Quantum Metric and Entanglement on Spin Networks

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arXiv:gr-qc/1703.05231 , arXiv:gr-qc/1703.06415
with G. Chirco, D. Oriti and P. Vitale

Introduction and Motivations

Background-indep. approaches to QG (e.g., LQG, SF and GFT) share a very radical picture of the microscopic quantum structure of spacetime. At the Planck scale, space and time dissolve into pre-geometric, combinatorial and algebraic objects (spin networks).

$\rightarrow$ How can spacetime emerge from its fundamental constituents?

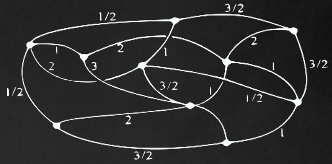
Entanglement is expected to play a key role in the reconstruction of spacetime geometry!

- AdS/CFT: Ryu-Takayanagi formula ('06 arXiv:hep-th/0603001), bulk space from boundary entanglement (Van Raamsdonk '10 arXiv:hep-th/1005.3035);
- entanglement from gluing of spin networks (Donnelly '08 arXiv:gr-qc/0802.0880);
- reconstructing quantum geometry from quantum information (Livine, Terno '06 arXiv:gr-qc/0603008);
- spin networks as generalized tensor networks (Chirco, Oriti, Zhang '17 arXiv:gr-qc/1701.01383).

Spin Network States of Quantum Geometry

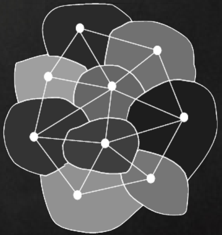
Spin network basis $\equiv$ graphs with links labelled by $SU(2)$ irreps and nodes by invariant tensors (intertwiners) ensuring gauge invariance of the states:

$$\psi_{\Gamma,\vec{j},\vec{i}}[A] = \bigotimes_{\ell=1}^{L} D^{(j_\ell)}(h_\ell(A)) \cdot \bigotimes_{v=1}^{V} i_v$$



Spin networks diagonalize geometric observables such as area and volume which admit a discrete spectrum, e.g.:

$$\hat{A}(S)\,\psi_{\Gamma,\vec{j},\vec{i}}[A] = \sum_{\ell \in S \cap \Gamma} \hbar \sqrt{\gamma^2\, j_\ell(j_\ell + 1)}\;\psi_{\Gamma,\vec{j},\vec{i}}[A]$$



$\Rightarrow$ Quanta of (Space) Geometry !

Geometric Quantum Mechanics: Pure States

For a given quantum system, the space of pure states $\mathcal{D}^1(\mathcal{H})$ (identified with the complex projective space $\mathbb{C}P(\mathcal{H})$) naturally inherits a Kähler structure from $\mathcal{H}_0 = \mathcal{H} - \{\mathbf{0}\}$.

Indeed, by means of the momentum map

$$\mu : \mathcal{H}_0 \longrightarrow \mathfrak{u}^*(\mathcal{H}) \supset \mathcal{D}^1(\mathcal{H}) \quad , \quad |\psi\rangle \longmapsto \rho = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$$

we define

$$\underbrace{\boxed{\mathcal{K} = \mathrm{Tr}(\rho d\rho \otimes d\rho)}}_{\textit{Hermitian (0,2)-tensor}} \overset{\mu^*}{\dashrightarrow} \underbrace{\boxed{\frac{\langle d\psi \otimes d\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle\psi|d\psi\rangle}{\langle\psi|\psi\rangle} \otimes \frac{\langle d\psi|\psi\rangle}{\langle\psi|\psi\rangle}}}_{\textit{Fubini-Study tensor}}$$

whose real and imaginary parts define a metric (quantum Fisher information metric) and a symplectic structure, respectively.

Tensorial Characterization of Entanglement

For a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \cong \mathbb{C}^n \otimes \mathbb{C}^n$, we identify orbit submanifolds of unitarily related quantum states (i.e., with fixed amount of entanglement):

$$\mathcal{O} := \left\{ \rho(g) = U(g)\rho_0 U^{-1}(g)\,,\; U(g) = (U_A(g_A) \otimes \mathbb{1}) \cdot (\mathbb{1} \otimes U_B(g_B)) \right\}$$

The pulled-back Hermitian tensor encodes all the information about entanglement:

$$\mathcal{K}_{jk} = \mathcal{K}_{(jk)} + i\mathcal{K}_{[jk]} = \left(\begin{array}{c|c} A & C \\ \hline C & B \end{array} \right) + i \left(\begin{array}{c|c} D_A & 0 \\ \hline 0 & D_B \end{array} \right)$$

$$\boxed{\rho_0 \text{ separable } \Leftrightarrow \; C = 0} \quad , \quad \boxed{\rho_0 \text{ max. ent. } \Leftrightarrow \; D_{A,B} = 0}$$

