# The Phase Diagram of the BMN Matrix Model

**Denjoe O'Connor**

*School of Theoretical Physics*
*Dublin Institute for Advanced Studies*
*Dublin, Ireland*

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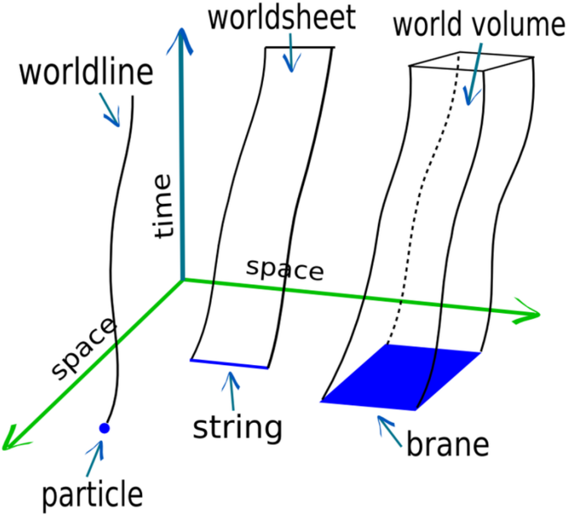
**Background:** V. Filev and D.O'C. [1506.01366 and 1512.02536]
Y. Asano, V. Filev, S. Kováčik and D.O'C. [1605.05597, 1612.09281].

# A particle, a string and a membrane



The action functional

$$S_{particle} = -m \int d\tau_{proper} = m \int dt \sqrt{1 - v^2}.$$

For small (non-relativistic) velocities this gives

$$S_{particle} = -m \int dt + \int dt \frac{mv^2}{2}$$

# Movement in a generic background

$$S_{particle} = -m \int dt \sqrt{-g_{\mu\nu} \frac{dX^\mu}{dt} \frac{dX^\nu}{dt}}$$

this is the Nambu-Goto form of the action. If we rewrite the action using the Lagrange multiplier $h$ as

$$S = \frac{1}{2} \int dt \{ h^{-1} g_{\mu\nu} \frac{dX^\mu}{dt} \frac{dX^\nu}{dt} - hm^2 \}$$

we have the Polyakov form of the action. Eliminating $h$ with its saddle point, $h = -\sqrt{-g}/m$, recovers the Nambu form. The equations of motion give us the geodesic equation.

# Coupling to an electromagnetic field

$$S_{charged-particle} = -m \int dt \sqrt{-g_{\mu\nu} \frac{dX^\mu}{dt} \frac{dX^\nu}{dt}} - q \int \frac{dX^\mu}{dt} A_\mu dt$$

# We can repeat this exercise for a string

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d\sigma d\tau\sqrt{-detG} \qquad G_{\mu\nu} = \partial_\mu X^M \partial_\nu X^N g_{MN}$$

or the Polyakov form, with the Lagrange multiplier metric $h_{\mu\nu}$,

$$S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma d\sigma d\tau\sqrt{-h}h^{\mu\nu}G_{\mu\nu}$$

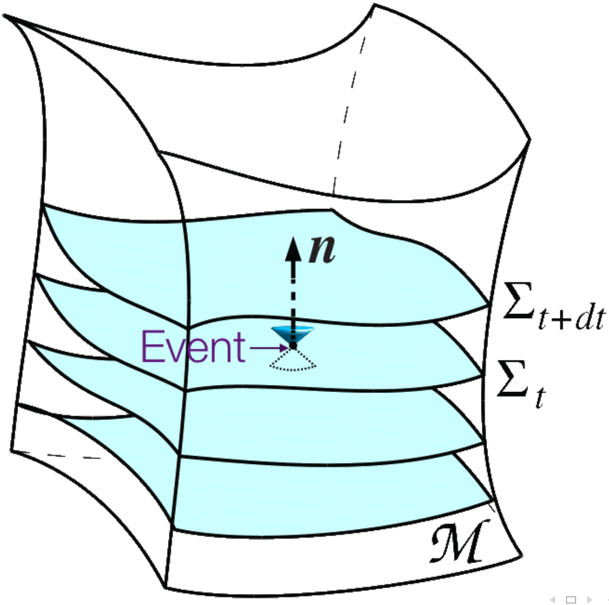
The string is very special in that it is a conformally invariant action. Again one can couple the string to e.g. an RR 2-form to get the

$$S_{NG} - q\int \partial_\mu X^M \partial_\nu X^N \epsilon^{\mu\nu} B_{MN}$$

# Quantisation

We can quantise the particle or string in either a path integral or Hamiltonian formulation and the results are well appreciated. Both can be generalised to supersymmetric versions with the string leading to string theory and conformal field theory.

