Enhancing Tensor Field Theories (renormalizable ϕ^4 melonic case)

Reiko Toriumi

Radboud University, Nijmegen, the Netherlands

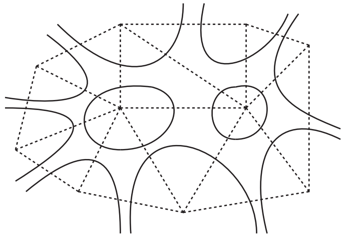
with Joseph Ben Geloun
(Labotatoire d’Informatique de Paris Nord, Université Paris 13)
arXiv:1709.05141

Tensor model approach to Quantum Gravity

$$\mathcal{Z}_{\text{matrix model}} = \int \mathcal{D}M \, e^{-S[M]}, \qquad S[M] = \frac{1}{2}\,\mathrm{Tr}\,M^2 - \frac{\lambda}{\sqrt{N}}\,\mathrm{Tr}\,M^3$$

't Hooft 1974, David 1985, Kazakov, Kostov, Migdal 1985, Ambjorn, Durhuus, Frohlich 1985, Kazakov 1986, Distiler, Kawai 1989, Di Francesco, Ginsparg, Zinn-Justin 1995, Brezin, Kazakov 1990, Douglas, Shenker 1990, Gross, Migdal 1990, …

… generates triangulated 2-dimensional surfaces

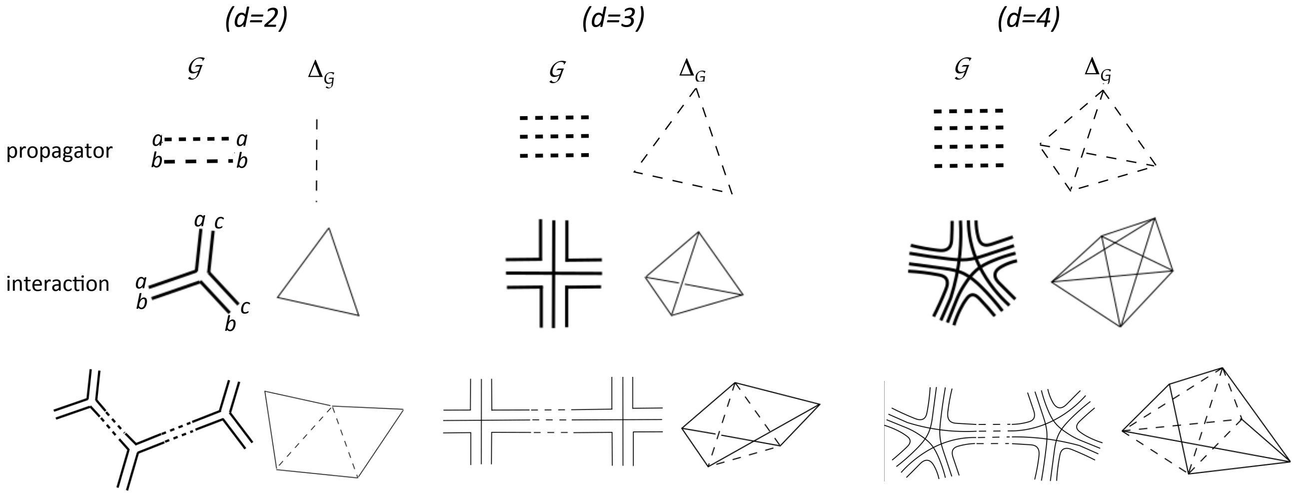


… a statistical model for infinitely refined triangulations, when tuned to the criticality (*N* → *Infinity*, $\lambda \to \lambda_c$)

$$\lambda \leftrightarrow \exp\left(-\frac{\Lambda\, a}{G}\right), \qquad N \leftrightarrow \exp\left(\frac{4\pi}{G}\right)$$

$$\mathcal{Z}_{\text{matrix model}} = \sum_{\Delta_{\mathcal{G}}} \lambda^{T(\Delta_{\mathcal{G}})} N^{\chi(\Delta_{\mathcal{G}})} \quad \leftrightarrow \quad \mathcal{Z}_{2d-\text{QG}} = \int \mathcal{D}g_{\mu\nu}\, e^{-\frac{1}{16\pi G}\int_s d^2x\sqrt{g}(2\Lambda - R)}$$

$$\varphi \quad \to \quad M_{ab},\ T_{abc\ldots}$$



Rank d tensor models generate Feynman graphs dual to d dim. triangulated surfaces.

