Magnetic monopoles in (noncommutative quantum) mechanics

Samuel Kováčik

Dublin Institute for Advanced Studies

SK, Prešnajder: arXiv/1604.05968

Corfu, 17.9.2017

1. Magnetic monopoles

Maxwell equations

$$\begin{aligned}
\text{div } \vec{E}(\vec{r},t) &= 4\pi\rho_E(\vec{r},t), \\
\text{div } \vec{B}(\vec{r},t) &= 0, \\
\text{rot } \vec{E}(\vec{r},t) &= -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}, \\
\text{rot } \vec{B}(\vec{r},t) &= \frac{\partial \vec{E}(\vec{r},t)}{\partial t} + 4\pi \vec{J}_E(\vec{r},t).
\end{aligned}$$

1. Magnetic monopoles

Maxbetter equations

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\text{rot } \vec{E}(\vec{r},t) &= -\frac{\partial\vec{B}(\vec{r},t)}{\partial t} - 4\pi\vec{J}_M(\vec{r},t),\\
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Dirac: Quantised Singularities in the Electromagnetic Field, (Proc. Roy. Soc. 60 (1931))

't Hooft: Magnetic Monopoles in Unified Gauge Theories, (Nucl.Phys. B79 (1974))

Preskill: Magnetic Monopoles, (Ann.Rev.Nucl.Part.Sci. 34 (1984))

Brief summary of quantum theory

Zwanziger: Exactly soluble nonrelativistic model of particles with both electric and magnetic charge, Phys. Rev. 176 (1968)

$$
\begin{aligned}
\left[\hat{x}_i, \hat{x}_j\right] &= 0 &\leftrightarrow\quad& [\hat{X}_i, \hat{X}_j] = \lambda^2 \varepsilon_{ijk} \hat{L}_k, \\
\left[\hat{x}_i, \hat{p}_j\right] &= i\delta_{ij} &\leftrightarrow\quad& [\hat{X}_i, \hat{V}_j] = i\delta_{ij}\left(1 - \lambda^2 \hat{H}_0\right), \\
\left[\hat{p}_i, \hat{p}_j\right] &= i\mu\varepsilon_{ijk}\frac{\hat{x}_k}{r^3} &\leftrightarrow\quad& \left[\hat{V}_i, \hat{V}_j\right] = i\frac{-\kappa}{2}\varepsilon_{ijk}\frac{\hat{X}_k}{\hat{r}(\hat{r}^2 - \lambda^2)}. \\
\hat{C}_1 &= -q\mu &\leftrightarrow\quad& \hat{C}_1 = \frac{\kappa}{2}q, \\
\hat{C}_2 = q^2 &+ (\mu)^2(-2E) &\leftrightarrow\quad& \hat{C}_2 = q^2 + \left(\frac{\kappa}{2}\right)^2(-2E + \lambda^2 E^2).
\end{aligned}
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In quantum theory $\mu = eg \in \mathbb{Z}/2$ (Dirac Quantisation Condition)

2. (Even more) noncommutative quantum mechanics

NC relations

$$\boxed{[x_i, x_j] = 2i\lambda\varepsilon_{ijk}x_k}$$

- $i, j = 1, 2, 3$ and $\lambda \approx l_{Planck}$.
- Absence of very short distances and (dual to it) very large energies.
- Bosonic construction of NC coordinates $x_i = \lambda\sigma^i_{\alpha\beta}a^+_\alpha a_\beta$, $r = \lambda(a^+_\alpha a_\alpha + 1)$ where σ^i, the c/a operators satisfy

$$[a_\alpha, a^+_\beta] = \delta_{\alpha\beta}, \quad [a_\alpha, a_\beta] = [a^+_\alpha, a^+_\beta] = 0$$

Jabbari et. al. arXiv [hep-th] : 0110291
Gáliková, SK, Prešnajder: arXiv [math-ph] : 1510.04496

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2. Noncommutative quantum mechanics

Hilbert space $\mathcal{H}$

- Spanned on analytic functions equipped with a norm

$$\Psi \;=\; \Psi(a^+, a), \;\; \|\Psi\|^2 \;=\; 4\pi\,\lambda^2\,\mathrm{Tr}[\hat{r}\,\Psi^\dagger\,\Psi].$$

