An Introduction to Nonassociative Physics

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cost Action MP 1405
Quantum Structure of Spacetime

HORIZON 2020

Dualities and Generalized Geometries
Corfu Summer Institute

September 13, 2018

Outline

- Introduction: A brief history of nonassociativity in physics
- Magnetic Poisson brackets
- Classical & quantum dynamics in fields of magnetic charge
- Closed strings in locally non-geometric flux backgrounds
- M2-branes in locally non-geometric flux backgrounds
- M-waves in locally non-geometric KK-monopole backgrounds

Jordanian Quantum Mechanics

If A, B are Hermitian operators, then so is $A \circ B = \frac{1}{2}(A\,B + B\,A)$ (Jordan '32):

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- Only 3×3 Hermitian matrices over octonions $\mathbb{O}$ are non-special (Jordan, von Neumann & Wigner '34; Zelmanov '84; . . .)
- "Octonionic quantum mechanics" satisfies von Neumann axioms, no Hilbert space formulation (Günaydin, Piron & Ruegg '78)

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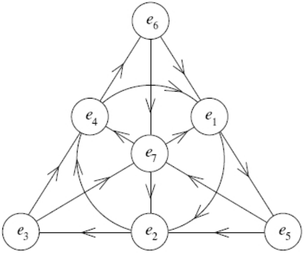
- Octonions $\mathbb{O}$: $a_0 + a_1 e_1 + a_2 e_2 + \cdots + a_7 e_7$ noncommutative Jordan algebra with multiplication given by Fano plane:



$e_1 e_5 = e_7 = -e_5 e_1$ etc.

$e_A^2 = -1$

$e_A e_B = -\delta_{AB} + \eta_{ABC} e_C$

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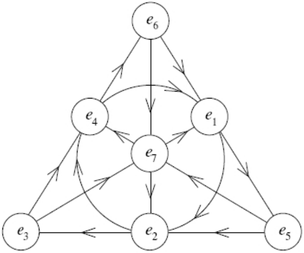

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- Rewrite $e_4, e_5, e_6 = f_1, f_2, f_3$:

$$[e_i, e_j] = 2\,\varepsilon_{ijk}\, e_k \,, \qquad [e_7, e_i] = 2\, f_i$$

$$[f_i, f_j] = -2\,\varepsilon_{ijk}\, e_k \,, \qquad [e_7, f_i] = -2\, e_i$$

$$[e_i, f_j] = 2\,(\delta_{ij}\, e_7 - \varepsilon_{ijk}\, f_k)$$

- Jacobiator: $[e_A, e_B, e_C] = -12\,\eta_{ABCD}\, e_D = 6\,((e_A\, e_B)\, e_C - e_A\,(e_B\, e_C))$

Nambu Dynamics

- Bi-Hamiltonian dynamics with Nambu–Poisson 3-bracket on $\mathbb{R}^3$:

$$\{f, g, h\} := \varepsilon^{ijk}\, \partial_i f\, \partial_j g\, \partial_k h$$

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- Euler equations $\dot{\vec{L}} + \vec{\omega} \times \vec{L} = \vec{0}$ for rotating rigid body in $\mathbb{R}^3$ equivalent to bi-Hamiltonian equations:

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- Nambu later suggests to use nonassociative algebras to quantize 3-bracket as a Jacobiator:

$$[A, B, C] \ := \ [[A, B], C] + [[C, A], B] + [[B, C], A]$$

Related to formulating nonassociative quantum mechanics

Nonassociativity in String/M-Theory

- Closed string field theory: L_∞-algebras (Strominger '87; Zwiebach '93)
- D-branes in curved backgrounds: $H = \mathrm{d}B$ controls Jacobiator (Cornalba & Schiappa '01; Herbst, Kling & Kreuzer '01)
- Topological T-duality of principal torus bundles (Mathai & Rosenberg '04; Bouwknegt, Hannabuss & Mathai '06; Brodzki, Mathai, Rosenberg & Sz '08)

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- Multiple M2-branes & 3-algebras (Basu & Harvey '05; Bagger & Lambert '07)
- Open M2-branes in C-field backgrounds: Quantization of Nambu–Poisson 3-brackets (Bergshoeff, Berman, van der Schaar & Sundell '00; Kawamoto & Sasakura '00; Chu & Smith '09; Sämann & Sz '12)
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- In these lectures, we focus on two related occurences:
  - Magnetic monopoles (Jackiw '85; Günaydin & Zumino '85)
  - Locally non-geometric string & M-theory backgrounds (Blumenhagen & Plauschinn '10; Lüst '10; Mylonas, Schupp & Sz '12; Günaydin, Lüst & Malek '16; Kupriyanov & Sz '17; Lüst, Malek & Sz '17; Freidel, Leigh & Minic '17; ...)

