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Dualities & Generalised Geometry
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Title: An Introduction to Generalised Dualities & Their Applications.

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- caveat emptor! This talk / transcription is not likely to have correct signs or factors of π. Please consult literature. Also bibliography here is - with apologies - partial.
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1. Introduction.

Dualities not just impotant in strings but also gauge theory, electromagnetism, statistical & condensed matter physics

e.g. bosonisation $\psi \leftrightarrow \phi$
Thirring $\leftrightarrow$ Sine-Gordon,

Interesting extension to fermionic 2d system

with non-Abelian Global symtry: Witten WZW

Here some hierachy of "duality" transf. in ST.

(1) Abelian. $R \leftrightarrow \frac{1}{R}$.

- $U(1)$ action $V = \partial_\theta$
- isometry conserved noether cnt.

$$d * J = 0 .$$

(2) Non-Abelian. [quevedo de la Ossa]

- G action $V_a = V_a{}^i \partial_i$
$$[\mathcal{L}_{V_a}, \mathcal{L}_{V_b}] = f_{ab}{}^c \mathcal{L}_{V_c}$$

- isometry $\Rightarrow$ conserved cnt.

$$d * J_a = 0$$

(3) Poisson-Lie. [klimcik severa]

- G action $V_a = V_a{}^i \partial_i$

- non-isometric. $\Rightarrow$ non-conserved cmt
but special structure

$$d * J_a = \tilde{f}^{bc}{}_a (*J)_b \wedge (*J)_c$$

$\tilde{f}$ "dual Lie algebra"

Still dualisable!

Both Non-Ab T & PL are <u>old</u> ideas from mid 90s. Why talking about them now?

⊕ Applications of Non-Ab T to holography (see talks Lozano; Nunez)

⊕ Applications of PL to integrability (notable Delduc et al) & new integrable models

- η [Klimcik; Delduc Magro Vicedo, ...]
- λ [Sfetsos, Hollowood Miramontes .....].

Again I reiterate apologies for an incomplete biblio.

2. Non-Abelian T-duality

We are looking at strong σ-models

$$S = \int_\Sigma d^2\sigma \; \partial_+ x^i \left(G_{ij}(x) + B_{ij}(x)\right)\partial_- x^j$$

light cone $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$

x^i coordinates on target space with <u>metric</u> G & NS two-form B (perhaps locally define), with $H = dB$.

case where target space has G-action

for today <u>simplest example</u>. toy model target space is a group mfd G.

instead of x^i better to work with group valued map $g: \Sigma \longrightarrow G$.

Principal Chiral Model.

(1) $$S = \frac{-\kappa^2}{4\pi} \int d^2\sigma \; \mathrm{Tr}\left(g^{-1}\partial_+ g, g^{-1}\partial_- g\right)$$

$$[T_a \; T_b] = f_{ab}{}^c T_c \qquad \mathrm{Tr}\, T_a T_b = -\delta_{ab}.$$

target space is G (no B-field for now).

Very special theory

① $G_L \times G_R$ global symmetry.

② non-conformal κ runs & is Asym. Free; in IR theory has a dynamic mass scale. 2d "proto-QCD".

③ Can be a subsector of a ST. eg $G = SU(2)$ PCM $\subset$ $AdS_3 \times S^3$ + RR flux.

④ <u>Integrable</u>!

Classical (& that is what I use here) integrability $\Rightarrow$ $\exists$ Lax formulation

$$(\Delta) \begin{cases} \text{eqm} & \partial_+ J_- + \partial_- J_+ = 0 \\ \text{bianch} & \partial_+ J_- - \partial_- J_+ = -[J_+ J_-] \end{cases}$$

with $J_\pm = g^{-1} \cdot \partial_\pm g$

encoded in $G^{\mathbb{C}}$ valued

$$\underbrace{\mathcal{L}_\pm(z)}_{\text{spectral p-r}} = \frac{z}{z \pm 1} J_\pm$$

$$(\Delta) \Leftrightarrow [\partial_+ + \mathcal{L}_+ , \partial_- + \mathcal{L}_-] = 0 \quad \forall z$$

$T(z) = \mathrm{Pexp} \int \mathcal{L}_\sigma$ conserved.

$\Rightarrow \infty$ conserved charges.

(actually Yangian x Yangian).
(also $\exists$ h.s. local conserved charges which in some way are sufficient for integrability)
see Goldschmidt Witten.