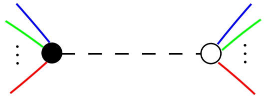
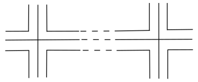
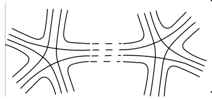
Quantum Gravity a la tensor models/tensor field theories

$$\mathcal{Z} = \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi}\; e^{-(S^{\text{kinetic}} + S^{\text{interaction}})}$$

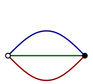
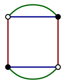
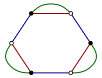
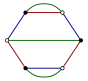
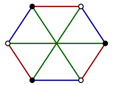
$\varphi,\ \bar{\varphi}$ are rank *d* tensors/tensor fields:

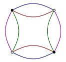
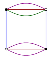
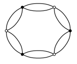
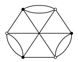
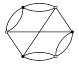
$$\varphi_{i_1 i_2 \cdots i_d} \to \varphi(g_1, g_2, \cdots, g_d)$$
$$\bar{\varphi}_{\tilde{i}_1 \tilde{i}_2 \cdots \tilde{i}_d} \to \bar{\varphi}(\tilde{g}_1, \tilde{g}_2, \cdots, \tilde{g}_d)$$

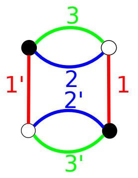
$g,\ \tilde{g} \in G$

$S^{\text{kinetic}}[\bar{\varphi}, \varphi] = \mathrm{Tr}_2(\bar{\varphi} \cdot K \cdot \varphi) + \mu\, \mathrm{Tr}_2(\varphi^2)$ =  ($\overset{d=3}{=}$  $\overset{d=4}{=}$ )

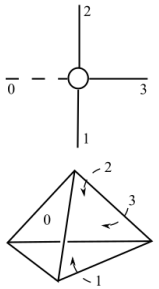
$S^{\text{interation}}[\bar{\varphi}, \varphi] = \sum_{n_b} \lambda_{n_b} \mathrm{Tr}_{n_b}(\bar{\varphi}^{n_b} \cdot \mathcal{V}_{n_b} \cdot \varphi^{n_b})$

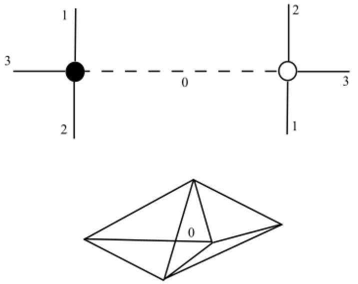
$\overset{d=3}{=}$ $\lambda_2^{(3)}$  $+ \lambda_4^{(3)}$  $+ \lambda_{6,1}^{(3)}$  $+ \lambda_{6,2}^{(3)}$  $+ \lambda_{6,3}^{(3)}$  $+ \cdots$

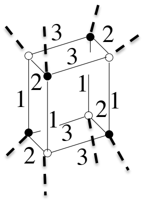
$\overset{d=4}{=}$ $\lambda_2^{(4)}$  $+ \lambda_{4,1}^{(4)}$  $+ \lambda_{4,2}^{(4)}$  $+ \lambda_{6,1}^{(4)}$  $+ \lambda_{6,2}^{(4)}$  $+ \lambda_{6,3}^{(4)}$  $+ \cdots$

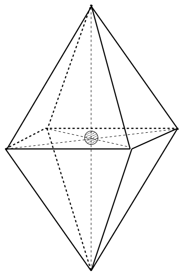
e.g., 

$$\mathrm{Tr}_{4;1}(\varphi^4) = \sum_{i_1, i_2, i_3, i'_1, i'_2, i'_3} \varphi_{i_1 i_2 i_3}\, \bar{\varphi}_{i'_1 i_2 i_3}\, \varphi_{i'_1 i'_2 i'_3}\, \bar{\varphi}_{i_1 i'_2 i'_3}$$









A *(d+1)*-colored Feynman tensor graph is a *d*-dimensional triangulation of a (pseudo) manifold with a boundary.