Magnetic Poisson Brackets

- $M = \mathbb{R}^d$ configuration space x^i, M^* momentum space p_i, $\mathcal{M} = T^*M = M \times M^*$ phase space $X^I = (x^i, p_i)$, with canonical symplectic 2-form $\omega_0 = \mathrm{d}p_i \wedge \mathrm{d}x^i$

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- H-twisted Poisson structure on $\mathcal{M}$ with $H = \mathrm{d}B$ 'magnetic charge' $[\theta_B, \theta_B]_{\mathrm{S}} = \bigwedge^3 \theta_B^\sharp(\mathrm{d}\omega_B)$ gives nonassociative algebra with Jacobiators $\{f, g, h\}_B = [\theta_B, \theta_B]_{\mathrm{S}}^{IJK}\, \partial_I f\, \partial_J g\, \partial_K h$:

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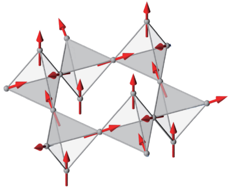
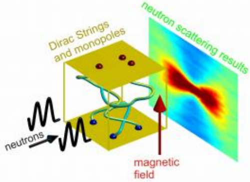
- In the lab: Neutron scattering off spin ice pyrochlore lattices

(Castelnovo, Moessner & Sondhi '08; Morris *et al.* '09; ...)





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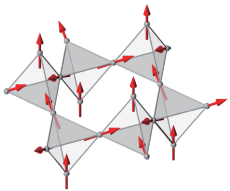

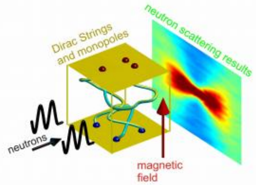

- Smooth $H = \mathrm{d}B \neq 0$ gives smooth distributions $\vec{\nabla}\cdot\vec{B} \neq 0$ of magnetic charge

Locally Non-Geometric Fluxes

- Born reciprocity $(x, p) \longmapsto (p, -x)$ preserves ω_0, maps $B \in \Omega^2(M) \longmapsto \beta \in \Omega^2(M^*)$ with twisted Poisson brackets:

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- $M = T^3$ with H-flux gives geometric and non-geometric fluxes via T-duality (Shelton, Taylor & Wecht '05)

$$H_{ijk} \xleftrightarrow{\mathsf{T}_i} f^i{}_{jk} \xleftrightarrow{\mathsf{T}_j} Q^{ij}{}_k \xleftrightarrow{\mathsf{T}_k} R^{ijk}$$

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- Sends string winding $(w^i)\in H_1(T^3,\mathbb{Z})\;=\;\mathbb{Z}^3$ to momenta (p_i)

- In Double Field Theory: $H_{ijk}\;=\;\partial_{[i}B_{jk]}\;\xleftrightarrow{\;\mathsf{T}_{ijk}\;}\;R^{ijk}\;=\;\hat\partial^{[i}\beta^{jk]}$
(Andriot, Hohm, Larfors, Lüst & Patalong '12)

Classical Motion in Fields of Magnetic Charge

- For $d = 3$, motion in magnetic field $\vec{B}$ (with or without sources) governed by Lorentz force

$$\dot{\vec{p}} = \frac{e}{m}\,\vec{p}\times\vec{B} \quad , \quad \vec{p} = m\,\dot{\vec{x}}$$

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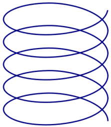
- $\vec{B} = $ constant:

Motion follows helical trajectory with uniform velocity along $\vec{B}$-direction



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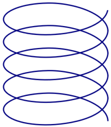
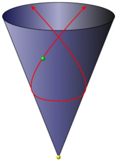


- Dirac monopole field $\vec{B} \;=\; \vec{B}_{\rm D}$: (Bakas & Lüst '13)

Conservation of Poincaré vector $\vec{K}$ confines motion to surface of cone, electric charge never reaches magnetic monopole and nonassociativity plays no role



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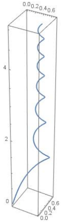

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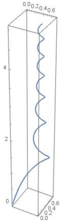

- $\vec{B} = \frac{1}{3}\,\rho\,\vec{x}$, constant magnetic charge ρ: Motion is no longer integrable or confined, equivalent to motion in Dirac monopole field $\vec{B}_{\mathrm{D}}$ with additional frictional forces (Bakas & Lüst '13)

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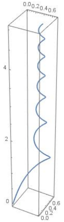

- $\vec{B} = \frac{1}{3}\,\rho\,\vec{x}$, constant magnetic charge ρ: Motion is no longer integrable or confined, equivalent to motion in Dirac monopole field $\vec{B}_{\rm D}$ with additional frictional forces (Bakas & Lüst '13)

- **Questions:**
  - What substitutes for canonical quantization of locally non-geometric closed strings?
  - Is there a sensible nonassociative quantum mechanics?