Quant integrable $\rightarrow$ [Zamolodchikov]
exact S-matrix obeying factorised scattering axioms.
Very nice result using Bethe ansatz vs perturbation theory with background field [hassenfratz Niedermeyer] exact mass gap M in terms of Λ!

Can we dualise?

idea: $J_\mu = \varepsilon_{\mu\nu} \partial^\nu \Phi$.

can find

$$S' = -\kappa^2 \int \mathrm{Tr}(\partial\Phi)^2 + [\Phi \, \partial_\mu \Phi] \partial_\nu \Phi \, \varepsilon^{\mu\nu}$$

same eqm but [Nappi]

1. not a canonical trans
2. κ doesn't have same RG
3. not quant integrable (particle production).

Can we do better? Yes [BUSCHER]

3 step procedure.

STEP 1) Gauge G_L global sym

$$\partial g \rightarrow D g = \partial g + A g$$

st invt $\quad g \mapsto h^{-1}(\sigma,\tau)\, g\, h$
$\quad A \mapsto h^{-1} A h + h^{-1} dh$.

STEP 2) Enforce flat connection.

$F_{+-} = [D_+ \; D_-]$ field strength

add lagrange multiplier

$$\mathrm{Tr}\, v \cdot F_{+-} = v_a F^a_{+-}$$

idea int out $v \Rightarrow F=0 \Rightarrow A = h^{-1}dh$

$\Rightarrow g^{-1}Dg = (hg)^{-1}\partial(hg)$

redefine (gauge fix) $hg = g'$. then recover original PCM (1).

note in this route

(2) $$A_\pm = g^{-1}\partial_\pm g .$$

STEP 3) int by parts & solve for A gauge fix $g = e$

(3) $$\begin{cases} A_+ = M^{-T}\partial_+ v \\ A_- = -M^{-1}\partial_- v \end{cases}$$

with $M_{ab} = \kappa^2 \delta_{ab} + f_{ab}{}^c v_c$

find dual model

(NABT) $$\hat{S} = \int \partial_+ v_a (M^{-1})^{ab} \partial_- v_b$$

<u>Comments</u>

- combine (2) & (3)

(4) $$\begin{aligned} A_+ &= g^{-1}\partial_+ g = M^{-T}\partial_+ v \\ A_- &= g^{-1}\partial_- g = -M^{-1}\partial_- v \end{aligned}$$

This defines a canonical trans. (symplectomorphism). Worldsheet derivatives mapped.

- Thus classical integrability holds in $\hat{S}$ (also true for coset & super cosets [Sfetsos; Borsato Wulff].
Indeed the Lax for the dual is just

$$\hat{\mathcal{L}}_\pm = \frac{2}{z \pm 1} A_\pm$$ ← on-shell use (3).

- Isometries are typically lost; anything that doesn't commute with G (including eg supersym) but recovered non-locally.

- Note map of worldsheet derivatives (not coordinates). Also see left & right (ie ∂_+ & ∂_-) transform differently.

<u>Example</u> round S^3 = PCM on SU(2)

dual G_L ⇒ 3 lag multiple $v_1 v_2 v_3$ nicer in spherical coordinates (r, θ, ϕ).

$$\hat{ds} = \frac{dr^2}{\kappa^2} + \frac{r^2\kappa^2}{r^2+\kappa^4} ds_2^2$$ ← residual S^2

$$\hat{B} = \frac{r^3}{r^2+\kappa^4} \sin\theta \, d\theta \wedge d\phi$$ ← unbroken G_R isometry

$$\hat{\Phi} = \phi_0 - \frac{1}{2}\log(r^2+\kappa^4)$$ ← from gaussian elimination $A_+ M A_- =$

Open Questions

- robust in α'?
- robust in g_s?
- non-compactness ("r direction in example)

actually Giveon & Rocek ⇒ holonomies of
Verlinde & Rocek A around cycles on Σ, prevent an exact CFT equivalence.

(or at least to make NabT an exact CFT equivalence would require some additional input).

but still an interesting map perhaps from CFT → CFT'.

3. Holographic Applications

holography gives a second life to SUGRA as defining gauge theories,

Now since g_s correction are like $1/N$, and we are working in a SUGRA limit the first two open questions are not relevant.

Perhaps this transformation still has utility?

Nice precedent – that motivated me – was

"fermionic-T-duality" used by Berkovits & Maldacena (Alday-Maldacena) to provide origin of Wilson loop/amplitude @ strong coupling. There the same issues arise.

Idea use Non-Ab-T as a solution generating technique in type II SUGRA.