Special attention is paid to Ψ_0 that has

$$\#a = \#a^+$$

(makes sense for $\Psi(x) = \Psi(a^+\sigma a)$)

Operators in $\mathcal{H}$

- $\hat{H}_0\Psi = \frac{1}{2m\lambda r}\,[a^+_\alpha, [a_\alpha, \Psi]]$
- $\hat{L}_i\Psi = \frac{1}{2\lambda}[x_i, \Psi], \;\; [\hat{L}_i, \hat{L}_j] = i\varepsilon^{ijk}\hat{L}_k$
- $\hat{X}_i\Psi = \frac{1}{2}(x_i\Psi + \Psi x_i), \;\; \hat{r}\Psi = \frac{1}{2}\,(r\Psi + \Psi r)$

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2. Noncommutative quantum mechanics

Some of our results

H-atom:	$\left(\hat{H}_0 - \frac{q}{r}\right)\Psi_0 = E\Psi_0$	$\rightarrow E_n\lambda^2 = 1 - \sqrt{1 + (\alpha\lambda/n)^2}$
Planewaves:	$\hat{V}_3\Psi_0 = k\Psi_0$	$\rightarrow \Psi_0 \propto e^{ix_3 q}, k = \lambda^{-1}\sin q\lambda$
SO(4) structure:	$\hat{V}_4 = \lambda^{-1} - \lambda\hat{H}_0$	$\rightarrow \hat{V}_i^2 + \hat{V}_4^2 = \lambda^{-2}$
LHO:	$\left(\hat{H}_0 + \alpha r^2\right)\Psi_0 = E\Psi_0$	$\rightarrow$???
NC black holes	$\rho \propto e^{-r\lambda^{-1}}$	$\rightarrow$ Minimal r_h, maximal T_H

arXiv[math-ph]: 1302.4623, 1309.4614

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arXiv[quant-ph]: 1603.00292

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arXiv[math-ph]: 1309.4592

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arXiv[gr-qc]: 1504.04460

3. Generalized states

Hilbert space $\mathcal{H}_\kappa$

Let us now consider a generalized class of states $\Psi_\kappa(a, a^+)$ that have

$$\#a \neq \#a^+$$

Example

$$\Psi_{\kappa=1} = \Psi(x_i)a_1^+.$$

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Example

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What difference does it make?

$$[x_i, r] = 0 \to [\Psi(x_i), r] = 0 \to \hat{r}_L = \hat{r}_R.$$

This has to be replaced with

$$\hat{r}_L - \hat{r}_R = \lambda\kappa \neq 0.$$

3. Generalized states

Relations between (physical) operators for generalized states

$$
\begin{aligned}
\left[\hat{V}_i, \hat{V}_j\right] &= 0 + i\frac{-\kappa}{2}\varepsilon_{ijk}\frac{\hat{X}_k}{\hat{r}(\hat{r}^2-\lambda^2)} = i\hat{F}_{ij}, \\
\hat{V}_i^2 + \hat{V}_4^2 + \hat{\varphi}\hat{F}_{ab}^2 &= \lambda^{-2}, \\
\hat{C}_1 &= \frac{\kappa}{2}q, \\
\hat{C}_2 &= q^2 + \left(\frac{\kappa}{2}\right)^2(-2E + \lambda^2 E^2), \\
\varepsilon_{ijk}[\hat{V}_i, [\hat{V}_j, \hat{V}_k]] &= 0.
\end{aligned}
$$