Quantization of Magnetic Poisson Brackets

- Quantization: Linear map $f \longmapsto \mathcal{O}_f$ on $f \in C^\infty(\mathcal{M})$:

$$[\mathcal{O}_f, \mathcal{O}_g] = \mathrm{i}\,\hbar\,\mathcal{O}_{\{f,g\}_B} + O(\hbar^2)$$

$$[\mathcal{O}_{x^i}, \mathcal{O}_{x^j}] = 0 \quad , \quad [\mathcal{O}_{x^i}, \mathcal{O}_{p_j}] = \mathrm{i}\,\hbar\,\delta^i{}_j \quad , \quad [\mathcal{O}_{p_i}, \mathcal{O}_{p_j}] = -\,\mathrm{i}\,\hbar\,B_{ij}(\mathcal{O}_x)$$

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- Magnetic translation operators $\mathcal{P}_v = \exp\big(\frac{\mathrm{i}}{\hbar}\,\mathcal{O}_{p\cdot v}\big)$:

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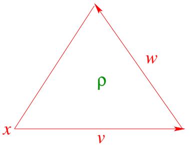
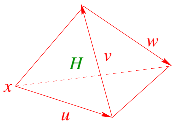
- Representation of translation group $\mathbb{R}^d$? (Jackiw '85)

$$\mathcal{P}_w\,\mathcal{P}_v = \mathrm{e}^{\,\mathrm{i}\,\Phi_2(x;v,w)}\,\mathcal{P}_{v+w} \quad , \quad \mathcal{P}_w\,(\mathcal{P}_v\,\mathcal{P}_u) = \mathrm{e}^{\,\mathrm{i}\,\Phi_3(x;u,v,w)}\,(\mathcal{P}_w\,\mathcal{P}_v)\,\mathcal{P}_u$$





Quantization with $\mathrm{d}B = 0$

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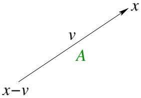

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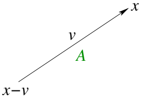

$$(\mathcal{P}_v\psi)(x) = \exp\Big(-\frac{\mathrm{i}}{\hbar}\int_{\triangle^1(x;v)} A\Big)\,\psi(x-v)$$

- Defines weak projective representation of translation group $\mathbb{R}^d$ on $\mathcal{H}$:

$$(\mathcal{P}_w\,\mathcal{P}_v\psi)(x) = \omega_{v,w}(x)\ (\mathcal{P}_{v+w}\psi)(x)$$

$$\omega_{v,w}(x) = \exp\Big(-\frac{\mathrm{i}}{\hbar}\int_{\triangle^2(x;w,v)} B\Big) \qquad \Big(= \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}B(v,w)} \text{ for } B \text{ constant}\Big)$$

2-cocycle on $\mathbb{R}^d$ with values in $C^\infty(M, U(1))$

Quantization with $\mathrm{d}B = 0$

- Magnetic Weyl correspondence $f \in C^\infty(\mathcal{M}) \longmapsto \mathcal{O}_f \in \mathrm{End}(\mathcal{H})$:

$$W(x,p) : \mathcal{H} \longrightarrow \mathcal{H} \quad , \quad \big(W(x,p)\psi\big)(y) = \mathrm{e}^{\,\frac{\mathrm{i}\,\hbar}{2}\,p\cdot x}\,\mathrm{e}^{\,-\,\mathrm{i}\,p\cdot y}\,(\mathcal{P}_x\psi)(y)$$

$$\mathcal{O}_f = \int_{\mathcal{M}} \bigg(\int_{\mathcal{M}} \mathrm{e}^{\,\mathrm{i}\,\omega_0(X,Y)}\, f(Y)\, \frac{\mathrm{d}Y}{(2\pi)^d} \bigg)\, W(X)\, \frac{\mathrm{d}X}{(2\pi)^d}$$

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- Magnetic Moyal–Weyl star product $\mathcal{O}_{f\star_B g} = \mathcal{O}_f\,\mathcal{O}_g$;
  e.g. for B constant:

$$(f\star_B g)(X) = \frac{1}{(\pi\,\hbar)^d}\int_{\mathcal{M}}\int_{\mathcal{M}} \mathrm{e}^{\,-\,\frac{2\,\mathrm{i}}{\hbar}\,\omega_B(Y,Z)}\,f(X-Y)\,g(X-Z)\,\mathrm{d}Y\,\mathrm{d}Z$$

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- Canonical quantum mechanics $\Longrightarrow$ phase space quantum mechanics:
  - Observables/states: (real) functions on phase space
  - Operator product: star product , Traces: integration
  - State function (density matrix): $S \geqslant 0$, $\int_{\mathcal{M}} S = 1$
  - Expectation values: $\langle \mathcal{O} \rangle = \int_{\mathcal{M}} \mathcal{O} \star_B S$ ...