Need to couple to RR fields. Do either

GS ~ quite a mess [Borsato Wulff]
Pure Spinor ~ requires some sophistik [Benichou; Sfetsos Siampos DCT]

or "boot strap" from NS sector [Sfetsos Thompson]. We mimic what happens for Abelian T-duality [Hassan]

• idea: left & right movers couple to different frame fields for same geometry.

$$e_L = -M^{-1}dv \qquad e_R = M^{-T}dv.$$

(see eqn (4)).

<u>But</u> these must be related!

$$e_L = \Lambda e_R \qquad \Lambda = -M^{-1}M^T$$

$$\Lambda^T = \Lambda^{-1} \quad \rightsquigarrow \quad \text{Lorentz transf}^n.$$

Acts on spinors via Clifford

$$\Omega^{-1} \Gamma \Omega = \Lambda^i{}_j \Gamma^j$$

RR fields in democratic form

IIA $\not{F} = F_1 + F_3 + F_5 + *F_3 + *F_7$

IIB $\not{F} = F_0 + F_2 + F_4 + \text{duals}$.

Turn into bi-spinors eg

$$\not{F}_3 = F^{(3)}_{ijk} \Gamma^{ijk}.$$

Then T-duality rules:

$$e^{\hat{\Phi}} \hat{\not{F}} = e^{\Phi} \not{F} \Omega.$$

(only need to hit one spinor index because only one is associate to right movers).

can extract dual fields.

eg in our example

$$\Omega \sim \frac{\Gamma^{123} + v_a \Gamma^a}{\sqrt{1 + v^2}}$$

cf abelian T-duality $\Omega \sim \Gamma_\theta \Rightarrow F_p \to F_{p+1}$, $F_p \to F_{p-1}$

here Γ^{123} $F_p \to F_{p+3}$, $F_p \to F_{p-3}$!

so in eg $AdS_3 \times S^3$ $(\times T^4)$ in IIB we start with

$$F_3 = \text{vol}(AdS) + \text{vol}(S^3).$$

$$\not{F}_3 = \Gamma^{\tilde{1}\tilde{2}\tilde{3}} + \Gamma^{123}$$

$F_3 \to \hat{F}_0$ (mass term)

$F_3 \to \hat{F}_2 \sim \frac{r^3}{1+r^2} \text{vol}(S^2)$.

$F_3 \to F_4$, $F_7 \to F_4$: $F_4 \sim r\,dr \wedge \text{vol}\, AdS + \text{vol}\, T_4 + \text{hodge duals}$

in general lots of flux are active. solutions that are hard to guess.

notice $\det \Lambda = \pm 1$ depending on $\dim G$.

here $\dim G = \text{odd} \Rightarrow$ changed chirality.

Opens door to landscape of solutions

~ AdS 3 4 5 6

prototype Giatto Maldacena

eg first new AdS_6 sols (Lozano O Colgain Rodriguez-Gomez Montero).

~ G-structure classifications eg novel dynamical $SU(2)$ structures.

most tricky part: interpret apparent non-compactness.

Typical quiver a never ending linear chain:

$$(N) - (2N) - (3N) - \cdots - (pN) - \cdots ?$$

at $N = \infty$ ok I suppose but for N finite it looks like an ∞ central charge.

Suggestion: Non-Ab T-duality is incomplete & requires some geometrical completion.

e.g. by terminating quiver with a flavour group.

Non-Ab T geometries appear to encode some penrose limit capturing generic feature of these long quivers.

More see talks Lozano Nunez.

interesting conjecture that the finite N effect is compensating for g_s corrections.

4. A comment on Doubled Worldsheet

DFT/EFT key theme of meeting. Nice connection here.

Revisit Buscher start with more general PCM

$$S = \int L_+^a E_{ab} L_-^b$$

$$E_{ab} = G_{ab} + B_{ab} \quad \text{const}$$

$$L_\pm = L_\pm^a T_a = g^{-1}\partial_\pm g \quad \text{left inv. 1-forms.}$$

instead of gauge fixing $g = e$ keep g and v & partial fix on A:

$$A_+ = A_- = \alpha \quad (\text{Rocek Tseytlin Abelian})$$

[Lorentz covariance broken but not much worse than in say axial gauge. However a covariant PST approach does exist (~~...~~ Driezen Sevrin Thompson) but we ignore for simplicity.]