Our Problem

Melons are branched polymers.

V. Bonzom, R. Gurau, A. Riello and V. Rivasseau, "Critical behavior of colored tensor models in the large N limit," Nucl. Phys. B 853, 174 (2011)

R. Gurau and J. Ryan, "Melons are branched polymers," Annales Henri Poincare 15, no. 11, 2085 (2014)

Want to find a way to escape from the branched polymer phase from more physical phase with large and smooth structure of our universe.

Proposal:

Enhance non-melonic graphs with derivative couplings in tensor field theories

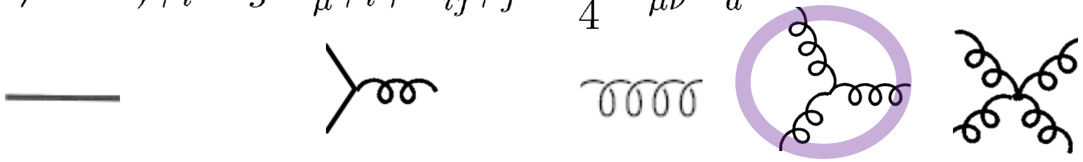
Enhancing tensor models by statistical weights

V. Bonzom, T. Delepouve, V. Rivasseau, "Enhancing non-melonic triangulations: A tensor model mixing melonic and planar maps," Nucl. Phys. B 895, 161 (2015)

Derivative couplings are quite natural in field theories. *e.g.,* in Yang-Mills theory,

$$\mathcal{L}_{QCD} = \bar{\psi}_i (i\partial\!\!\!/ - m)\psi_i - g\, A^a_\mu \bar{\psi}_i \gamma^\mu t^a_{ij} \psi_j - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\nu - g f^{abc} A^b_\mu A^c_\nu$$



Our immediate goal is to launch the program of enhancing non-melonic graphs with derivative couplings in a field theory setting in a systematic way. Namely, the first step is to find renormalizable models.

Our enhanced models (quartic melonic interactions)

Set $G = U(1)^D$

Introduce a complex function $\varphi : (U(1)^D)^{\times d} \to \mathbb{C}$

Work on Fourier component $\varphi_{\mathbf{P}}$ where $\mathbf{P} = (p_1, p_2, \ldots, p_d)$ with $p_s = (p_{s,1}, p_{s,2}, \ldots, p_{s,D})\,,\ \ p_{s,i} \in \mathbb{Z}$

model +

$$\begin{cases} S_+^{\text{interaction}}[\bar\varphi, \varphi] = \dfrac{\lambda}{2}\,\mathrm{Tr}_4(\varphi^4) + \dfrac{\eta_+}{2}\,\mathrm{Tr}_4(p^{2a}\,\varphi^4) + CT_{2;b}[\bar\varphi,\varphi] + CT_{2;a}[\bar\varphi,\varphi] + CT_2[\bar\varphi,\varphi] \\ S_+^{\text{kinetic}}[\bar\varphi, \varphi] = \mathrm{Tr}_2(p^{2b}\varphi^2) + \mathrm{Tr}_2(p^{2a}\varphi^2) + \mu\mathrm{Tr}_2(\varphi^2) \end{cases}$$

model x

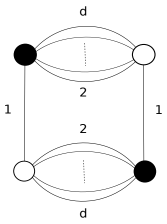
$$\begin{cases} S_\times^{\text{interaction}}[\bar\varphi, \varphi] = \dfrac{\lambda}{2}\,\mathrm{Tr}_4(\varphi^4) + \dfrac{\eta_\times}{2}\,\mathrm{Tr}_4([p^{2a}p'^{2a}]\,\varphi^4) + CT_{2;b}[\bar\varphi,\varphi] + CT_{2;a}[\bar\varphi,\varphi] + CT_{2;2a}[\bar\varphi,\varphi] + CT_2[\bar\varphi,\varphi] \\ S_\times^{\text{kinetic}}[\bar\varphi, \varphi] = \mathrm{Tr}_2(p^{2b}\varphi^2) + \mathrm{Tr}_2(p^{2a}\varphi^2) + \mathrm{Tr}_2(p^{4a}\varphi^2) + \mu\mathrm{Tr}_2(\varphi^2) \end{cases}$$