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  - Magnetic Poisson brackets are associative on $M^\circ = \mathbb{R}^3 \setminus \{0\}$, $B_{\rm D} = \mathrm{d}A_{\rm D}$ locally
  - Quantum Hilbert space is $\mathcal{H} = \mathrm{L}^2(M^\circ, L)$ for a non-trivial line bundle $L \longrightarrow M^\circ$ iff Dirac charge quantization: $\frac{2\,e\,g}{\hbar} \in \mathbb{Z}$ (Wu & Yang '76)
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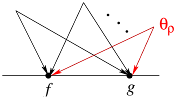
  - Magnetic Weyl correspondence on $\mathcal{M}^\circ$ induces associative phase space star product (Soloviev '17)
- For generic smooth distributions $H \in \Omega^3(M)$, standard canonical quantization breaks down
- Can be studied perturbatively in H using noncommutative Jordan algebra of quantum moments (Bojowald, Brahma, Büyükçam & Strobl '14)

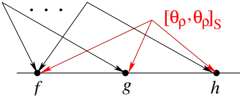
Deformation Quantization

- For any $H = \mathrm{d}B \in \Omega^3(M)$, Kontsevich formality provides noncommutative and nonassociative star product on $C^\infty(\mathcal{M})[[\hbar]]$:

$$f \star_H g = f\, g + \frac{\mathrm{i}\,\hbar}{2}\, \{f,g\}_B + \sum_{n\geqslant 2} \frac{(\mathrm{i}\,\hbar)^n}{n!}\ \mathtt{b}_n(f,g)$$



$$[f,g,h]_{\star_H} = -\hbar^2\, \{f,g,h\}_B + \sum_{n\geqslant 3} \frac{(\mathrm{i}\,\hbar)^n}{n!}\ \mathtt{t}_n(f,g,h)$$



where $\mathtt{b}_n = U_n(\theta_B,\dots,\theta_B)$, $\mathtt{t}_n = U_{n+1}([\theta_B,\theta_B]_{\mathrm{S}},\theta_B,\dots,\theta_B)$ are bi-/tri-differential operators (Mylonas, Schupp & Sz '12)

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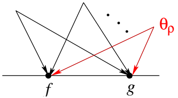

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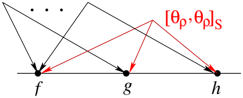

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- *R*-flux model: Expectation values of oriented volume uncertainty operators $V^{ijk} \;=\; \big\langle \frac{1}{2}\,[\Delta x^i, \Delta x^j, \Delta x^k]_{\star_R} \big\rangle$ give quantum of volume

$$V^{ijk} \;=\; \tfrac{1}{2}\,\ell_s^3\,R^{ijk}$$

For $d = 3$, no D0-branes in locally non-geometric string backgrounds (T-dual to Freed–Witten anomaly for D3-branes on T^3 with H-flux)
(Wecht '07)

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$$\mathcal{H} = \tfrac{1}{m}\,p_I\,\eta^{IJ}\,p_J \quad , \quad p_I = (p_i, \tilde{p}_i) \quad , \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

    reproduces Lorentz force $\dot{\vec{p}} = \frac{e}{m}\,\vec{p}\times\vec{B}$, $\vec{p} = m\,\dot{\vec{x}}$

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- **Transgression:** Send field strength of gerbe on M to field strength of line bundle on loop space $C^\infty(S^1, M)$ (Sämann & Sz '12)

How do Closed Strings see Nonassociativity?

- Configuration space triproducts in R-flux model (Aschieri & Sz '15)

$$(f \vartriangle g \vartriangle h)(x) \ = \ \big(f(x) \star_R g(x)\big) \star_R h(x)\big|_{p=0}$$

$$= \int_{k,k',k''} \tilde{f}(k)\,\tilde{g}(k')\,\tilde{h}(k'')\ \mathrm{e}^{-\frac{\mathrm{i}\,\ell_s^3}{12}\,R(k,k',k'')}\ \mathrm{e}^{\,\mathrm{i}\,(k+k'+k'')\cdot x}$$

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$$(f \vartriangle g \vartriangle h)(x) \ = \ \big(f(x) \star_R g(x)\big) \star_R h(x)\big|_{p=0}$$

$$= \int_{k,k',k''} \tilde{f}(k)\, \tilde{g}(k')\, \tilde{h}(k'')\ \mathrm{e}^{-\frac{\mathrm{i}\,\ell_s^3}{12}\, R(k,k',k'')}\ \mathrm{e}^{\,\mathrm{i}\,(k+k'+k'')\cdot x}$$