# Membrane propagating in spacetime



# Membrane Actions

## Nambu Goto—the simplest

$S_{NG} = \int_{\mathcal{M}} d^{p+1}x\sqrt{-detG} \qquad G_{\mu\nu} = \partial_\mu X^M \partial_\nu X^N g_{MN}$

## Higher form gauge field on the world volume

$$S_{p-form} = -\int_{\mathcal{M}} \frac{1}{(p+1)!} \epsilon^{\mu_1 \dots \mu_{p+1}} C_{\mu_1 \dots \mu_{p+1}}$$

$$C_{\mu_1 \dots \mu_{p+1}} = \partial_{\mu_1} X^{M_1} \dots \partial_{\mu_{p+1}} X^{M_{p+1}} C_{M_1 \dots M_{p+1}}$$

We can add

- an anti-symmetric part to $G_{\mu\nu}$ to get a Dirac-Born-Infeld action.
- extrinsic curvature terms.

Supersymmetric $S_{NG}$ exist only in 4, 5, 7 and 11 dim-spacetime.

# From Membranes to Matrices (à la Hoppe)

**The Membrane action, Polyakov form**

$$S = -\frac{T}{2}\int_{\mathcal{M}} d^3\sigma\sqrt{-h}\left(h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \Lambda\right)$$

Choose $\Lambda = 1$ (rescale $X^a$ and $T$), and for membrane topology $\mathbb{R}\times\Sigma$ use the gauge $h_{0i} = 0$ and $h_{00} = -\frac{4}{\rho}\mathrm{det}(h_{ij})$.

The action becomes

$$S = \frac{T\rho}{4}\int dt\int_\Sigma d^2\sigma\left(\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu} - \frac{4}{\rho^2}\mathrm{det}(h_{ij})\right)$$

Noting that

$$\det(\partial_i X^a \partial_j X^b h_{ab}) =$$
$$\frac{1}{p!}\{X^{a_1}, X^{a_2} \ldots, X^{a_p}\}\{X^{b_1}, X^{b_2} \ldots, X^{b_p}\} h_{a_1 b_1} h_{a_2 b_2} \ldots h_{a_p b_p}$$
$$\{X^{a_1}, X^{a_2} \ldots, X^{a_p}\} := \epsilon^{j_1, j_2, \ldots, j_p} \partial_{j_1} X^{a_1} \partial_{j_2} X^{a_2} \ldots \partial_{j_p} X^{a_p}$$

$$S = \frac{T\rho}{4} \int dt \int_\Sigma d^2\sigma \left( \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - \frac{4}{\rho^2} \det(h_{ij}) \right)$$

becomes

$$S = \frac{T\rho}{4} \int dt \int_\Sigma d^2\sigma \left( \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - \frac{4}{p!\rho^2} \{X^{a_1}, X^{a_2} \ldots, X^{a_p}\}^2 \right)$$

In 2-dim $det(h_{ij})$ can be rewritten using $\{f,g\} = \epsilon^{ij}\partial_i f \partial_j g$ as

$$S = \frac{T\rho}{4}\int dt \int_\Sigma d^2\sigma \left(\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - \frac{4}{\rho^2}\{X^\mu, X^\nu\}^2\right)$$

and the constraints become

$$\dot{X}^\mu \partial_i X_\mu = 0 \implies \{\dot{X}^\mu, X_\mu\} = 0$$

$$\text{and} \qquad \dot{X}^\mu \dot{X}_\mu = -\frac{2}{\rho^2}\{X^\mu, X^\nu\}\{X_\mu, X_\nu\}\,.$$

Using lightcone coordinates with $X^\pm = (X^0 \pm X^{D-1})/\sqrt{2}$ with $X^+ = \tau$ we can solve the constraint for $\dot{X}^-$ and Legendre transform to the Hamiltonian to find

$$S = -T\int \sqrt{-G} \longrightarrow H = \int_\Sigma (\frac{1}{\rho T}P^a P^a + \frac{T}{2\rho}\{X^a, X^b\}^2)$$

With the remaining constraint $\{P^a, X^a\} = 0$.