result; quite messy

$$S_{double} = \int - \mathbb{L}_\sigma^A \mathcal{H}_{AB} \mathbb{L}_\sigma^A + \mathbb{L}_\sigma^A \eta_{AB} \mathbb{L}_\tau^B + \mathbb{L}_\sigma^A \Omega_{AB} \mathbb{L}_\tau^B$$

with $\mathcal{H} = \begin{pmatrix} G - BG^{-1}B & -BG^{-1} \\ G^{-1}B & G^{-1} \end{pmatrix}$

$\eta = \begin{pmatrix} 0 & 1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix} \qquad \Omega = \begin{pmatrix} 0 & 1\!\!1 \\ -1\!\!1 & 0 \end{pmatrix}$

ingredients of parahermitian geometry (Lust Osten ; Hassler ; Pezzella in this workshop)

here $\mathbb{L}^A = \begin{pmatrix} L^a \\ (g^{-1} dv\, g)_a \end{pmatrix}$

$= D_a{}^b(g^{-1})\, dV_b$

with $D_{ab}(g) = \mathrm{Tr}\, g^{-1} T_a g T_b$.

We have a doubled world sheet from which eqm are 1st order & can be integrated to give either S or $\hat{S}$!

now lets consider $\tilde{T}^a$ be $u(1)^{\dim G}$ generators and let

$\tilde{g} = \exp(v_a \tilde{T}^a)$

then

$l = \tilde{g} \cdot g$

gives $l^{-1} dl = \mathbb{L}^A T_A$

with $T_A = (T_a, \tilde{T}^a)$

we have an important structure: the (classical) Drinfeld double. (semi-abelian)

- Lie algebra $\mathfrak{D}$
- decomposition $\mathfrak{D} = \mathfrak{g} \oplus \underbrace{u(1)^{\dim \mathfrak{g}}}_{\tilde{\mathfrak{g}}}$
- inner product ad-invt.

$\eta_{AB} = \langle T_A, T_B \rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow$ $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are max. isotropic.

here $\tilde{\mathfrak{g}} = u(1)^{\dim \mathfrak{g}}$ but lets be open minded about this for later!

one final interesting feature. $[T_A\, T_B] = F_{AB}{}^C T_C$

$d(\Omega_{AB}\, \mathbb{L}^A \wedge \mathbb{L}^B) = F_{ABC}\, \mathbb{L}^A \wedge \mathbb{L}^B \wedge \mathbb{L}^C$

(that Ω is global seems to be only possible for semi-abelian case)

So more generally consider cf. Hull Reid-Edwards, Klimcik Severa

(5) $S_{\mathcal{H}} = \int_\Sigma -\mathcal{H}\, \mathbb{L}_\sigma \mathbb{L}_\sigma + \eta\, \mathbb{L}_\sigma \mathbb{L}_\tau$

$+ \int_{M_3} \underbrace{F_{ABC}}_{\text{WZ term}} \mathbb{L}^A \wedge \mathbb{L}^B \wedge \mathbb{L}^C$

$\partial M_3 = \Sigma$

encodes structure consts.

~~one~~ nice thing Derendinger Prezas [Sfetsos Siampos Thompson]

β-fn for $\mathcal{H}$:

$$\frac{d\,\mathcal{H}_{AB}}{d\log\mu} = (\mathcal{H}\mathcal{H} - \eta\eta)_{AC\,BF}(\mathcal{H}\mathcal{H} - \eta\eta)^{KD\,HE} \times F_{KH}{}^{C} F_{DE}{}^{F}.$$

RHS reminiscent of eqn of S.S. reduced DFT i.e. gauge SUGRA [see talk Hassler].

5 Poisson-Lie Duality

σ-model with G action but not iso-metric instead

(6) $$d * J_a = \tilde{f}^{bc}{}_a * J_b \wedge * J_c.$$

for G action gen'ed by $V_a = v_a{}^i \partial_i$

"noether current"

$$J_{\pm a} = v_a{}^i (G_{ij} \pm B_{ij}) \partial_\pm x^j .$$

(6) requires that

(7) $$\mathcal{L}_{V_a} E_{ij} = \tilde{f}^{bc}{}_a v_b{}^m v_c{}^n E_{mi} E_{jn}$$

now $[\mathcal{L}_{V_a}, \mathcal{L}_{V_b}] = f_{ab}{}^c \mathcal{L}_{V_c}$

requires that - hitting (7) with another $\mathcal{L}_{V_a}$:

(8) $$\tilde{f}^{ac}{}_k f_{fa}{}^l - \tilde{f}^{al}{}_k f_{fa}{}^c - \tilde{f}^{ac}{}_f f_{ka}{}^l + \tilde{f}^{al}{}_f f_{ka}{}^c - \tilde{f}^{\,cl}{}_a f_{fk}{}^a = 0$$

ugh $\rightarrow$ what does this mean?

