The Geometric Structure of Double Field Theory

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Motivation.

Can we reconcile the structures in double field theory (DFT) with Courant algebroid structures in generalized geometry, knowing that DFT reduces to Courant algebroid after imposing the strong constraint?

Definition.
Let $E \xrightarrow{\pi} M$ be a vector bundle.
Let $[\cdot,\cdot]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$.
Let $\langle \cdot,\cdot \rangle_E : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$ be a symmetric $C^\infty(M)$-bilinear non-degenerate form.
Anchor map $\rho : E \to TM$.

$\Rightarrow (E, [\cdot,\cdot]_E, \langle \cdot,\cdot \rangle_E, \rho)$ is a **Courant algebroid**, satisfying 5 properties.

(Liu, Weinstein and Xu, arXiv:dg-ga/9508013)

Here we consider: $E = TM \oplus T^*M$

Courant algebroid

1. Jacobi: $[[A,B],C] + \text{cyclic} = \frac{1}{3}\,\mathcal{D}\langle [A,B],C\rangle + \text{cyclic}$
2. Homomorphism: $\rho[A,B] = [\rho(A),\rho(B)]$
3. Leibniz: $[A, f\,B] = f\,[A,B] + \big(\rho(A)f\big)\,B - \langle A,B\rangle\,\mathcal{D}f$
4. *Strong-Constraint-related:*
   $\rho\circ\mathcal{D} = 0 \quad \Longleftrightarrow \quad \langle \mathcal{D}f, \mathcal{D}g\rangle = 0$
5. Compatibility:
   $\rho(C)\langle A,B\rangle = \langle [C,A] + \mathcal{D}\langle C,A\rangle, B\rangle + \langle A, [C,B] + \mathcal{D}\langle C,B\rangle\rangle$

where the differential operator $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is defined by

$$\langle \mathcal{D}f, A\rangle = \frac{1}{2}\,\rho(A)f \ ,$$

for any $A, B, C \in \Gamma(E)$ and $f \in C^\infty(M)$.

Courant algebroid (CA) properties in local expressions

Introduce a local basis for sections of E, e^I where $I = 1, \ldots, 2d$:

Operations:

$[e^I, e^J] = \eta^{IK}\eta^{JL}T_{KLM}e^M$ (twisted bracket), $\quad \langle e^I, e^J\rangle = \frac{1}{2}\eta^{IJ}$,

$\rho(e^I)f = \eta^{IJ}\rho^i{}_J\partial_i f, \quad Df = D_I f\, e^I = \rho^i{}_I\partial_i f\, e^I$,

where components of anchor ρ, $(\rho^i{}_J) = (\rho^i{}_j, \rho^{ij})$ for $i = 1, \ldots, d$.

In local coordinates, the properties of CA are

$$
\begin{aligned}
\eta^{IJ}\,\rho^i{}_I\,\rho^j{}_J &= 0\,,\\
\rho^i{}_I\,\partial_i\rho^j{}_J - \rho^i{}_J\,\partial_i\rho^j{}_I - \eta^{KL}\,\rho^j{}_K\,T_{LIJ} &= 0\,,\\
4\,\rho^i{}_{[L}\,\partial_i T_{IJK]} + 3\,\eta^{MN}\,T_{M[IJ}\,T_{KL]N} &= 0\,.
\end{aligned}
$$

An algebroid for Double Field Theory

Strategy:

Double the canonical Courant algebroid and *Project*

(a) Doubling the target space

Consider a target space T^*M with local coordinates (x^i, p_i), thus a map

$$\mathbb{X} : \Sigma_3 \longrightarrow T^*M \ .$$

The components of this map are

$$\mathbb{X} = (\mathbb{X}^I) = (\mathbb{X}^i, \mathbb{X}_i) =: (X^i, \widetilde{X}_i) \ .$$

The fields X^i and $\widetilde{X}_i$ are identified with the pullbacks of the coordinate functions, $X^i = \mathbb{X}^*(x^i)$ and $\widetilde{X}_i = \mathbb{X}^*(p_i)$.

For worldvolume description of DFT:-
Larisa Jonke's talk on Saturday at 11.

