudo-Riemannian spectral triple with signature $(4,4)$ or $(0,0)$.
The full Dirac operator $\mathcal{D}$:

$$\mathcal{D} = \begin{pmatrix} 0 & d & d^* & 0 \\ d^* & 0 & 0 & c \\ d & 0 & 0 & c^* \\ 0 & -c^* & -c & 0 \end{pmatrix},$$

where c, d are arbitrary complex numbers.

$$A_f = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \qquad H_f = (H_l \oplus H_q) \oplus (H_{\bar{l}} \oplus H_{\bar{q}})$$

$$H_l = \langle \{\nu_R, e_R, (\nu_L, e_L)\} \rangle$$

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$$\pi(\lambda, h, m) = \bar{\lambda} \text{ on } H_{\bar{l}} \text{ and } 1_4 \otimes m \text{ on } H_{\bar{q}}$$

$$D_f = \begin{pmatrix} S & T^\dagger \\ T & \bar{S} \end{pmatrix}, \qquad S = \begin{pmatrix} S_l & \\ & S_q \otimes 1_3 \end{pmatrix}$$

$$T\nu_R = Y_R \bar{\nu}_R$$

$$S_l = \left[\begin{array}{c|c|c|c} & & Y_\nu^\dagger & \\ \hline & & & Y_e^\dagger \\ \hline Y_\nu & & & \\ \hline & Y_e & & \end{array}\right], \quad S_q = \left[\begin{array}{c|c|c|c} & & Y_u^\dagger & \\ \hline & & & Y_d^\dagger \\ \hline Y_u & & & \\ \hline & Y_d & & \end{array}\right]$$

γ_f - chirality grading
J_f - real structure

The Standard Model

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- The existence of right neutrinos implies nonorientability of the geometry
- It is well known that the above Dirac operator is not unique within the model-building scheme of noncommutative geometry. Even the introduction of more constraints, like the second-order condition or Hodge-duality does not allow to exclude the terms, which would introduce the couplings between lepton and quarks and lead to the leptoquark fields

There exists 0-cycle

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$(A_f, H_f, D_f, \gamma_f, J_f, \beta)$ could be seen as a Riemannian restriction of a real even pseudo-Riemannian spectral triple of signature $(0, 2)$.

Take as a Hilbert space $H \cong F \oplus F^*$ with

$$F \ni v = \begin{bmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{bmatrix} \in M_4(\mathbb{C}).$$

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We can identify $\mathrm{End}_{\mathbb{C}}(H)$ with $M_4(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_4(\mathbb{C})$ and denote by e_{ij} a matrix with the the 1 in position (i, j) and zero everywhere else.

Elements of the algebra $A = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ are represented by

$$\left[\begin{array}{cc|c} \lambda & & 0 \\ & \bar{\lambda} & \\ \hline \multicolumn{2}{c|}{0} & q \end{array}\right] \otimes e_{11} \otimes 1 + \left[\begin{array}{c|c} \lambda & 0 \\ \hline 0 & m \end{array}\right] \otimes e_{22} \otimes 1,$$

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The Dirac operator is of the form

$$D = D_0 + D_1,$$

where $D_1 = JD_0J^{-1}$.

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We would like to have a spectral triple of KO-dimension 6, with a selfadjoint Dirac operator, but such that commutes with a suitable β that represents the shadow of a pseudo-Riemannian structure. Let us now take the general form of a Dirac operator that satisfies an order-one condition. We have

$$\begin{aligned} D_0 = & \begin{bmatrix} & M \\ M^\dagger & \end{bmatrix} \otimes e_{11} \otimes e_{11} + \begin{bmatrix} & N \\ N^\dagger & \end{bmatrix} \otimes e_{11} \otimes (1 - e_{11}) + \\ & + \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \otimes e_{12} \otimes e_{11} + \begin{bmatrix} A^\dagger & 0 \\ B^\dagger & 0 \end{bmatrix} \otimes e_{21} \otimes e_{11}, \end{aligned}$$

where M, N, A, B are 2×2 complex matrices.

Possible pseudo-Riemannian structures for the Standard Model

We look for a β that is a 0-cycle, i.e. a sum of elements of the form

$$\beta = \pi(\lambda_1, q_1, m_1) J \pi(\lambda_2, q_2, m_2) J^{-1},$$

with $\lambda_1, \lambda_2 \in \mathbb{C},\ q_1, q_2 \in \mathbb{H},\ m_1, m_2 \in M_3(\mathbb{C})$.

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Finally, with the $\beta = \pi(1, 1, -1) J \pi(1, 1, -1) J^{-1}$ we have no restriction whatsoever for M, N while then $B = 0$ and A needs to satisfy: $A = A \cdot \mathrm{diag}(1, -1)$. That leaves the possibility that A_{11} and A_{21} coefficients are present, providing no significant physical effects, and in particular leading only to terms involving a sterile neutrino.

Summary

- We proposed new definition of the finite pseudo-Riemannian spectral triples
- We proposed an alternative explanation of the observed quarks-leptons symmetry which prevents the $SU(3)$-breaking, as a shadow of the pseudo-Riemannian structure
- We proposed that the consistent model-building for the physical interactions and possible extensions of the Standard Model within the noncommutative geometry framework should use possibly the pseudo-Riemannian extension of finite spectral triples. We demonstrated that the pseudo-Riemannian framework allows for more restrictions and, in the discussed case introduces an extra symmetry grading, which we interpreted as the lepton-quark symmetry