Nicer treatment. consider

$$\delta : g \longrightarrow g \wedge g$$

$$\delta(T_a) \longmapsto \tilde{f}^{bc}{}_a T_b \wedge T_c.$$

$\leftarrow$ skew symtric tensor product $a\otimes b - b\otimes a$.

Then (8) says δ is a <u>one-cocyle</u> valued in $g \wedge g$:

i.e. $0 = d\delta \equiv ad_x \delta(y) - ad_y \delta(x) - \delta([x\,y])$

ext deriv acting in a Lie algebra

so now δ is a one-cocyle for g & obeys its co-jacobi (Jacobi for $\tilde{f}$) so

(g, δ) is a Lie-bialgebra.

Now a <u>poisson-Lie</u> group is a group equipped with a Poisson Bracket obeying a comptability with group action $\Rightarrow$ extra structure on $T^*_e G$.

infinitesimal version of PL group
= Lie bialgebra.

Hence the name P.L. symmetry.

alternatively consider $T_A = (T_a \; \tilde{T}^a)$

with $[T_a \; T_b] = f_{ab}{}^c T_c$
$[\tilde{T}^a \; \tilde{T}^b] = \tilde{f}^{ab}{}_c \tilde{T}^c$

then $\mathfrak{D}$ $= \mathrm{span}_{\mathbb{R}}(\{T_A\})$ has a Lie algebra
$[T_A \; T_B] = F_{AB}{}^C \, T_C$ with mixed structure constants uniquely determined by demanding ad-invariance of

$$\eta_{AB} = \langle T_A \; T_B \rangle = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

Then co-cycle condition (8) is just a consequence of JACOBI for $F_{AB}{}^C$.

Nice structure but ... do examples exist. Yes!

Let $g \in G$ $(\tilde{g} \in \tilde{G}$ resp.$)$ $\dim G = d$
$L = g^{-1} dg$

Define viewing g as an element of $D = \exp \mathfrak{D}$

$$g^{-1} T_a \, g = a_a{}^b \, T_b$$

$$g^{-1} \tilde{T}^a g = b^{ab} \, T_b + (a^{-1})_b{}^a \, \tilde{T}^b$$

Define
$$\Pi^{ab} \equiv b^{ca} \, a_c{}^b$$

[for experts Π is the poisson bi-vector on G obeying

$$\Pi(gh) = a^T(h)\, \Pi(g)\, a(h) + \Pi(h) \;]$$

Let $E_0 = G + B$ be d^2 consts.

$$S = \int L_+^a \left(E_0^{-1} + \Pi(g) \right)^{-1}_{ab} L_-^b .$$

[Klimcik Severa] this obeys the PL condition!

Swap $\tilde{g}$ & g

$$\tilde{S} = \int \tilde{L}_+ \left(E_0 + \tilde{\Pi} \right)^{-1} \tilde{L}_-$$

also obeys PL condition with f & $\tilde{f}$ exchanged.

- S and $\tilde{S}$ are canonically equivalent & said to be PL duals.

- both S and $\tilde{S}$ ~~follow~~ follow from a 1st order action. for $l \in D$.

In fact we saw it already in eqn 5 !

scheme of canonical equivalences.

$$\begin{tikzcd}
 & S_{\mathcal{H}}[l] \arrow[dl, leftrightarrow, "l = g\tilde{g}"'] \arrow[dr, leftrightarrow, "l = \tilde{g}g"] & \\
S[g] \arrow[rr, leftrightarrow, "\text{PL dual}"'] & & \tilde{S}[\tilde{g}]
\end{tikzcd}$$

Downside: typical geometries are rather messy.

As an example of a drinfeld double:

$g^{\mathbb{C}}$ viewed as a real lie-algebra has a iwsawa decomposition

$$g^{\mathbb{C}} = \underset{\text{max compact sub.}}{g} + (a + n)$$

so $D = G \underbrace{AN}_{}$ typically some triangular matrices.

6. Yang-Baxter & η-models.

Some nice application & simple examples of PL models.