(a) Doubling the target space

The vector bundle is

$$E = \mathbb{T}(T^*M) := T(T^*M) \oplus T^*(T^*M) \; .$$

The sections of the bundle: $(\mathbb{A}^{\hat{I}}) = (\mathbb{A}^I, \widetilde{\mathbb{A}}_I) = (\mathbb{A}^i, \mathbb{A}_i, \widetilde{\mathbb{A}}_i, \widetilde{\mathbb{A}}^i)$ and $\mathbb{A} = \mathbb{A}_V + \mathbb{A}_F := \mathbb{A}^I \, \partial_I + \widetilde{\mathbb{A}}_I \, \mathrm{d}\mathbb{X}^I$.
The basis vectors on T^*M: $(\partial_I) = (\partial/\partial X^i, \partial/\partial \widetilde{X}_i) =: (\partial_i, \tilde{\partial}^i)$
The basis forms on T^*M: $(\mathrm{d}\mathbb{X}^I) := (\mathrm{d}X^i, \mathrm{d}\widetilde{X}_i)$
For the anchor $\rho^I{}_{\hat{J}}$, we have $(\rho^I{}_J, \widetilde{\rho}^{IJ})$ of E.

$\therefore$ This defines a 'large' Courant algebroid: $(E, [\,\cdot\,,\,\cdot\,], \langle\,\cdot\,,\,\cdot\,\rangle, \rho)$.

Intermediate step: Introduce a decomposition

Before projecting, first introduce

$$\mathbb{A}^I_\pm = \tfrac{1}{2}\left(\mathbb{A}^I \pm \eta^{IJ}\,\widetilde{\mathbb{A}}_J\right)\ , \quad e^\pm_I = \partial_I \pm \eta_{IJ}\,\mathrm{d}\mathbb{X}^J\ .$$

For the anchor $\rho^I{}_{\hat{J}} = (\rho^I{}_J, \widetilde{\rho}^{IJ})$ of E, define

$$(\rho_\pm)^I{}_J = \rho^I{}_J \pm \eta_{JK}\,\widetilde{\rho}^{IK}\ .$$

An $O(d,d)$ metric η is involved.
$\therefore$ The generalized tangent bundle is thus decomposed as

$$E = \mathbb{T}(T^*M) = T(T^*M) \oplus T^*(T^*M) = L_+ \oplus L_-\ ,$$

where $L_\pm$ is the bundle whose space of sections is spanned locally by $e^\pm_I$.

Hint: Reduce the $\mathbb{A}^{\hat{I}}$ field from 4d to 2d.

Goal: By projections from the 'large' Courant algebroid (CA), reproduce the DFT data:

(1) DFT vector
(2) C-bracket of DFT vectors
(3) Generalized Lie derivative in DFT $\rightarrow$ strong constraint

(b) Projecting from the 'large' Courant algebroid

The generalized vector of the 'large' CA ($E = L_+ \oplus L_-$) is given as

$$\mathbb{A} = \mathbb{A}^I \,\partial_I + \widetilde{\mathbb{A}}_I \,\mathrm{d}\mathbb{X}^I = \mathbb{A}_+^I \, e_I^+ + \mathbb{A}_-^I \, e_I^- \ .$$

By *setting* the components $\mathbb{A}_-^I = 0$ and renaming $\mathbb{A}_+^I = A^I$, we obtain a special generalized vector

$$A = A_i \left(\mathrm{d}X^i + \tilde{\partial}^i\right) + A^i \left(\mathrm{d}\widetilde{X}_i + \partial_i\right) \ .$$

This is a **DFT vector** (Deser and Saemann, arXiv:1611.02772 [hep-th]).

(b) Projecting from the 'large' Courant algebroid

The generalized vector of the 'large' CA ($E = L_+ \oplus L_-$) is given as

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$$A = A_i \left(\mathrm{d}X^i + \tilde{\partial}^i\right) + A^i \left(\mathrm{d}\widetilde{X}_i + \partial_i\right) \ .$$

This is a **DFT vector** (Deser and Saemann, arXiv:1611.02772 [hep-th]).
This is systematically achieved by a **projection** to the subbundle L_+ of E by introducing the bundle map

$$\mathrm{p}_+ : E \longrightarrow L_+ \ , \quad (\mathbb{A}_V, \mathbb{A}_F) \longmapsto \mathbb{A}_+ := A \ .$$

(b) Projecting from the 'large' Courant algebroid

Applying the projection twice to the standard Courant bracket,

$$\begin{aligned} \mathsf{p}_+\big([\mathsf{p}_+(\mathbb{A}), \mathsf{p}_+(\mathbb{B})]_E\big) &= \big(A^K\,\partial_K B^J - \tfrac{1}{2}\,A^K\,\partial^J B_K - \{A \leftrightarrow B\}\big)e^+_J \\ &= [\![A, B]\!]_{L_+} \ , \end{aligned}$$

we obtain the **C-bracket of DFT vectors**.