- Quantizes 3-bracket $[f,g,h]_\vartriangle \ = \ \mathrm{Asym}(f \vartriangle g \vartriangle h)$ (Takhtajan '94)

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- Agrees with multiplication of tachyon vertex operators $V_k(z,\bar{z}) = :\mathrm{e}^{\,\mathrm{i}\,k\cdot X(z,\bar{z})}:$ in CFT scattering of momentum states in R-flux background: $\big\langle V_k\, V_{k'}\, V_{k''} \big\rangle_R \sim \exp\big(-\frac{\mathrm{i}\,\ell_s^3}{12} R(k,k',k'')\big)$

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- On-shell associativity of CFT amplitudes: $\int f \vartriangle g \vartriangle h \; = \; \int f\, g\, h$

Nonassociative Gravity?

- Nonassociative (Riemannian) differential geometry can be developed using quasi-Hopf algebra 2-cochain (coboundary is a 3-cocyle) twist deformation techniques (Mylonas, Schupp & Sz '13; Aschieri & Sz '15; Barnes, Schenkel & Sz '15; Blumenhagen & Fuchs '16; Aschieri, Dimitrijević-Ćirić & Sz '17)

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- Metric formulation of nonassociative gravity on phase space:

  Ricci tensor, unique metric-compatible torsion-free connection, non-trivial real deformation of spacetime Ricci tensor:

$$
\begin{aligned}
\mathrm{Ric}^{\circ}_{ij} = \mathrm{Ric}_{ij} + \frac{\ell_s^3}{12}\, R^{abc} \Big(& \partial_k \big(\partial_a g^{kl} \, (\partial_b g_{lm})\, \partial_c \Gamma^m_{ij}\big) - \partial_j \big(\partial_a g^{kl} \, (\partial_b g_{lm})\, \partial_c \Gamma^m_{ik}\big) \\
& + \partial_c g_{mn} \big(\partial_a (g^{lm}\, \Gamma^k_{lj})\, \partial_b \Gamma^n_{ik} - \partial_a (g^{lm}\, \Gamma^k_{lk})\, \partial_b \Gamma^n_{ij} \\
& + (\Gamma^l_{ik}\, \partial_a g^{km} - \partial_a \Gamma^l_{ik}\, g^{km})\, \partial_b \Gamma^n_{lj} - (\Gamma^l_{ij}\, \partial_a g^{km} - \partial_a \Gamma^l_{ij}\, g^{km})\, \partial_b \Gamma^n_{lk}\big) \Big)
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- **Question:** Is there an equivalent $O(d,d)$-invariant (off-shell) nonassociative version of the closed string effective action?

$$S = \frac{1}{16\pi\, G} \int_M \sqrt{g}\, \Big(\mathrm{Ric} - \frac{1}{12}\, \mathrm{e}^{\,-\phi/3}\, H_{ijk}\, H^{ijk} - \frac{1}{6}\, \partial_i \phi\, \partial^i \phi + \cdots\Big)$$

  [cf. Invariance of noncommutative Yang–Mills theory of D-branes on T^d under open string $SO(d,d;\mathbb{Z})$ T-duality]

M-Theory Lift of the R-Flux Model

(Günaydin, Lüst & Malek '16; Lüst, Malek & Syväri '17)

$$\begin{array}{ccc} S^1 & \hookrightarrow & \widetilde{M} \\ & & \big\downarrow \\ & & M \end{array}$$

S^1 radius λ $\longrightarrow$ string coupling g_s

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- Generate string R-flux starting from twisted torus $M = \tilde{T}^3$:

$$f^i{}_{jk} \xrightarrow{T_{jk}} R^{ijk} = R\,\varepsilon^{ijk}$$

- Lift to M-theory on $\widetilde{M} = M \times S^1_{x^4}$: T-duality $\Longrightarrow$ U-duality

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- Lift to M-theory on $\widetilde{M} = M \times S^1_{x^4}$: T-duality $\Longrightarrow$ U-duality
- Sends membrane wrapping $(w^{ij}) \in H_2(\widetilde{M}, \mathbb{Z})$ to momenta (p_i)
- In $SL(5)$ Exceptional Field Theory: $C_{\mu\nu\rho} \xrightarrow{U_{\mu\nu\rho}} \Omega^{\mu\nu\rho}$ with $R^{\mu,\nu\rho\alpha\beta} = \hat{\partial}^{\mu[\nu}\Omega^{\rho\alpha\beta]}$ (**Not** a 5-vector!) (Blair & Malek '14)
- Choice $R^{4,\mu\nu\alpha\beta} = R\,\varepsilon^{\mu\nu\alpha\beta}$ breaks $SL(5) \longrightarrow SO(4)$