Define R matrix solution to.

$$\text{mCYBE}: \quad [Rx, Ry] - R\left([Rx\, y] + [x\, Ry]\right) = -c^2 [x, y] \quad \forall x, y \in g$$

- wlog $c^2 = -1, 0, 1$.
- here R assumed to be skew.

important eqn; when solve it says:

$$[x, y]_R \equiv [Rx, y] + [x, Ry]$$

obeys Jacobi & defines a second bracket structure over g, (as a vector space). again see a bi-algebra structure.

another way of thinking: algebra homomorphism

$$[(R \pm c)x, (R \pm c)y] = (R \pm c)[x, y]_R$$

Notation sometimes natural to work with

$r \in g \wedge g$ $\qquad r_{12} = r^{ab}(T_a \otimes T_b - T_b \otimes T_a)\frac{1}{2}$

with $R(T_a) = R_a{}^b T_b$ related by

$$R(x) = \mathrm{Tr}_2\,(r \cdot \mathbb{1} \times x)$$

trace in second part of tensor product.

Casimir t s.t. $\mathrm{Tr}_2(t \cdot (\mathbb{1} \times x)) = x$

then

$$r^\pm = r \pm t.$$

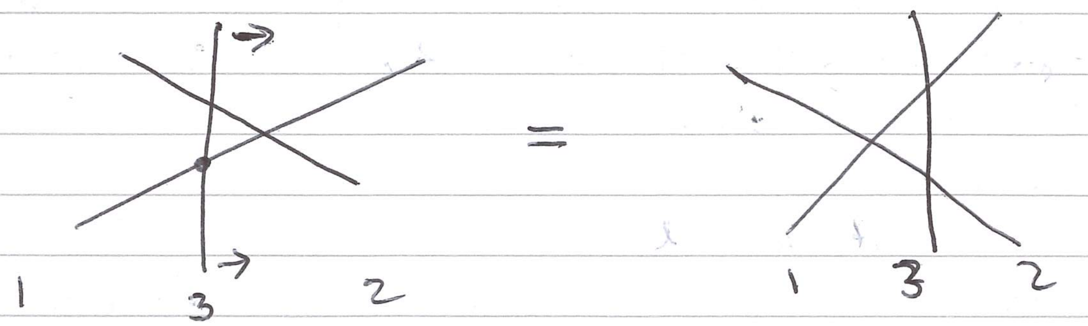
MCYBE is equ. to

$$0 = [r^\pm_{12}, r^\pm_{13}] + [r^\pm_{12}, r^\pm_{23}] + [r^\pm_{13}, r^\pm_{23}]$$

where $r_{13} = r^{ij}\, T_i \otimes \mathbb{1} \otimes T_j$ etc.

† aside where does CYBE come from.

Quantum YB asserts that scattering is factorisable and obeys



each scattering related to an "S-matrix" R

$$R_{12}\, R_{13}\, R_{23} = R_{23}\, R_{13}\, R_{12}.$$

This underpins quantum integrable systems.

suppose $R = \mathbb{1} + \hbar\, r + O(\hbar^2)$

then expanding QYB gives CYB $(c=0)$. †

YB model:
Klimcik proposed.

$$(9) \quad S_\eta = \frac{1}{t} \int \left(\partial_+ g g^{-1}, \frac{1}{\mathbb{1} - \eta R}\, \partial_- g g^{-1}\right)$$

real p-r

Integrable (Klimcik $c^2 = -1$ but Yoshida et al for $c^2 = +1, 0$) also.

Lax

$$\mathcal{L}_\pm(z) = \frac{1}{1 \pm z}\left(1 \mp \frac{\eta z}{1 \pm \eta R}(-c^2 \eta \pm R)\right) \partial_\pm g g^{-1}.$$

* case $c^2 = -1$

simplest e.g. $g = su(2)$

$R: (T^1, T^2, T^3) \rightarrow (-T^2, T^1, 0).$
$R^3 = -R.$

target space is a squashed S^3

$$ds^2 = R_1^2 + R_2^2 + (1 + \eta^2) R_3^2$$

right invt forms!

- G_L broken to $u(1)$ but...

recovered non-locally as a classical (p.b. version) of Quantum Group

re $\exists$ charges Q^3 (local noether for $U(1)$)
& $Q^\pm$ non-local expressions

such that

$$\{Q^+ \; Q^-\} = \frac{q^{Q^3} - q^{-Q^3}}{q - q^{-1}}$$

with $q \in \mathbb{R} \sim \exp(t\eta)$ ← RG invart!