In this scheme functions are approximated by $N \times N$ matrices, $f \to \mathrm{F}$, and $\int_\Sigma f \to \mathrm{TrF}$. The Hamiltonian becomes

$$\mathrm{H} = -\frac{1}{2}\nabla^2 - \frac{1}{4}\sum_{i,j=1}^{d} \mathrm{Tr}[X^i, X^j]^2$$

and describes a "fuzzy" relativistic membrane in $d+1$ dimensions. Note: Much of the classical topology and geometry are lost in the quantum theory.

Once we have the Hamiltonian $H$ we can consider thermal ensembles of membranes whose partition function is given by

$$Z = \mathrm{Tr}_{Phys}(\mathrm{e}^{-\beta H})$$

where the physical constraint means the states are $U(N)$ invariant. The simplest example of a quantum mechanical model with Gauss Law constraint in this class is a family of $p$ gauged Gaussians. Their Euclidean actions are

$$N \int_0^\beta \mathrm{Tr}(\frac{1}{2}(\mathcal{D}_\tau X^i)^2 + \frac{1}{2} m^2 (X^i)^2)$$

$\mathcal{D}_\tau X^i = \partial_\tau X^i - i[A, X^i]$.

# Properties of gauge gaussian models

- The eigenvalues of $X^i$ have a Wigner semi-circle distribution.
- At $T = 0$, we can gauged $A$ away, while for large $T$ we get a pure matrix model with $A$ one of the matrices.
- The entry of $A$ as an additional matrix in the dynamics signals a phase transition. In the Gaussian case with $p$ scalars it occurs at

$$T_c = \frac{m}{\ln p}$$

  The transition can be observed as centre symmetry breaking in the Polyakov loop.

Bosonic matrix membranes are approximately gauge gaussian models V. Filev and D.O'C. [1506.01366 and 1512.02536]. Note they are the zero volume limit of Yang-Mills compactified on $T^3$ and on closer inspection they exhibit two phase transitions, very close in temperature.

# Quantum Gravity

At short distances it is expected [Doplicher, Fredenhagen and Roberts, 1995] that spatial co-ordinates, $X^a$ should not commute $[X^a, X^b] \neq 0$ in analogy with $[x, p] = i\hbar$ in phase space, but $[X^a, X^b] = i\theta^{ab}$ breaks rotational invariance.

We only need the coordinates to commute at low energies.

*Hand waving à la Polchinski, 2014 (arXiv:1412.5704):*
Take each $X^a$ to be an $N \times N$ matrix and try

$$H_0 = \mathrm{Tr}\left(\frac{1}{2}\sum_{a=1}^{p}\dot{X}^a\dot{X}^a - \frac{1}{4}\sum_{a,b=1}^{p}[X^a, X^b][X^a, X^b]\right)$$

**The model describes membranes, Hoppe 1982.**

$$S = -T\int\sqrt{-G} \longrightarrow H = \int\left(\frac{1}{\rho T}P^aP^a + \frac{T}{2\rho}\{X^a, X^b\}^2\right)$$

With the remaining constraint $\{P^a, X^a\} = 0$.

At low energy, or the bottom of the potential $[X^a, X^b] = 0$.

# The BFSS model

$$S_{SMembrane} = \int \sqrt{-G} - \int C + \quad \text{Fermionic terms}$$

The susy version only exists in $4, 5, 7$ and $11$ spacetime dimensions.

**BFFS Model — The supersymmetric membrane à la Hoppe**

$$\mathrm{H} = \mathrm{Tr}\left(\frac{1}{2}\sum_{a=1}^{9} P^a P^a - \frac{1}{4}\sum_{a,b=1}^{9}[X^a, X^b][X^a, X^b] + \frac{1}{2}\Theta^T \gamma^a [X^a, \Theta]\right)$$

The model is claimed to be a non-perturbative 2nd quantised formulation of $M$-theory.

It also describes a system of N interacting D0 branes.

Note the flat directions.