M2-Brane Phase Space

- No D0-branes on $M \implies p_4 = 0$ along M-theory direction
- $R^{\mu,\nu\rho\alpha\beta}\, p_\mu = 0 \implies$ membrane has 7D phase space $\widetilde{\mathcal{M}}$:

$$
\begin{aligned}
\{x^i, x^j\} &= \tfrac{\ell_s^3}{3\hbar^2}\, R^{4,ijk4}\, p_k \ , \qquad \{x^4, x^i\} = \tfrac{\lambda\, \ell_s^3}{3\hbar^2}\, R^{4,1234}\, p^i \\
\{x^i, p_j\} &= \delta^i{}_j\, x^4 + \lambda\, \varepsilon^i{}_{jk}\, x^k \ , \qquad \{x^4, p_i\} = \lambda^2\, x_i \\
\{p_i, p_j\} &= -\lambda\, \varepsilon_{ijk}\, p^k \\
\{x^i, x^j, x^k\} &= \tfrac{\ell_s^3}{3\hbar^2}\, R^{4,ijk4}\, x^4 \ , \quad \{x^i, x^j, x^4\} = -\tfrac{\lambda^2\, \ell_s^3}{3\hbar^2}\, R^{4,ijk4}\, x_k \\
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\{p_i, p_j, p_k\} &= 0
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- Originates from nonassociative, alternative, octonion algebra $\mathbb{O}$:

$$
(x^A) = (x^i, x^4, p_i) = \Lambda(e_A) = \frac{1}{2}\Big(\sqrt{\lambda\, \ell_s^3\, R/3}\ f_i\, ,\ \sqrt{\lambda^3\, \ell_s^3\, R/3}\ e_7\, ,\ -\lambda\, e_i\Big)
$$

- Reduces to magnetic Poisson brackets for closed strings at $\lambda = 0$ (with $x^4 = 1$ central)

Quantization of M2-Brane Phase Space (Kupriyanov & Sz '17)

- G_2-structure: $\mathbb{R}^7$ carries cross product $(\vec{k} \times_\eta \vec{p})_A = \eta_{ABC}\, k_B\, p_C$, invariant under $G_2 \subset SO(7)$
- Represented on $\mathbb{O}$ through $X_{\vec{k}} = k^A\, e_A$: $X_{\vec{k} \times_\eta \vec{k}'} = \frac{1}{2}\big[X_{\vec{k}}, X_{\vec{k}'}\big]$

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- Alternativity $(X\,X)\,X = X\,(X\,X)$ defines octonion exponential:

$$\mathrm{e}^{X_{\vec{k}}} = \cos|\vec{k}| + \frac{\sin|\vec{k}|}{|\vec{k}|}\; X_{\vec{k}}$$

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- BCH formula $\mathrm{e}^{\,X_{\vec{k}}}\,\mathrm{e}^{\,X_{\vec{k}'}} = \mathrm{e}^{\,X_{\vec{\mathcal{B}}_\eta(\vec{k},\vec{k}'\,)}}$ can be computed explicitly giving octonionic Weyl correspondence and star product:

$$(f \star_\lambda g)(\vec{x}\,) = \int_{\vec{k},\vec{k}'} \tilde{f}(\vec{k}\,)\,\tilde{g}(\vec{k}'\,)\,\mathrm{e}^{\,\mathrm{i}\,\vec{\mathcal{B}}_\eta(\Lambda\,\vec{k},\Lambda\,\vec{k}'\,)\cdot\Lambda^{-1}\,\vec{x}}$$

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  such that $(f \star_\lambda g)(\vec{x}\,) \xrightarrow{\lambda\to 0} (f \star_R g)(x)$
- Triproduct $\big(f \vartriangle_\lambda g \vartriangle_\lambda h\big)(\vec{x}\,) = \big((f \star_\lambda g) \star_\lambda h\big)(x^\mu, p_i)\big|_{p=0}$ quantizes the 3-Lie algebra A_4:

$$[x^\mu, x^\nu, x^\alpha]_{\vartriangle_1} = \ell_s^3\, R\, \varepsilon^{\mu\nu\alpha\beta}\, x^\beta$$

Magnetic Monopoles & Quantum Gravity

- Apply canonical transformation $(x,p) \longmapsto (p,-x)$, $\ell_s^3\, R \longmapsto \hbar^2\, e\, \rho$:

$$
\begin{aligned}
[x^i, x^j] &= -\,\mathrm{i}\,\hbar\,\lambda\,\varepsilon^{ijk}\,x^k \\
[p_i, p_j] &= \mathrm{i}\,\hbar\, e\,\rho\,\varepsilon_{ijk}\,x^k \ , \qquad [p_4, p_i] \ = \ \mathrm{i}\,\hbar\,\lambda\, e\,\rho\, x_i \\
[x_i, p_j] &= \mathrm{i}\,\hbar\,\delta_{ij}\, p_4 + \mathrm{i}\,\hbar\,\lambda\,\varepsilon_{ijk}\, p_k \ , \qquad [x^i, p_4] \ = \ -\,\mathrm{i}\,\hbar\,\lambda^2\, x^i
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  Reduces to magnetic brackets for electric charges at $\lambda = 0$