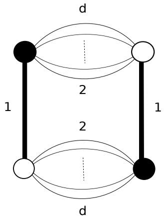
$$\mathrm{Tr}_4(\varphi^4) := \sum_{p_s, p'_s \in \mathbb{Z}^D} \varphi_{p_1\,p_2 \ldots p_d}\,\bar\varphi_{p'_1\,p_2\,p_3 \ldots p_d}\,\phi_{p'_1\,p'_2\,p'_3 \ldots p'_d}\,\bar\varphi_{p_1\,p'_2\,p'_3 \ldots p'_d} + \mathrm{Sym}(1 \to 2 \to \cdots \to d)$$

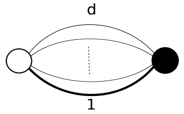
$$\mathrm{Tr}_4(p^{2a}\,\varphi^4) := \sum_{p_s, p'_s \in \mathbb{Z}^D} |p_1|^{2a}\,\varphi_{p_1\,p_2 \ldots p_d}\,\bar\varphi_{p'_1\,p_2\,p_3 \ldots p_d}\,\varphi_{p'_1\,p'_2\,p'_3 \ldots p'_d}\,\bar\varphi_{p_1\,p'_2\,p'_3 \ldots p'_d} + \mathrm{Sym}(1 \to 2 \to \cdots \to d)$$

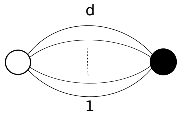
$$\mathrm{Tr}_4([p^{2a}p'^{2a}]\,\varphi^4) := \sum_{p_s, p'_s \in \mathbb{Z}^D} \left(|p_1|^{2a}|p'_1|^{2a}\right)\varphi_{p_1\,p_2 \ldots p_d}\,\bar\varphi_{p'_1\,p_2\,p_3 \ldots p_d}\,\varphi_{p'_1\,p'_2\,p'_3 \ldots p'_d}\,\bar\varphi_{p_1\,p'_2\,p'_3 \ldots p'_d} + \mathrm{Sym}(1 \to 2 \to \cdots \to d)$$

Our enhanced model x



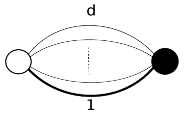


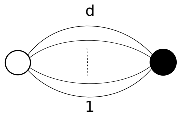




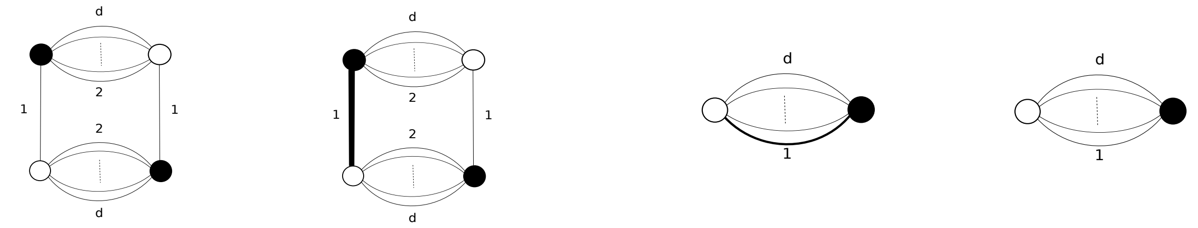
model x

$$\begin{cases} S_{\times}^{\text{interaction}}[\bar{\varphi},\varphi] = \dfrac{\lambda}{2}\,\mathrm{Tr}_4(\varphi^4) + \dfrac{\eta_{\times}}{2}\,\mathrm{Tr}_4([p^{2a}p'^{2a}]\,\varphi^4) + \displaystyle\sum_{\xi=a,2a,b} CT_{2;\xi}[\bar{\varphi},\varphi] + CT_2[\bar{\varphi},\varphi] \\ S_{\times}^{\text{kinetic}}[\bar{\varphi},\varphi] = \displaystyle\sum_{\xi=a,2a,b} \mathrm{Tr}_2(p^{2\xi}\varphi^2) + \mu\mathrm{Tr}_2(\varphi^2) \end{cases}$$