# Finite Temperature Model

The partition function and Energy of the model at finite temperature is

$$Z = Tr_{Phys}(\mathrm{e}^{-\beta\mathcal{H}}) \qquad \text{and} \qquad E = \frac{Tr_{Phys}(\mathcal{H}\mathrm{e}^{-\beta\mathcal{H}})}{Z} = \langle\mathcal{H}\rangle$$

The 16 fermionic matrices $\Theta_\alpha = \Theta_{\alpha A} t^A$ are quantised as

$$\{\Theta_{\alpha A}, \Theta_{\beta B}\} = 2\delta_{\alpha\beta}\delta_{AB}$$

The $\Theta_{\alpha A}$ are $2^{8(N^2-1)}$ and the Fermionic Hilbert space is

$$\mathcal{H}^F = \mathcal{H}_{256} \otimes \cdots \otimes \mathcal{H}_{256}$$

with $\mathcal{H}_{256} = \mathbf{44} \oplus \mathbf{84} \oplus \mathbf{128}$ suggestive of the graviton ($\mathbf{44}$), anti-symmetric tensor ($\mathbf{84}$) and gravitino ($\mathbf{128}$) of $11 - d$ SUGRA.

For an attempt to find the ground state see: J. Hoppe et al arXiv:0809.5270

# Lagrangian formulation

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The BFSS matrix model is also the dimensional reduction of ten dimensional supersymmetric Yang-Mills theory down to one dimension:

$$\begin{aligned} S_M &= \frac{1}{g^2}\int dt\,\mathrm{Tr}\left\{\frac{1}{2}(\mathcal{D}_0 X^i)^2 + \frac{1}{4}[X^i, X^j]^2 \right. \\ &\quad \left. -\frac{i}{2}\Psi^T C_{10}\,\Gamma^0 D_0\Psi + \frac{1}{2}\Psi^T C_{10}\,\Gamma^i[X^i,\Psi]\right\} , \end{aligned}$$

where $\Psi$ is a thirty two component Majorana–Weyl spinor, $\Gamma^\mu$ are ten dimensional gamma matrices and $C_{10}$ is the charge conjugation matrix satisfying $C_{10}\Gamma^\mu C_{10}^{-1} = -\Gamma^{\mu\,T}$.

# The BMN or PWMM

The supermembrane on the maximally supersymmetric plane wave spacetime

$$ds^2 = -2dx^+dx^- + dx^a dx^a + dx^i dx^i - dx^+ dx^+ ((\frac{\mu}{6})^2 (x^i)^2 + (\frac{\mu}{3})^2 (x^a)^2)$$

with

$$dC = \mu dx^1 \wedge dx^2 \wedge dX^3 \wedge dx^+$$

so that $F_{123+} = \mu$. This leads to the additional contribution to the Hamiltonian

$$\begin{aligned} \Delta \mathrm{H}_\mu &= \frac{N}{2} \mathbf{Tr} \Big( (\frac{\mu}{6})^2 (X^a)^2 + (\frac{\mu}{3})^2 (X^i)^2 \\ &\qquad + \frac{2\mu}{3} i \epsilon_{ijk} X^i X^j X^k + \frac{\mu}{4} \Theta^T \gamma^{123} \Theta \Big) \end{aligned}$$

$$
\begin{aligned}
\Delta S_\mu &= -\frac{1}{2g^2}\int_0^\beta d\tau \, \mathbf{Tr}\left( (\frac{\mu}{6})^2 (X^a)^2 + (\frac{\mu}{3})^2 (X^i)^2 \right. \\
&\qquad \left. + \frac{2\mu}{3} i \epsilon_{ijk} X^i X^j X^k + \frac{\mu}{4} \Psi^T \gamma^{123} \Psi \right)
\end{aligned}
$$

# The gravity dual and its geometry

Gauge/gravity duality predicts that the strong coupling regime of the theory is described by $II_A$ supergravity, which lifts to 11-dimensional supergravity.

The bosonic action for eleven-dimensional supergravity is given by

$$S_{11D} = \frac{1}{2\kappa_{11}^2} \int [\sqrt{-g}R - \frac{1}{2} F_4 \wedge *F_4 - \frac{1}{6} A_3 \wedge F_4 \wedge F_4]$$

where $2\kappa_{11}^2 = 16\pi G_N^{11} = \frac{(2\pi l_p)^9}{2\pi}$.