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- Quaternion subalgebra: Setting $\rho = 0$ reveals noncommutative associative deformation of spacetime with $[p_i, p_j] = 0$; restrict to $\lambda^2\,\vec{p}^{\,2} + p_4^2 = 1$: (Freidel & Livine '05)

$$[x^i, x^j] = -\mathrm{i}\,\hbar\,\lambda\,\varepsilon^{ijk}\,x^k\ , \ [x_i, p_j] = \mathrm{i}\,\hbar\,\sqrt{1-\lambda^2\,\vec{p}^{\,2}}\ \delta_{ij} + \mathrm{i}\,\hbar\,\lambda\,\varepsilon_{ijk}\,p_k$$

  Ponzano–Regge spin foam model of 3D quantum gravity

- **Uncontracted octonion algebra is related to monopoles in the spacetime of 3D quantum gravity, with $\boldsymbol{\lambda = \ell_{\rm P}/\hbar}$!**

M-Wave Phase Space (Lüst, Malek & Sz '17)

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$$\mathrm{d}s_{11}^2 = \mathrm{d}s_7^2 + U\,\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x} + U^{-1}\big(\mathrm{d}x^4 + \vec{A}\cdot\mathrm{d}\vec{x}\big)^2$$

$$\vec{\nabla}\times\vec{A} = \vec{\nabla}U \quad , \quad \vec{\nabla}^2 U = \rho$$

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$$\vec{\nabla}\times\vec{A} = \vec{\nabla}U \quad , \quad \vec{\nabla}^2 U = \rho$$

- Parameters: $\ell_s^2 = \ell_{\rm P}^3/R_{11}$, $g_s = \left(R_{11}/\ell_{\rm P}\right)^{3/2}$ (Witten '95)
- M-theory $\longrightarrow$ string theory: $g_s, R_{11} \longrightarrow 0$ with ℓ_s finite

$$\Longleftrightarrow \lambda \sim \ell_{\rm P} \sim R_{11}^{1/3} \longrightarrow 0$$

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- **Non-geometric KK-monopole:** $\rho =$ smearing of Dirac monopoles, no local expression for $\vec{A}$ and metric

- Taub-NUT $\xrightarrow{S^1} \mathbb{R}^3 \Longrightarrow S^1$-gerbe over $\mathbb{R}^3$

M-Theory Phase Space 3-Algebra

- *Spin*(7)-structure: $[\xi_{\hat{A}}, \xi_{\hat{B}}, \xi_{\hat{C}}]_\phi = \phi_{\hat{A}\hat{B}\hat{C}\hat{D}}\,\xi_{\hat{D}}$ for $\xi = (\xi_0, \vec{\xi}\,) = (1, e_A)$, where $\phi_{0ABC} = \eta_{ABC}$ and $\phi_{ABCD} = \eta_{ABCD}$; Symmetry: $Spin(7) \subset SO(8)$

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- 8D phase space coordinates $X = (x^\mu, p_\mu) = (\Lambda\, \vec{\xi}, -\frac{\lambda}{2}\, \xi_0)$ have $SO(4)\times SO(4)$-symmetric 3-brackets:

$$
\begin{aligned}
[x^i, x^j, x^k]_\phi &= -\tfrac{\ell_s^3}{2}\, R^{4,ijk4}\, x^4 \ , \quad [x^i, x^j, x^4]_\phi = \frac{\lambda^2\, \ell_s^3}{2}\, R^{4,ijk4}\, x_k \\
[p^i, x^j, x^k]_\phi &= -\frac{\lambda^2\, \ell_s^3}{2}\, R^{4,ijk4}\, p_4 - \frac{\lambda\, \ell_s^3}{2}\, R^{4,ijk4}\, p_k \\
[p_i, x^j, x^4]_\phi &= -\frac{\lambda^2\, \ell_s^3}{2}\, R^{4,1234}\, \delta_i^j\, p_4 - \frac{\lambda^2\, \ell_s^3}{2}\, R^{4,ijk4}\, p_k \\
[p_i, p_j, x^k]_\phi &= \tfrac{\lambda^2}{2}\, \varepsilon_{ij}{}^k\, x^4 + \tfrac{\hbar^2\, \lambda}{2}\, \big(\delta_j^k\, x_i - \delta_i^k\, x_j\big) \\
[p_i, p_j, x^4]_\phi &= -\tfrac{\hbar^2\, \lambda^3}{2}\, \varepsilon_{ijk}\, x^k \ , \quad [p_i, p_j, p_k]_\phi = -2\hbar^2\, \lambda\, \varepsilon_{ijk}\, p_4 \\
[p_4, x^i, x^j]_\phi &= \frac{\lambda\, \ell_s^3}{2}\, R^{4,ijk4}\, p_k \ , \quad [p_4, x^i, x^4]_\phi = -\frac{\lambda^2\, \ell_s^3}{2}\, R^{4,1234}\, p^i \ , \\
[p_4, p_i, x^j]_\phi &= -\tfrac{\hbar^2\, \lambda}{2}\, \delta_i^j\, x^4 - \tfrac{\hbar^2\, \lambda^2}{2}\, \varepsilon_i{}^{jk}\, x_k \\
[p_4, p_i, x^4]_\phi &= -\tfrac{\hbar^2\, \lambda^3}{2}\, x_i \ , \quad [p_4, p_i, p_j]_\phi = -\tfrac{\hbar^2\, \lambda^2}{2}\, \varepsilon_{ijk}\, p^k
\end{aligned}
$$