- Delduc Magro Vicedo.
  ↳ true for $\mathbb{Z}_2$ graded cosets and $\mathbb{Z}_4$ graded semi symmetric space i.e. $AdS_5 \times S^5$ superstring.

Lots of excitement: a quantum group deformation of holography?!

caveat (see talk Wulff & Hoare)

$$I^c = f_{ab}{}^c r^{ab} \neq 0$$

in this case g_R is said to be non-unimodular.

η-deformed is not solution of sugra. → but a solution of a very constrained modification to SUGRA

why – η theory scale but not Weyl invt [Roiban Hoare Tseytlin Arutyunov]

Connection to PL? Yes – write (9) in terms of left invt forms & it is a PL model

$$\Pi = ad_g \circ R \circ ad_{g^{-1}} - R$$

$$E_0 = \frac{1}{\eta} \mathbb{1} - R.$$

Double is $g^{\mathbb{C}}$!

notice that $\delta = dr$ "co-boundary" Lie-bialgebra structure.

$c^2 = 0$

here r solves CYBE.

- very nice encodes things like TsT deformations!
  in general the setting here is actually semi-abelian double so some connection to (Non-abelian) T-duality?

r^{ab} is invertible on a sub-algebra & define $\omega_{ab} \in H^2(g, \mathbb{R})$

CYBE $\Rightarrow$ ω is a 2-cocycle.

the YB defmtn: effectively turns on a ~~large~~ gauge transfmtn.

$$\delta B = \underset{\substack{\wr \\ \text{defmtn} \\ \text{p-r related to } \frac{1}{\eta}}}{\beta}\ \omega_{ab}\, L^a \wedge L^b$$

T-duality with a large gauge transfmtn generates the β defmtn!
[Borsato Wulff]

equivalently: ω_{ab} labels central extensions:

$$[t_a\ t_b] = f_{ab}{}^c\, t_c + \omega_{ab} \underset{\substack{\wr \\ \text{central} \\ \text{elemt.}}}{Z}$$

[Tseytlin Hoare]; T-dualise with a central extension.

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case $c^2 > 0$ doesn't exist real R for G compact.

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7. λ-models.

Something apparently quite different.

A defmtion of NabT:

motivation: regulate the non-compactness
outcome: a class of integrable models.

$$\underset{\lambda = 0}{\text{WZW}} \xrightarrow[\text{RG}]{\qquad} \underset{\lambda \to 1}{\text{NabT. of PCM}}.$$

- encode quantum group defmtions at root of unity.
- extend to $\mathbb{Z}_2$ & $\mathbb{Z}_4$ graded symmetric & semi symmetric spaces. (Hollowood et al)
  applicat to $PSU(2,2|4)$
- multi-pr defmtions
- METHOD (ok for bosonic case, modificat needed for super cosets).

Revisit Buscher.

(1) PCM $\quad S_{PCM}[\tilde{g}] = \kappa^2 \int \tilde{g}^{-1}\partial_+\tilde{g} \cdot \tilde{g}^{-1}\partial_-\tilde{g}$

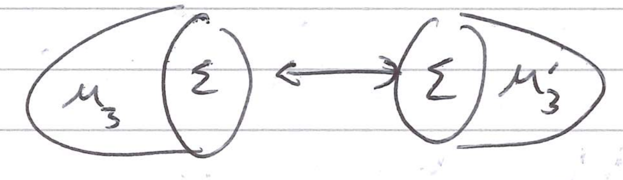
gauge G_L $\quad \tilde{g} \to h\tilde{g}$ $\quad \partial \to D = \partial + A$

(2) Add not lag multiplier but gauge WZW

a second group elt

$$S_{WZW}[g] = k \int_\Sigma (g^{-1}\partial_+ g, g^{-1}\partial_- g) + k \int_{M_3} (g^{-1}dg)^3$$

[recall with correct normalisation (not these ones!) k is quantised to be integer or half-integer, because we demand $e^{xp\, i S_{WZW}}$ is unambiguous under choice of M_3 st $\partial M_3 = \Sigma$.



join into a ball

$$\Delta S = S_{WZ}[g, M_3] - S_{WZ}[g\ M_3']$$

$$\sim k \int_B (g^{-1}dg)^3 .$$ should be integ multiple of 2π.]

