Generalized Kähler Geometry and current algebras in $N=2$ superconformal WZW model.

S.E. Parkhomenko

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Conformal supersymmetric $\sigma$-models with extended supersymmetry are important in the construction of superstring compactification.

$N = 2$ world-sheet supersymmetry $\Rightarrow$ complex Kähler $\sigma$-model target space.

In a more general case the background geometry may include an antisymmetric $B$-field. The corresponding 2-dimensional supersymmetric $\sigma$-model have a second supersymmetry when the target-space has a bi-Hermitian geometry (Gates-Hull-Roček geometry).

Bi-Hermitian geometry $\Leftrightarrow$ generalised Kähler geometry (GKG).

Geometric data of bi-Hermitian geometry:

\[
\{M, g, B, J_l, J_r\} \quad J_l^2 = J_r^2 = -1 \\
g(J_l, J_l) = g(J_r, J_r) = g(\cdot, \cdot)
\] (1)
• The $N = 2$ supersymmetric WZW models on the compact groups provide a large class of exactly solvable quantum conformal $\sigma$-models whose targets supports simultaneously GKG geometry and affine Kac-Moody superalgebra structure causing their exact solution. Therefore it is important to know the exact relation between the GK geometry data and affine Kac-Moody superalgebra conserved currents. In a more general context it would be also important to see if there are GKG targets which allow the $W$-superalgebra conserved currents. Perhaps Kazama-Suzuki coset models can be related to such targets.
1. Bi-Hermitian data and double Lie group structure in $N = 2$ supersymmetric WZW model on the group $G$.

- There is a left and right actions of the complex groups $G_{l,r}^\pm$ on the group $G$ so the elements of $G$ can be parametrized by the elements from the complex group $G_{l}^\pm$ (or $G_{r}^\pm$):

$$\{G^C, G_l^+, G_l^-\} \Leftrightarrow \{M = G, g, B, J_l, J_r\} \Leftrightarrow \{G^C, G_r^+, G_r^-\}$$ (2)

- Thus, the compact group $G$ is endowed with natural complex coordinates. We employ these complex coordinates to rewrite the WZW action on $G$ in the form of supersymmetric $\sigma$-model action on the super-world-sheet parametrized by $(\sigma_{0,1}, \theta_{0,1})$:

$$S = \frac{k}{2} \int d^2\sigma d^2\theta E_{ij} \rho^i_+ \rho^j_-, \ E_{ij} = i\Omega_{ik}(J_r)^k_j$$ (3)

- $\Omega_{ij}$ is the Semenov-Tian-Shansky simplectic form, $\rho_\pm = D_\pm h^+(h^+)^{-1}$, $h \in G_r^+$ and

$$g_{ij} = \frac{1}{2}(E_{ij} + E_{ji}), \ B_{ij} = -\frac{1}{2}(E_{ij} - E_{ji})$$ (4)
2. Canonical variables and Kac-Moody superalgebra currents.

- Having the action, Hamiltonian formalism allows to find out the canonical coordinates and momenta and express the conserved Kac-Moody superalgebra currents by the canonical variables:

\[ L^i = -\frac{1}{2}(\Pi(\rho^i) + \nu k J_r \rho^i), \quad R^i = -\frac{1}{2}(\Pi(\tilde{\rho}^i) - \nu k J_l \tilde{\rho}^i) \]

\[ \rho_1 = D_1 h^+(h^+)^{-1}, \quad \tilde{\rho}_1 = D_1 \tilde{h}^+ (\tilde{h}^+)^{-1}, \quad \tilde{h} \in G_f^+, \]

\[ \Pi = \Pi^{ij} \rho_i^* \rho_j^* = \tilde{\Pi}^{ij} \tilde{\rho}_i^* \tilde{\rho}_j^* = \Omega^{-1}, \]

\[ D_1 = D_+ + D_- \quad (5) \]

\( \rho_i^* \) is a canonically conjugated to \( \rho^i \),

\( \tilde{\rho}_i^* \) is a canonically conjugated to \( \tilde{\rho}^i \).
\[ [\rho_i^*(\sigma_1, \theta_1), \rho_j^*(\sigma'_1, \theta'_1)] = \delta(\sigma_1 - \sigma'_1)\delta(\theta_1 - \theta'_1)\phi^k_{ij}\rho_k^*(\sigma'_1, \theta'_1) \]
\[ [\rho_i^*(\sigma_1, \theta_1), \rho_j^*(\sigma'_1, \theta'_1)] = D_1\delta(\sigma_1 - \sigma'_1)\delta(\theta_1 - \theta'_1)\delta^j_i - \delta(\sigma_1 - \sigma'_1)\delta(\theta_1 - \theta'_1)\phi^j_{ik}\rho^k(\sigma'_1, \theta'_1) \] (6)

\( \phi^k_{ij} \) are the structure constants of the Lie algebra of the group \( G^+ \times G^- \).