The relevant solution to eleven dimensional supergravity for the dual geometry to the BFSS model corresponds to $N$ coincident $D0$ branes in the IIA theory. It is given by

$$ds^2 = -H^{-1}dt^2 + dr^2 + r^2 d\Omega_8^2 + H(dx_{10} - Cdt)^2$$

with $A_3 = 0$
The one-form is given by $C = H^{-1} - 1$ and $H = 1 + \frac{\alpha_0 N}{r^7}$ where $\alpha_0 = (2\pi)^2 14\pi g_s l_s^7$.

# Including temperature

The idea is to include a **black hole** in the gravitational system.

The Hawking termperature provides the temperature of the system.

### Hawking radiation

We expect difficulties at low temperatures, as the system should Hawking radiate. It is argued that this is related to the flat directions and the propensity of the system to leak into these regions.

# The black hole geometry

$$ds_{11}^2 = -H^{-1}Fdt^2 + F^{-1}dr^2 + r^2 d\Omega_8^2 + H(dx_{10} - Cdt)^2$$

Set $U = r/\alpha'$ and we are interested in $\alpha' \to \infty$
$H(U) = \frac{240\pi^5\lambda}{U^7}$ and the black hole time dilation factor
$F(U) = 1 - \frac{U_0^7}{U^7}$ with $U_0 = 240\pi^5\alpha'^5\lambda$. The temperature

$$\frac{T}{\lambda^{1/3}} = \frac{1}{4\pi\lambda^{1/3}} H^{-1/2} F'(U_0) = \frac{7}{2^4 15^{1/2} \pi^{7/2}} \left(\frac{U_0}{\lambda^{1/3}}\right)^{5/2}.$$

**From black hole entropy we obtain the prediction for the Energy**

$$S = \frac{A}{4G_N} \sim \left(\frac{T}{\lambda^{1/3}}\right)^{9/2} \implies \frac{E}{\lambda N^2} \sim \left(\frac{T}{\lambda^{1/3}}\right)^{14/5}$$

# Checks of the predictions

We found excellent agreement with this prediction V. Filev and D.O'C. [1506.01366 and 1512.02536].
The best current results (Berkowitz et al 2016) consistent with gauge gravity give

$$\begin{aligned}\frac{1}{N^2}\frac{E}{\lambda^{1/3}} &= 7.41\left(\frac{T}{\lambda^{1/3}}\right)^{\frac{14}{5}} - (10.0 \pm 0.4)\left(\frac{T}{\lambda^{1/3}}\right)^{\frac{23}{5}} \\ &\quad + (5.8 \pm 0.5)\, T^{\frac{29}{5}} + \ldots \\ &\quad - \frac{5.77\, T^{\frac{2}{5}} + (3.5 \pm 2.0)\, T^{\frac{11}{5}}}{N^2} + \ldots\end{aligned}$$

Using $D_4$ branes as probes (these adds new fundamental matter).

See: M. Berkooz and M. R. Douglas, "Five-branes in M(atrix) theory," [hep-th/9610236].
In IIA string theory this describes a $D0 - D4$ system.

# The D4-brane as a probe of the geometry.

The dual adds $N_f$ D4 probe branes. In the probe approximation $N_f \ll N_c$, their dynamics is governed by the Dirac-Born-Infeld action:

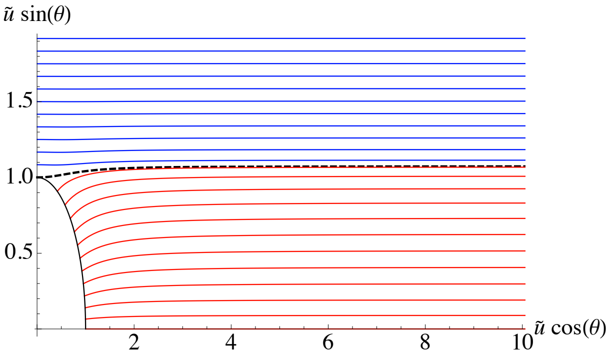
$$S_{\mathrm{DBI}} = -\frac{N_f}{(2\pi)^4\,\alpha'^{5/2}\,g_s}\int d^4\xi\,e^{-\Phi}\sqrt{-\det||G_{\alpha\beta}+(2\pi\alpha')F_{\alpha\beta}||}\;,$$

where $G_{\alpha\beta}$ is the induced metric and $F_{\alpha\beta}$ is the $U(1)$ gauge field of the D4-brane. For us $F_{\alpha\beta}=0$.