Now gauge in WZW → careful how to gauge; not minimal coupling but anomaly free vector action

$$g \longrightarrow h^{-1} g h$$

$$S_{gWZW}[g, A] = S_{WZW}[g] + k \int A_- \partial_+ g g^{-1} - A_+ g^{-1} \partial_- g + A_- g A_+ g^{-1} - A_- A_+$$

(reader should check these signs done from memory sorry) →

consider now

$$S_{gPCM}(\tilde{g}, A) + S_{WZW}(g, A).$$

invt under $\tilde{g} \to h\tilde{g}$ $\quad g \to h g h^{-1}$.

(B) gauge fix $g = e$ (k always appears with equations!)

& int out $A_+ A_-$: $\boxed{\lambda = \frac{k}{k + \kappa^2}}$

$$S_\lambda[g] = S_{WZW}[g] + k \int \partial_+ g g^{-1} \frac{1}{\lambda^{-1} - Ad_g} g^{-1} \partial_- g .$$

- Integrable $\forall\ \lambda \in 0, 1$

lax $\quad \mathcal{L}_\pm = \frac{1}{1+\lambda} \frac{A_\pm}{1 \mp z}$ on-shell

let $\quad O_g = \frac{1}{1 - \lambda Ad_g}$

$$ds^2 = k \left(O_{g^{-1}} + O_g - 1 \right)_{ab} \ell^a \otimes \ell^b$$ left int

$$B = B_{WZW} + \frac{k}{2} \left(O_{g^{-1}} - O_g \right)_{ab} \ell^a \wedge \ell^b$$

$dB_{WZW} \sim k f_{abc} \ell^a \wedge \ell^b \wedge \ell^c$.

from gaussian ↪ $\Phi = \phi_0 + \frac{1}{2} \log \det O_{g^{-1}}$

- small λ :

$$S \sim S_{WZW} + \varepsilon \int J_+^a J_-^a$$

current-current perturbation
marginally relevant (schwarz).

seen before TSEYTLIN UNIVERSAL CLASS.

- $\lambda \to 1$ $\quad g = \mathbb{1} + \frac{v^a T_a}{k} + O\left(\frac{1}{k^2}\right)$

$k \to \infty$ κ^2 fixed.

$$S_\lambda \longrightarrow$$ Non-abelian T-dual of PCM !! of page 7.

- Quantum group deformation with

$$q = \exp i\pi/k$$

evidence in S-matrix proposal Hollowood Miramontes Schmidt matching TBA & QISM

reason is $\exists$ two commuting classical KM currents

$$\{ J^a_{\pm}(x) \, J^b_{\pm}(y) \} = f^{ab}{}_c J^c_{\pm} \, \delta(x-y) \pm \frac{k}{2\pi} \delta^{ab} \delta'(x-y) .$$

obtained from expansion of lax around special points. Note the currents are not holomorphic when $\lambda \neq 0$.

Extension to AdS_5 superstring - the deformation is 1-loop marginal (Appadu) (higher loops unknown). And gives a background that solves SUGRA (no modification) (Sfetsos Thompson Borsato Wulff Hoare Siebold etc).

Warning the backgrounds even for undeformed gWZW are ugly.

8. $\eta - \lambda$ Connection

At first sight quite different. but "scheme"

- Hoare Tseytlin
- Klimcik.

$$\begin{tikzcd}
\eta \arrow[d, leftrightarrow, "\text{PL-dual}"] & \\
\lambda^* \arrow[r, leftrightarrow, "\text{analytic}", "\text{continuation}"'] & \lambda
\end{tikzcd}$$

this is just the dual of the η mode continuation acts on same Euler angles & coupling prs :

$$\eta \longrightarrow i\,\frac{(1-\lambda)}{1+\lambda}$$
$$t \longrightarrow \;-\;-\;-$$

notice

$$q = \exp(\eta t) \longrightarrow q \exp\frac{i\pi}{k}.$$

More algebraic

$$\mathcal{D} = \mathfrak{g}^{\mathbb{C}} \longrightarrow \mathcal{D}' = \mathfrak{g} + \mathfrak{g}$$
η - model $\qquad\qquad\qquad$ λ - model.

universal perspective is the "$\mathcal{E}$ model". where $\mathcal{E} = \mathcal{R}\eta$.

Klimcik.

Notice $\mathcal{D}' = \mathfrak{g} + \mathfrak{g}$ here with $\mathfrak{g}_{diag}$ an isotropic subgroup; the decomp

$$\mathcal{D} = \mathfrak{g}_{diag} + \mathfrak{g}^{\perp}$$

$\mathfrak{g}^{\perp}$ is isotropic but <u>not</u> a sub so beyond drinfeld double.