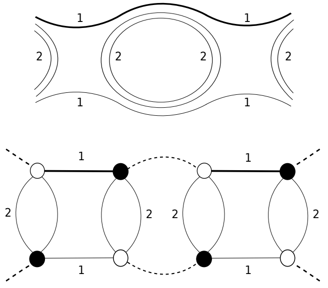
Our enhanced model +

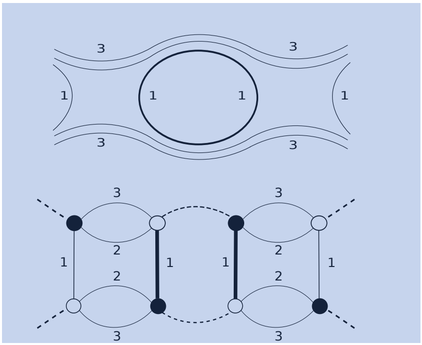


model +

$$\begin{cases} S_{+}^{\text{interaction}}[\bar{\varphi},\varphi] = \dfrac{\lambda}{2}\,\mathrm{Tr}_4(\varphi^4) + \dfrac{\eta_+}{2}\,\mathrm{Tr}_4(p^{2a}\,\varphi^4) + \displaystyle\sum_{\xi=a,b} CT_{2;\xi}[\bar{\varphi},\varphi] + CT_2[\bar{\varphi},\varphi] \\ S_{+}^{\text{kinetic}}[\bar{\varphi},\varphi] = \displaystyle\sum_{\xi=a,b} \mathrm{Tr}_2(p^{2\xi}\varphi^2) + \mu\mathrm{Tr}_2(\varphi^2) \end{cases}$$

e.g.,





a melonic Feynman graph

a non-melonic Feynman graph

Power Counting is achieved

$$C_{\bullet}(\mathbf{P};\mathbf{P}') = \frac{1}{\sum_\xi \mathbf{P}^{2\xi} + \mu}\,\boldsymbol{\delta}_{\mathbf{P},\mathbf{P}'}$$

$$V_{4;s}(\mathbf{P};\mathbf{P}';\mathbf{P}'';\mathbf{P}''') = \frac{\lambda}{2}\delta_{4;s}(\mathbf{P};\mathbf{P}';\mathbf{P}'';\mathbf{P}''')\,,$$

$$V_{+;4;s}(\mathbf{P};\mathbf{P}';\mathbf{P}'';\mathbf{P}''') = \frac{\eta_+}{2}|p_s|^{2a}\,\delta_{4;s}(\mathbf{P};\mathbf{P}';\mathbf{P}'';\mathbf{P}''')\,,$$

$$V_{\times;4;s}(\mathbf{P};\mathbf{P}';\mathbf{P}'';\mathbf{P}''') = \frac{\eta_\times}{2}|p_s|^{2a}|p'_s|^{2a}\,\delta_{4;s}(\mathbf{P};\mathbf{P}';\mathbf{P}'';\mathbf{P}''')$$

Amplitude: $A_{\mathcal{G}} = \sum_{\mathbf{P}_v}\prod_{l\in\mathcal{L}} C_{\bullet;l}(\mathbf{P}_{v(l)};\mathbf{P}'_{v'(l)})\prod_{v\in\mathcal{V}}(-V_v(\{\mathbf{P}_v\}))$

With multi-scale analysis, we optimally bound the amplitude and get:

V. Rivasseau, "From perturbative to constructive renormalization," Princeton series in Physics,1991

$$\omega_{\mathrm{d};+}(G^i_k) = -2bL(G^i_k) + DF_{\mathrm{int}}(G^i_k) + 2a\rho_+(G^i_k) + \sum_{\xi=a,b} 2\xi\rho_{2;\xi}(G^i_k)$$

$$\omega_{\mathrm{d};\times}(G^i_k) = -2b\,L(G^i_k) + DF_{\mathrm{int}}\,(G^i_k) + 2a\rho_\times(G^i_k) + \sum_{\xi=a,2a,b} 2\xi\rho_{2;\xi}(G^i_k)$$