$$d\Omega_8^2 = d\theta^2 + \cos^2\theta\,d\Omega_3^2 + \sin^2\theta\,d\Omega_4^2$$

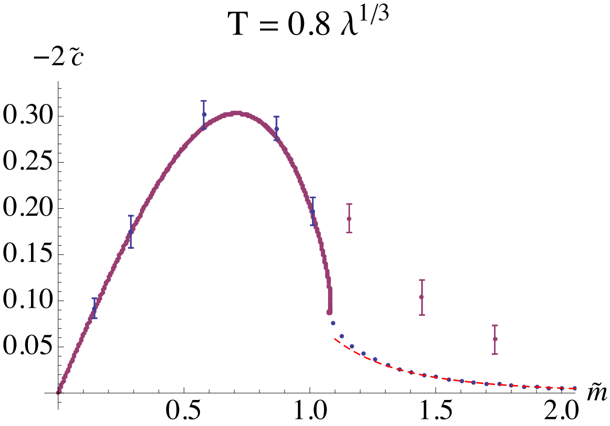
and taking a D4-brane embedding extended along: $t$, $u$, $\Omega_3$ with a non-trivial profile $\theta(u)$.

# Embeddings




$$\tilde{u} \sin\theta = m + \frac{\tilde{c}}{\tilde{u}^2} + \ldots .$$

# The condensate and the dual prediction




V. Filev and D. O'C. arXiv 1512.02536.

The data overlaps surprisingly well with the gravity prediction in the region where the $D4$ brane ends in the black hole.

# The BMN model

## The BMN action

$$
\begin{aligned}
S_{BMN} \;=\; & \frac{1}{2g^2}\int dt\,\mathrm{Tr}\Big\{(\mathcal{D}_0 X^i)^2 - (\frac{\mu}{6})^2 (X^a)^2 - (\frac{\mu}{3})^2 (X^i)^2 \\
& -i\Psi^T C_{10}\,\Gamma^0 D_0 \Psi - \frac{\mu}{4}\Psi^T \gamma^{123}\Psi \\
& +\frac{1}{4}[X^i, X^j]^2 - \frac{2\mu}{3} i\epsilon_{ijk} X^i X^j X^k - \frac{1}{2}\Psi^T C_{10}\,\Gamma^i [X^i, \Psi]\Big\}\ ,
\end{aligned}
$$

# Large mass expansion

For large $\mu$ the model becomes the supersymmetric Gaussian model

## Finite temperature Euclidean Action

$$S_{BMN} = \frac{1}{2g^2}\int_0^\beta d\tau\,\mathrm{Tr}\left\{(\mathcal{D}_\tau X^i)^2 + (\frac{\mu}{6})^2(X^a)^2 + (\frac{\mu}{3})^2(X^i)^2 \right.$$
$$\left. \Psi^T D_\tau \Psi + \frac{\mu}{4}\Psi^T\gamma^{123}\Psi\right\}$$

This model has a phase transition at $T_c = \frac{\mu}{12\ln 3}$

# Perturbative expansion in large $\mu$.

Three loop result of *Hadizadeh, Ramadanovic, Semenoff and Young* [hep-th/0409318]

$$T_c = \frac{\mu}{12\ln 3}\left\{1 + \frac{2^6 \times 5}{3^4}\frac{\lambda}{\mu^3} - (\frac{23 \times 19927}{2^2 \times 3^7} + \frac{1765769 \ln 3}{2^4 \times 3^8})\frac{\lambda^2}{\mu^6} + \cdots\right\}$$

# Gravity prediction at small $\mu$

Costa, Greenspan, Penedones and Santos, [arXiv:1411.5541]

$$\lim_{\frac{\lambda}{\mu^2}\to\infty} \frac{T_c^{\mathrm{SUGRA}}}{\mu} = 0.105905(57)\,.$$

The prediction is for low temperatures and small $\mu$ the transition temperature approaches zero linearly in $\mu$.