M-Theory Phase Space 3-Algebra

- *Spin*(7)-structure: $[\xi_{\hat A},\xi_{\hat B},\xi_{\hat C}]_\phi = \phi_{\hat A\hat B\hat C\hat D}\,\xi_{\hat D}$ for $\xi = (\xi_0,\vec\xi\,) = (1,e_A)$, where $\phi_{0ABC} = \eta_{ABC}$ and $\phi_{ABCD} = \eta_{ABCD}$; Symmetry: $Spin(7)\subset SO(8)$
- 8D phase space coordinates $X = (x^\mu,p_\mu) = (\Lambda\,\vec\xi,-\frac{\lambda}{2}\,\xi_0)$ have $SO(4)\times SO(4)$-symmetric 3-brackets:

$$
\begin{aligned}
[x^i,x^j,x^k]_\phi &= -\tfrac{\ell_s^3}{2}\,R^{4,ijk4}\,x^4\ , \quad [x^i,x^j,x^4]_\phi = \tfrac{\lambda^2\,\ell_s^3}{2}\,R^{4,ijk4}\,x_k \\
[p^i,x^j,x^k]_\phi &= -\tfrac{\lambda^2\,\ell_s^3}{2}\,R^{4,ijk4}\,p_4 - \tfrac{\lambda\,\ell_s^3}{2}\,R^{4,ijk4}\,p_k \\
[p_i,x^j,x^4]_\phi &= -\tfrac{\lambda^2\,\ell_s^3}{2}\,R^{4,1234}\,\delta_i^j\,p_4 - \tfrac{\lambda^2\,\ell_s^3}{2}\,R^{4,ijk4}\,p_k \\
[p_i,p_j,x^k]_\phi &= \tfrac{\lambda^2}{2}\,\varepsilon_{ij}{}^k\,x^4 + \tfrac{\hbar^2\,\lambda}{2}\,(\delta_j^k\,x_i - \delta_i^k\,x_j) \\
[p_i,p_j,x^4]_\phi &= -\tfrac{\hbar^2\,\lambda^3}{2}\,\varepsilon_{ijk}\,x^k\ , \quad [p_i,p_j,p_k]_\phi = -2\hbar^2\,\lambda\,\varepsilon_{ijk}\,p_4 \\
[p_4,x^i,x^j]_\phi &= \tfrac{\lambda\,\ell_s^3}{2}\,R^{4,ijk4}\,p_k\ , \quad [p_4,x^i,x^4]_\phi = -\tfrac{\lambda^2\,\ell_s^3}{2}\,R^{4,1234}\,p^i\ , \\
[p_4,p_i,x^j]_\phi &= -\tfrac{\hbar^2\,\lambda}{2}\,\delta_i^j\,x^4 - \tfrac{\hbar^2\,\lambda^2}{2}\,\varepsilon_i{}^{jk}\,x_k \\
[p_4,p_i,x^4]_\phi &= -\tfrac{\hbar^2\,\lambda^3}{2}\,x_i\ , \quad [p_4,p_i,p_j]_\phi = -\tfrac{\hbar^2\,\lambda^2}{2}\,\varepsilon_{ijk}\,p^k
\end{aligned}
$$

- $[f,g]_G := [f,g,G]_\phi$ for any constraint $G(X) = 0$; breaks $Spin(7)\longrightarrow G_2$, $\mathbf{8}=\mathbf{7}\oplus\mathbf{1}$
- $G(X) = p_4$ gives M2-brane phase space, $G(X) = x^4$ gives M-wave phase space; Born reciprocity is a *Spin*(7)-transformation