Non-locality of interactions are reflected in:

$$F_{\mathrm{int}} = -\frac{2}{(d-1)!}(\omega(\mathcal{G}_{\mathrm{color}})-\omega(\partial\mathcal{G}))-(C_{\partial\mathcal{G}}-1)-\frac{d-1}{2}N_{\mathrm{ext}}+(d-1)-\frac{d-1}{4}(4-2n)\cdot V$$

$\omega(\mathcal{G}_{\mathrm{color}}) = \sum_J g_J$ Degree of the colored tensor graph: extension of genus and allows large N expansion $A(\mathcal{G}) \sim N^{d-\frac{2}{(d-1)!}\omega(\mathcal{G}_{\mathrm{color}})}$

Gurau, R.: The complete 1/N expansion of colored tensor models in arbitrary dimension. Annales Henri Poincare 13, 399 (2012)

Classification of renormalizability

Superficial degree of divergence of a graph $\omega_d(\mathcal{G}) = \cdots + cV + \cdots$

- Super-renormalizable: *c < 0.* Amplitudes of graphs become more and more convergent as you go higher orders of perturbation theory, *i.e.,* only finite number of graphs are divergent.

- Just-renormalizable: *c = 0.* ω_d is independent of orders of perturbation, *i.e.,* infinite number of graphs are divergent.

- Non-renormalizable: c > 0. Amplitudes of graphs become more and more divergent as you go higher orders of perturbation theory.

Potentially just-renormalizable models

requiring just-renormalizability: $\omega_d(\mathcal{G})|_{N_{\rm ext}\geq 6} < 0$

c=0 in $\omega_d(\mathcal{G}) = \cdots + c\,V + \cdots$

model +

	$d=3$	$d=4$	$d=5$	$d=6$	$\cdots$
$D=1$	$0<a<\frac{2}{3}$ $\frac{1}{2}<b<\frac{5}{6}$	$0<a<\frac{7}{6}$ $\frac{3}{4}<b<\frac{4}{3}$	$0<a<\frac{5}{3}$ $1<b<\frac{11}{6}$	$0<a<\frac{13}{6}$ $\frac{5}{4}<b<\frac{7}{3}$	
$D=2$	$0<a<\frac{4}{3}$ $1<b<\frac{5}{3}$	$0<a<\frac{7}{3}$ $\frac{3}{2}<b<\frac{8}{3}$	$0<a<\frac{10}{3}$ $2<b<\frac{11}{3}$	$0<a<\frac{13}{3}$ $\frac{5}{2}<b<\frac{14}{3}$	
$D=3$	$0<a<2$ $\frac{3}{2}<b<\frac{5}{2}$	$0<a<\frac{7}{2}$ $\frac{9}{4}<b<4$	$0<a<5$ $3<b<\frac{11}{2}$	$0<a<\frac{13}{2}$ $\frac{15}{4}<b<7$	
$D=4$	$0<a<\frac{8}{3}$ $2<b<\frac{10}{3}$	$0<a<\frac{14}{3}$ $3<b<\frac{16}{3}$	$0<a<\frac{20}{3}$ $4<b<\frac{22}{3}$	$0<a<\frac{26}{3}$ $5<b<\frac{28}{3}$	
$\vdots$					

$\longrightarrow$

log-div. for 4-point function:

$\omega_{\rm d;+}(\mathcal{G}^{\rm non-melon})|_{N_{\rm ext}=4} = 0$

	$d=3$	$d=4$	$d=5$	$d=6$	$\cdots$
$D=1$	$a=\frac{1}{2}$ $b=\frac{3}{4}$	$a=1$ $b=\frac{5}{4}$	$a=\frac{3}{2}$ $b=\frac{7}{4}$	$a=2$ $b=\frac{9}{4}$	
$D=2$	$a=1$ $b=\frac{3}{2}$	$a=2$ $b=\frac{5}{2}$	$a=3$ $b=\frac{7}{2}$	$a=4$ $b=\frac{9}{2}$	
$D=3$	$a=\frac{3}{2}$ $b=\frac{9}{4}$	$a=3$ $b=\frac{15}{4}$	$a=\frac{9}{2}$ $b=\frac{21}{4}$	$a=6$ $b=\frac{27}{4}$	
$D=4$	$a=2$ $b=3$	$a=4$ $b=5$	$a=6$ $b=7$	$a=8$ $b=9$	
$\vdots$					