# Padé approximant prediction of $T_c$

$$T_c = \frac{\mu}{12 \ln 3} \left\{ 1 + r_1 \frac{\lambda}{\mu^3} + r_2 \frac{\lambda^2}{\mu^6} + \cdots \right\}$$

with

$$r_1 = \frac{2^6 \times 5}{3} \quad \text{and} \quad r_2 = -\left(\frac{23 \times 19927}{2^2 \times 3} + \frac{1765769 \ln 3}{2^4 \times 3^2}\right)$$

Using a Padé Approximant: $1 + r_1 g + r_2 g^2 + \cdots \to \frac{1+(r_1 - \frac{r_2}{r_1})g}{1 - \frac{r_2}{r_1} g}$

We have

$$T_c^{\text{Padé}} = \frac{\mu}{12 \ln 3} \frac{1 + (r_1 - \frac{r_2}{r_1}) \frac{\lambda}{\mu^3}}{1 - \frac{r_2}{r_1} \frac{\lambda}{\mu^3}}$$

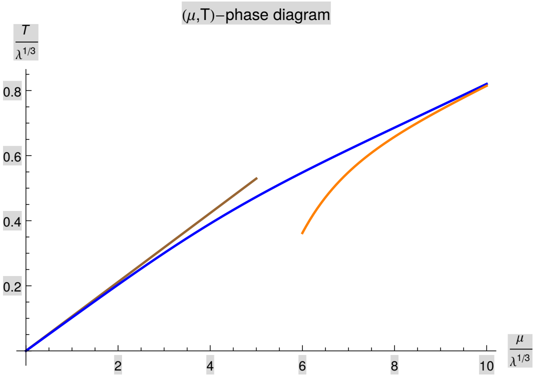
Now we can take the small $\mu$ limit to obtain a prediction that we can compare with supergravity

$$\lim_{\frac{\lambda}{\mu^2}\to\infty} \frac{T_c^{\text{Padé}}}{\mu} = \frac{1}{12\ln 3}(1-\frac{r_1^2}{r_2}) = 0.102353$$

This is to be compared with

$$\lim_{\frac{\lambda}{\mu^2}\to\infty} \frac{T_c^{\text{SUGRA}}}{\mu} = 0.105905(57)\,.$$

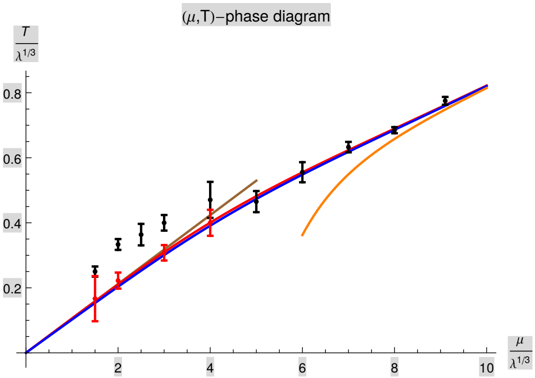
# An initial Phase diagram for the BMN model.




Orange Large mass expansion Hadizadeh, Ramadanovic, Semenoff, Young, [hep-th/0409318].

**Brown** Gravity prediction of Costa, Greenspan, Penedones, Santos, JHEP03(2015)069 [arXiv:1411.5541 [hep-th]].

Padé approximant: **Blue** uses only large mass expansion.




Orange Large mass expansion Hadizadeh, Ramadanovic, Semenoff, Young, [hep-th/0409318].

**Brown** Gravity prediction of Costa, Greenspan, Penedones, Santos, JHEP03(2015)069 [arXiv:1411.5541 [hep-th]].

Padé approximant: **Red** assumes Costa et al prediction, **Blue** uses only large mass expansion.

# Conclusions

- Bosonic membranes quantised a la Hoppe are well approximated as massive gauged gaussian models.
- Tests of the BFSS model against non-perturbative studies are in excellent agreement.
- It is useful to have probes of the geometry.
- The mass deformed model, i.e. the BMN model is more complicated. Initial phase diagrams indicate agreement with gravity predictions
- But ...
- More work is needed. A study of non-spherical type IIA black holes would be very useful.

Thank you for your attention!