The off-diagonal blocks allow to define an entanglement monotone interpreted as a distance with respect to the separable state:

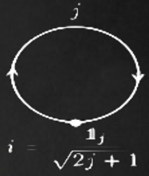
$$\boxed{\mathrm{Tr}(R^\dagger R) = \frac{1}{n^4}\mathrm{Tr}(C^T C) \quad , \quad R = \rho_0 - \rho_0^A \otimes \rho_0^B}$$

Local Correlations: Single Link Graph

For fixed j, we regard the single link Hilbert space as $\mathcal{H}_\gamma^{(j)} \cong \mathcal{V}^{(j)} \otimes \mathcal{V}^{(j)*}$ with $\mathcal{V}^{(j)} = span\{|j,m\rangle\}_{-j\leq m\leq j}$. Hence $\mathbb{G} \equiv SU(2) \times SU(2)$ and

$$C_{ab} = \mathrm{Tr}(\rho_0 J_a \otimes J_b) - \mathrm{Tr}(\rho_0 J_a \otimes \mathbb{1})\mathrm{Tr}(\rho_0 \mathbb{1} \otimes J_b)$$

- Maximally entangled state:



$$|0\rangle = \frac{1}{\sqrt{2j+1}} \sum_k |j,k\rangle \otimes \langle j,k|$$

$$\Rightarrow \quad \boxed{D_A = D_B = 0 \quad , \quad \mathrm{Tr}(C^T C) = \frac{1}{3}[j(j+1)]^2}$$

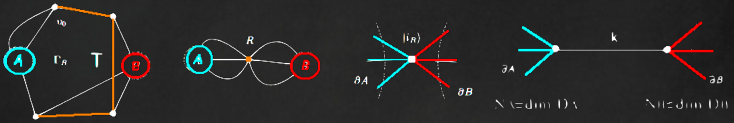
- Separable state:

$$|0\rangle = |j_1,k_1\rangle \otimes \langle j_2,k_2|$$

$$\Rightarrow \quad \boxed{D_A, D_B \neq 0 \quad , \quad C = 0 \quad \Rightarrow \quad \mathrm{Tr}(C^T C) = 0}$$

Correlations between Two Non-Adjacent Regions of a SN

Correlations induced by the intermediate region of quantum space modeled as a single node graph (no curvature case) intertwining the edges dual to the boundaries of the two regions



unfolded into two coupled N-level systems with N given by the degeneracies of the unfolded nodes.

$$\Rightarrow \qquad \rho_0 = \sum_{\alpha\alpha'\beta\beta'} c_{\alpha\beta}\bar{c}_{\alpha'\beta'}\tau^{(A)}_{\alpha\alpha'} \otimes \tau^{(B)}_{\beta\beta'} \qquad , \qquad \tau_{\alpha\alpha'} \equiv |\alpha\rangle\langle\alpha'|$$

ρ_0	$c_{\alpha\beta}$	$\mathrm{Tr}\left(\mathcal{K}^{(AB)\,T}\mathcal{K}^{(AB)}\right)$
separable	$\lambda_\alpha\lambda_\beta$	0
max. ent.	$\delta_{\alpha\beta}/\sqrt{N_<}$	$1-\frac{1}{N_<^2}$
entangled	$f(\alpha)\delta_{\alpha\beta}\ ,\ f(\alpha)\in\mathbb{C}$	$\sum_\alpha \bar{f}(\alpha)^2\sum_{\alpha'} f(\alpha')^2 - \sum_\alpha \lvert f(\alpha)\rvert^6$
	$f(\alpha)\delta_{\alpha\beta}\ ,\ f(\alpha)\in\mathbb{R}$	$1-\sum_\alpha \lvert f(\alpha)\rvert^6$

Conclusions

The main achievements of our work are:

- A purely relational interpretation of the link as an elementary process describing quantum correlations between its endpoints and thus generating the minimal element of geometry;
- A quantitative characterization of graph connectivity by means of the entanglement monotone constructed from the metric tensor;
- A preliminary connection between the GQM formalism and the (simplicial) geometric properties of SN states through entanglement.

Interpretation: Spin networks as information graphs whose connectivity encodes, both at the local and non-local level, quantum correlations between regions of space.

Future Perspectives

- Include curvature excitations: reduced graph with loopy degrees of freedom enconding information about a non-trivial topology of the region of space;
- Entanglement of mixed states: Quantum metric from relative entropy with possible application to Gibbs states for black holes;
- Semiclassical states: classical limit and further connection with the Fisher-Rao metric of Information Geometry;
- Analogies with General Boundary Formalism: Hermitian tensor as a (spin foam) path integral amplitude, i.e., a process generating a region of space-time.

Thank you for your attention!