model x

	$d=3$	$d=4$	$d=5$	$d=6$	$\cdots$
$D=1$	$0<a\leq\frac{1}{2}$ $\frac{1}{2}<b\leq 1$	$0<a\leq\frac{3}{4}$ $\frac{3}{4}<b\leq\frac{3}{2}$	$0<a\leq 1$ $1<b\leq 2$	$0<a\leq\frac{5}{4}$ $\frac{5}{4}<b\leq\frac{5}{2}$	
$D=2$	$0<a\leq 1$ $1<b\leq 2$	$0<a\leq\frac{3}{2}$ $\frac{3}{2}<b\leq 3$	$0<a\leq 2$ $2<b\leq 4$	$0<a\leq\frac{5}{2}$ $\frac{5}{2}<b\leq 5$	
$D=3$	$0<a\leq\frac{3}{2}$ $\frac{3}{2}<b\leq 3$	$0<a\leq\frac{9}{4}$ $\frac{9}{4}<b\leq\frac{9}{2}$	$0<a\leq 3$ $3<b\leq 6$	$0<a\leq\frac{15}{4}$ $\frac{15}{4}<b\leq\frac{15}{2}$	
$D=4$	$0<a\leq 2$ $2<b\leq 4$	$0<a\leq 3$ $3<b\leq 6$	$0<a\leq 4$ $4<b\leq 8$	$0<a\leq 5$ $5<b\leq 10$	
$\vdots$					

$\longrightarrow$

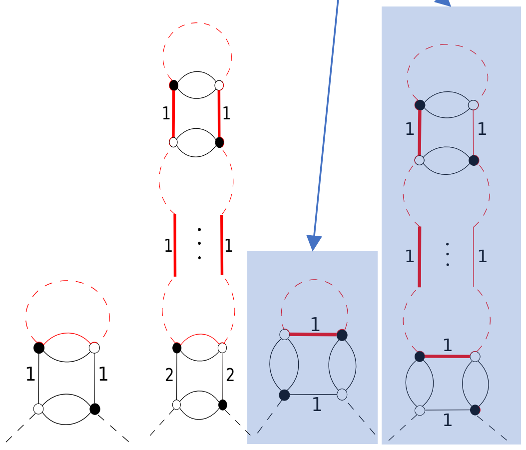
$$\omega_{\rm d;\times}(\mathcal{G})|_{N_{\rm ext}=4} < 0$$

$$\omega_{\rm d;\times}(\mathcal{G}^{\rm melon})|_{N_{\rm ext}=2} < \frac{D(d-1)}{2}$$

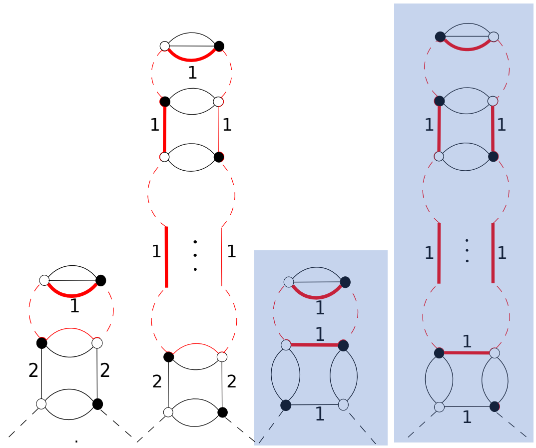
$$\omega_{\rm d;\times}(\mathcal{G}^{\rm non-melon})|_{N_{\rm ext}=2} \leq \frac{1}{2}D(3-d) \leq 0$$