By double projecting on the generalized Lie derivative,

$$\begin{aligned} \mathsf{p}_+\big(\mathbb{L}_{\mathsf{p}_+(\mathbb{A})}\mathsf{p}_+(\mathbb{B})\big) &= \big(A^I\,\partial_I B^J - B^I\,\partial_I A^J + B_I\,\partial^J A^I\big)e^+_J \\ &= \mathsf{L}_A B \ , \end{aligned}$$

we obtain the **generalized Lie derivative in DFT**.

Note that $[\mathsf{L}_C, \mathsf{L}_A] = \mathsf{L}_{[\![C,A]\!]_{L_+}}$ only if we impose the **strong constraint**: $\eta^{IJ}\,\partial_I f\,\partial_J g = 0$, for all fields f, g of DFT.

DFT algebroid

Definition.
Let L_+ be a vector bundle of rank $2d$ over T^*M.
Let $[\![\cdot , \cdot]\!]_{L_+} : \Gamma(L_+) \otimes \Gamma(L_+) \to \Gamma(L_+)$ be a skew-symmetric bracket (C-bracket).
Let $\langle \cdot , \cdot \rangle_{L_+} : \Gamma(L_+) \otimes \Gamma(L_+) \to C^\infty(T^*M)$ be a non-degenerate symmetric form.
Smooth bundle map $\rho_+ : L_+ \to T(T^*M)$.

$\Rightarrow (L_+, [\![\cdot , \cdot]\!]_{L_+}, \langle \cdot , \cdot \rangle_{L_+}, \rho_+)$ is a **Double Field Theory (DFT) algebroid**, satisfying 3 properties.

DFT algebroid properties

$$(1)\quad \langle \mathcal{D}_+ f, \mathcal{D}_+ g\rangle_{L_+} = \frac{1}{4}\,\langle \mathrm{d}f, \mathrm{d}g\rangle_{L_+} \quad \text{(strong-constraint-related)}$$

$$\begin{aligned}(2)\quad \rho_+(C)\langle A, B\rangle_{L_+} \quad &= \quad \langle [\![C, A]\!]_{L_+} + \mathcal{D}_+\langle C, A\rangle_{L_+}, B\rangle_{L_+} \\ &\quad\; +\langle A, [\![C, B]\!]_{L_+} + \mathcal{D}_+\langle C, B\rangle_{L_+}\rangle_{L_+}\end{aligned}$$

(compatibility condition)

$$(3)\quad [\![A, f\, B]\!]_{L_+} = f\,[\![A, B]\!]_{L_+} + \big(\rho_+(A)f\big)\,B - \langle A, B\rangle_{L_+}\,\mathcal{D}_+ f \quad \text{(Leibniz)}$$

for $A, B, C \in \Gamma(L_+)$, functions $f, g \in C^\infty(T^*M)$,
derivative $\mathcal{D}_+ : C^\infty(T^*M) \to \Gamma(L_+)$ defined through
$\langle \mathcal{D}_+ f, A\rangle_{L_+} = \frac{1}{2}\,\rho_+(A)f$.

DFT algebroid properties in local expressions

Introduce a local basis for sections of L_+, e_I where $I = 1, \ldots, 2d$:

Operations:
$[e_I, e_J] = \eta_{IK}\, \eta_{JL}\, T^{KLM}\, e_M, \quad \langle e_I, e_J \rangle = \eta_{IJ},$
$\rho(e_I) f = \rho^L{}_I\, \partial_L f, \quad \mathcal{D} f = \mathcal{D}^I f\, e_I = \frac{1}{2}\, \rho^K{}_L\, \partial_K f\, \eta^{LJ}\, e_J,$
where components of anchor ρ, $(\rho^I{}_J) = (\rho^i{}_j, \rho^{ij}, \rho_i{}^j, \rho_{ij})$ for $i = 1, \ldots, d$.

In local coordinates, the *strong-constraint-related* property is

$$\langle \mathcal{D} f, \mathcal{D} g \rangle_{L_+} = \frac{1}{4} (\rho^K{}_I\, \eta^{IJ}\, \rho^L{}_J)\, \partial_K f\, \partial_L g = \frac{1}{4} \eta^{KL}\, \partial_K f\, \partial_L g \ ,$$

where

$$\rho^K{}_I\, \eta^{IJ}\, \rho^L{}_J = \eta^{KL} \ .$$

Strong constraint

Homomorphism is modified.