3. Generalized states

Comparison

QM with MM ⠀⠀⠀ NC QM with $\kappa \neq 0$

$$
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\left[\hat{x}_i, \hat{x}_j\right] = 0 \quad &\leftrightarrow \quad [\hat{X}_i, \hat{X}_j] = \lambda^2 \varepsilon_{ijk} \hat{L}_k, \\
\left[\hat{x}_i, \hat{p}_j\right] = i\delta_{ij} \quad &\leftrightarrow \quad [\hat{X}_i, \hat{V}_j] = i\delta_{ij}\left(1 - \lambda^2 \hat{H}_0\right), \\
\left[\hat{p}_i, \hat{p}_j\right] = i\mu\varepsilon_{ijk}\frac{\hat{x}_k}{r^3} \quad &\leftrightarrow \quad \left[\hat{V}_i, \hat{V}_j\right] = i\frac{-\kappa}{2}\varepsilon_{ijk}\frac{\hat{X}_k}{\hat{r}(\hat{r}^2 - \lambda^2)}, \\
\hat{C}_1 = -\mu q \quad &\leftrightarrow \quad \hat{C}_1 = \frac{\kappa}{2} q, \\
\hat{C}_2 = q^2 + (\mu)^2(-2E) \quad &\leftrightarrow \quad \hat{C}_2 = q^2 + \left(\frac{\kappa}{2}\right)^2 (-2E + \lambda^2 E^2).
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Comparison

QM with MM $\quad\leftrightarrow\quad$ NC QM with $\kappa \neq 0$

$$\left[\hat{x}_i, \hat{x}_j\right] = 0 \quad \leftrightarrow \quad [\hat{X}_i, \hat{X}_j] = 0 + \lambda^2 \varepsilon_{ijk} \hat{L}_k,$$

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$$\hat{C}_1 = -\mu q \quad \leftrightarrow \quad \hat{C}_1 = \frac{\kappa}{2} q,$$

$$\hat{C}_2 = q^2 + (\mu)^2(-2E) \quad \leftrightarrow \quad \hat{C}_2 = q^2 + \left(\frac{\kappa}{2}\right)^2(-2E + \lambda^2 E^2).$$

$$\mu \in \mathbb{Z}/2 \leftrightarrow -\kappa/2 \in \mathbb{Z}/2.$$

Ordinary space (NNC)

$\mathbf{C}^2 \leftrightarrow \mathbf{R}^3$

- $a_\alpha^+, a_\alpha,\ \alpha = 1,2 \leftrightarrow x^i, i = 1,2,3$
- $\sqrt{\lambda} a_\alpha \to z_\alpha,\ \sqrt{\lambda} a_\alpha^+ \to \bar{z}_\alpha,\ [\ ,\ ] \to -i\{\ ,\ \}$
- $\bar{z}_\alpha, z_\alpha,\ \alpha = 1,2 \leftrightarrow x^i, i = 1,2,3$
- General: $\Psi(z,\bar{z})$, QM in R^3: $\psi(x_i) = \psi(\bar{z}\sigma^i z)$,
  QM with MM: $\Psi_\kappa = \psi(x_i) z_1^{\kappa_1} z_2^{\kappa_2},\ \kappa_1 + \kappa_2 = -\kappa$
- $\hat{V}^j \psi_\kappa = (-i\partial_j + \mathcal{A}^j)\psi_\kappa,\ \mathcal{A}_j = -\frac{i}{2r\xi}\sigma^j_{\gamma\delta} z_\delta (\partial_{z_\gamma}\xi)$

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$$
\begin{aligned}
\psi_0 &= \psi_0(\mathbf{x}) = \Phi(r,\varphi,\theta), \\
\psi_1 &= \psi_0(\mathbf{x})\bar{z}_1 = \Phi(r,\varphi,\theta)e^{-\frac{i}{2}\gamma}, \\
\psi_2 &= \psi_0(\mathbf{x})\bar{z}_1\bar{z}_2 = \Phi(r,\varphi,\theta)e^{-i\gamma}, \\
&\dots \\
\psi_\kappa &= \Phi(r,\varphi,\theta)e^{-i\frac{\kappa}{2}\gamma}.
\end{aligned}
$$

Summary

- In the bosonic approach to NC QM, you can describe monopoles states of **any** field strength by considering states with unequal number of c/a operators (for example of the form $\Psi = \varphi(x_i) a_1^{+\kappa_1} a_2^{+\kappa_2},\ x_i = \lambda a^+ \sigma^i a$)
- In the ordinary QM you can describe monopoles states of **any** field strength by formulating the theory on $\mathbb{C}^2$ instead of $\mathbb{R}^3$ and considering states with unequal number of $z, \bar{z}$ (for example of the form $\Psi = \varphi(x_i) \bar{z}_1^{+\kappa_1} \bar{z}_2^{+\kappa_2},\ x_i = \bar{z}\sigma^i z$)

Thank you for your attention.