Primitively divergent graphs

non-melonic graphs

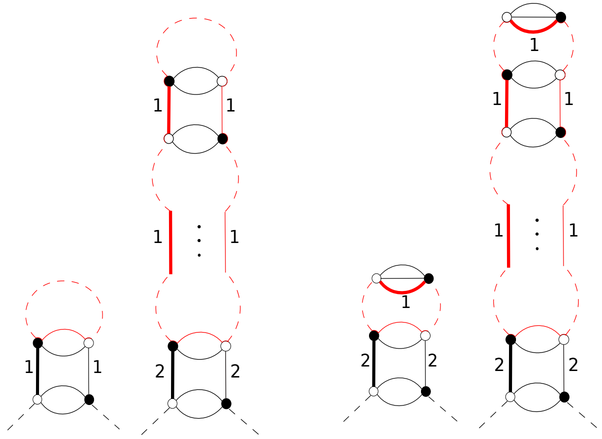


$\omega_{\mathrm{d};+} = \frac{1}{2}$



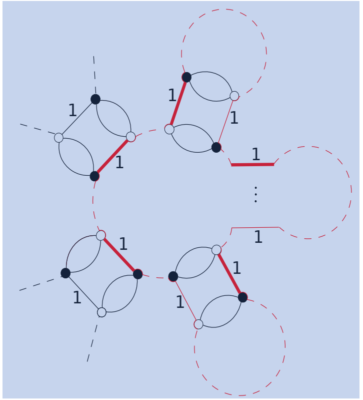
$\omega_{\mathrm{d};+} = 0$

renormalize mass



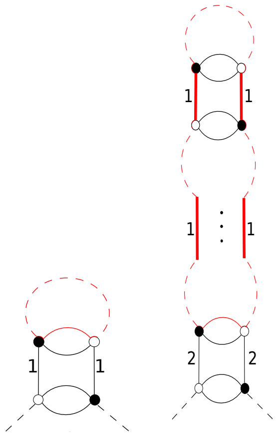
$\omega_{\mathrm{d};+} = \frac{1}{2}$ $\omega_{\mathrm{d};+} = 0$

renormalize Z_a (p^{2a} ϕ^2) coupling



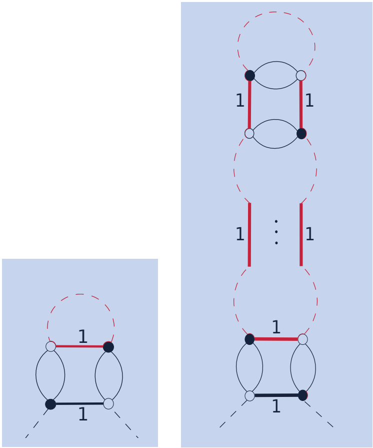
$\omega_{\mathrm{d};+} = 0$

renormalize λ (ϕ^4) and η_+ (p^{2a} ϕ^4) couplings



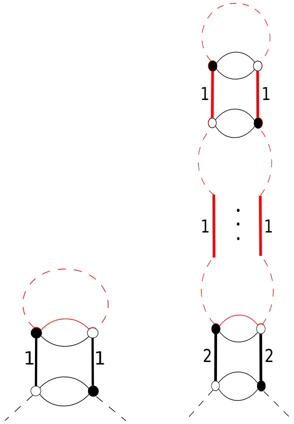
$\omega_{\mathrm{d};\times} = 0$

renormalize mass



$\omega_{\mathrm{d};\times} = 0$

renormalize Z_a (p^{2a} ϕ^2) coupling



$\omega_{\mathrm{d};\times} = 0$

renormalize Z_{2a} (p^{4a} ϕ^2) coupling

Nothing

renormalizes λ (ϕ^4) and $\eta_\times$ (p^{2a} ϕ^4) couplings

Conclusions and Outlook

- Model +: a just-renormalizable model

  Infinite number of graphs renormalizes λ (ϕ^4) and η_+ (p^{2a} ϕ^4) couplings.

- Model x: a new type of renormalizable model (neither just- nor super-)

  No graphs renormalize λ (ϕ^4) and η_x (p^{2a} ϕ^4) couplings.

  But infinite number of graphs renormalize mass, Z_a (p^{2a} ϕ^2) and Z_{2a} (p^{4a} ϕ^2) couplings.

- …We have established the mechanism of enhancing non-melonic graphs in a tensor field theory setting. These models are renormalizable. This is encouraging for analyses at next level…

- Beta functions of coupling constants can be computed.
- Non-perturbative analysis to be applied.
- Other types of interactions to be enhanced by derivative couplings.