Modification is controlled by the strong constraint:

$$\rho_+[\![A,B]\!]_{L_+} = [\rho_+(A), \rho_+(B)] + \mathrm{SC}_\rho(A,B) \ ,$$

where

$$\mathrm{SC}_\rho(A,B) = \big(\rho_{L[I}\,\partial^K \rho^L{}_{J]}\,A^I\,B^J + \tfrac{1}{2}\,(A^I\,\partial^K B_I - B^I\,\partial^K A_I)\big)\,\partial_K \ .$$

This SC_ρ term vanishes upon imposing the strong constraint, i.e. $\partial^K(\dots)\partial_K(\dots) = 0$.

Strong constraint

Jacobi is modified.

Modification is controlled by the strong constraint:

$$\begin{aligned} [\![[\![A, B]\!]_{L_+}, C]\!]_{L_+} + \text{cyclic} &= \mathcal{D}_+ \mathcal{N}_+(A, B, C) + \mathcal{Z}(A, B, C) \\ &\quad + \mathrm{SC}_{\mathrm{Jac}}(A, B, C) \ , \end{aligned}$$

where $\mathcal{N}_+(A, B, C) = \frac{1}{3} \langle [\![A, B]\!]_{L_+}, C \rangle_{L_+} + \text{cyclic}$,
and $\mathcal{Z}_{IJKL} = 4\, \rho^M{}_{[L}\, \partial_{\underline{M}} T_{IJK]} + 3\, \eta^{MN}\, T_{M[IJ}\, T_{KL]N}$, and

$$\begin{aligned} \mathrm{SC}_{\mathrm{Jac}}(A, B, C)^L &= -\tfrac{1}{2} \left(A^I\, \partial_J B_I\, \partial^J C^L - B^I\, \partial_J A_I\, \partial^J C^L \right) \\ &\quad - \rho_{I[J}\, \partial_M \rho^I{}_{N]} \left(A^J\, B^N\, \partial^M C^L \right. \\ &\quad \left. -\tfrac{1}{2}\, C^J\, A^K\, \partial^M B_K\, \eta^{NL} + \tfrac{1}{2}\, C^J\, B^K\, \partial^M A_K\, \eta^{NL} \right) \\ &\quad + \text{cyclic} \ , \end{aligned}$$

which vanishes upon imposing the strong constraint.

DFT properties in local expressions

Introduce a local basis for sections of E, e^I where $I = 1, \ldots, 2d$.

In local coordinates, the DFT properties are

$$
\begin{aligned}
\eta^{IJ}\, \rho^K{}_I\, \rho^L{}_J &= \eta^{KL} \,, \\
2\rho^L{}_{[I}\, \partial_{\underline{L}} \rho^K{}_{J]} - \eta^{MN}\, \rho^K{}_M\, \hat{T}_{NIJ} &= \rho_{L[I}\, \partial^K \rho^L{}_{J]} \,, \\
4\, \rho^M{}_{[L}\, \partial_{\underline{M}}\, \hat{T}_{IJK]} + 3\, \eta^{MN}\, \hat{T}_{M[IJ}\, \hat{T}_{KL]N} &= \mathcal{Z}_{IJKL} \,,
\end{aligned}
$$

where $\hat{T}_{IJK}$ gives the fluxes of Double Field Theory given

$$
\rho^I{}_J = \begin{pmatrix} \delta^i{}_j & \beta^{ij} \\ B_{ij} & \delta_i{}^j + \beta^{jk}\, B_{ki} \end{pmatrix} .
$$

Conclusion

Geometric origin of Double Field Theory.

Method of Doubling-Splitting-Projecting:

Large Courant algebroid $\xrightarrow{\mathrm{p}_+}$ Double field theory $\xrightarrow{\textit{strong constraint}}$ canonical Courant algebroid

- Double Field Theory is an example of the pre-DFT algebroid:

$$\text{Courant algebroid} \xrightarrow{\cancel{\mathit{Jacobi}}} \text{Pre-Courant algebroid} \xrightarrow{\cancel{\mathit{Homomorphism}}} \text{Ante-Courant algebroid} \xrightarrow{\cancel{\langle \mathcal{D}f, \mathcal{D}g \rangle = 0}} \text{Pre-DFT algebroid}$$

(Bruce and Grabowski, arXiv:1608.01585 [math-ph])
(Vaisman, arXiv:1203.0836 [math.DG])

Violating the properties simultaneously in a particular way:

$$\text{Double field theory} \xrightarrow{\mathit{strong\ constraint}} \text{Courant algebroid}$$

Thank You