All-loop non-Abelian Thirring model

K. SIAMPOS,
Albert Einstein Center for Fundamental Physics,
University of Bern,

based on works with
G. Georgiou, G. Itsios and K. Sfetsos
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Introduction and motivation

Exact β-functions and anomalous dimensions

1. In a renormalizable field theory, its quantum behaviour is depicted by:
   - The n-point correlation functions.
   - The dependence of the coupling with the energy scale.

2. Their dependence is encoded within the RG flow equations

$$\beta_\lambda = \frac{d\lambda}{d\ln\mu^2},$$

   which are usually determined perturbatively.

3. Can we obtain the all-loop β-function? New fixed points towards the IR?

4. Can we also calculate the all-loop correlators of various operators?

We study these aspects for the non-Abelian bosonized Thirring model.

Focal Points

- Non-Abelian Thirring model
- The resumed effective action
- The β-function
- Current correlators
- OPEs and equal-time commutators
- Conclusion and Outlook

Plan of the talk

Non-Abelian Thirring model

The resummed action

The β function

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NON-ABELIAN THIRRING MODEL

Consider the WZW action Witten (1983):

$$S_{\mathrm{WZW},k}(g) = -\frac{k}{4\pi}\int \mathrm{d}^2\sigma\,\mathrm{Tr}\left(g^{-1}\partial_+ g\, g^{-1}\partial_- g\right) + \frac{k}{24\pi}\int_B \mathrm{Tr}\left(g^{-1}\mathrm{d}g\right)^3 ,$$

invariant under the left-right current algebra symmetry: $g \mapsto \Omega^{-1}(\sigma_+)\, g\, \Omega(\sigma_-)$.

The holomorphic and anti-holomorphic currents obey the OPEs

$$J^a_\pm(z)J^b_\pm(0) = \frac{\delta_{ab}}{z^2} + \frac{f_{abc}\,J^c_\pm(0)}{\sqrt{k}\,z} + \text{regular}\,, \quad J^a_\pm(z)J^b_\mp(0) = \text{regular}\,,$$

$$J^a_+ = -i\,\mathrm{Tr}(t^a\,\partial_+ g\, g^{-1})\,, \quad J^a_- = -i\,\mathrm{Tr}(t^a\, g^{-1}\,\partial_- g)\,, \quad D_{ab} = \mathrm{Tr}(t^a g t^b g^{-1})\,,$$

where: $[t_a, t_b] = f_{abc} t_c$, $\mathrm{Tr}(t_a t_b) = \delta_{ab}$ and $f_{acd} f_{bcd} = -c_G\,\delta_{ab}$.

The non-abelian bosonized Thirring model is defined through

$$\boxed{S = S_{\mathrm{WZW},k} + k\,\frac{\lambda_{ab}}{2\pi}\int \mathrm{d}^2\sigma\, J^a_+\, J^b_-}$$

NON-ABELIAN THIRRING MODEL

Symmetries of the non-abelian bosonized Thirring model:

$$S = S_{\mathrm{WZW},k} + k\,\frac{\lambda_{ab}}{2\pi}\int \mathrm{d}^2\sigma\, J_+^a\, J_-^b\,.$$

1. It is invariant under the generalized parity symmetry:

$$\lambda \mapsto \lambda^T\,, \qquad g \mapsto g^{-1}\,, \qquad \sigma^{\pm} \mapsto \sigma^{\mp}\,.$$

2. The perturbation is not *exactly marginal*

Kutasov (1989) $$\beta_\lambda = -\frac{c_G\,\lambda^2}{2k\,(1+\lambda)^2} \leqslant 0\,, \quad \lambda_{ab} = \lambda\,\delta_{ab}\,.$$

3. The corresponding effective action is invariant under the inversion of the coupling:

Kutasov (1989) $$\lambda \mapsto \lambda^{-1}\,, \qquad k \mapsto -k\,, \qquad k \gg 1\,.$$

4. The left-right current algebra symmetry is broken for a generic matrix λ_{ab}.

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THE RESUMMED ACTION

By a gauging procedure we can construct the following action Sfetsos (2013)

$$S_{k,\lambda}(g) = S_{\mathrm{WZW},k} + \frac{k}{2\pi}\int \mathrm{d}^2\sigma\, J_+^a \left(\lambda^{-1}\mathbb{I} - D^T\right)^{-1}_{ab} J_-^b\,, \quad \lambda_{ab} = \lambda\,\delta_{ab}\,.$$

describes integrable interpolations from a WZW to (non-abelian T-duals) PCM models.
Sfetsos (2013), Itsios–Sfetsos–KS–Torrieli (2014)

Properties

- For $\lambda \ll 1$ we get the non-Abelian Thirring model.
- Invariance under the generalized parity symmetry: $g \mapsto g^{-1}\,, \sigma^{\pm} \mapsto \sigma^{\mp}$
- Weak-strong duality, $S_{-k,\lambda^{-1}}(g^{-1}) = S_{k,\lambda}(g)$, with the dressed currents given by

$$J_+^a(g)_{k,\lambda} = -\frac{i}{1+\lambda}(\mathbb{I} - \lambda D)^{-1}_{ab}\mathrm{Tr}(t^b\partial_+ g g^{-1})\,,$$
$$J_-^a(g)_{k,\lambda} = \frac{i}{1+\lambda}(\mathbb{I} - \lambda D^T)^{-1}_{ab}\mathrm{Tr}(t^b g^{-1}\partial_- g)\,,$$

where $J_\pm^a(g^{-1})_{-k,\lambda^{-1}} = \lambda^2 J_\pm^a(g)_{k,\lambda}$.

LIMITING CASES

There are two interesting limits:

1. *Zoom around* $\lambda = 1$*:*

$$\lambda = 1 - \frac{\kappa^2}{k} + \ldots\,, \quad g = \mathbb{I} + i\,\frac{v_a t^a}{k} + \ldots\,, \quad k \gg 1\,,$$

we get the non-abelian T-dual

$$S_{\text{non-Abel}} = \frac{1}{2\pi}\int \mathrm{d}^2\sigma\, \partial_+ v^a \left(\kappa^2 \mathbb{I} + f\right)^{-1}_{ab} \partial_- v^b\,, \quad f_{ab} := -i f_{abc}\, v^c\,,$$

of the PCM with respect to G_L or G_R

$$S_{\text{PCM}} = \frac{\kappa^2}{2\pi}\int \mathrm{d}^2\sigma\, \mathrm{Tr}\left(g^{-1}\partial_+ g\, g^{-1}\partial_- g\right)$$

2. *Zoom around* $\lambda = -1$*:*

$$\lambda = -1 + \frac{1}{b^{2/3}\, k^{1/3}} + \ldots\,, \quad g = \mathbb{I} + i\,\frac{v_a t^a}{k^{1/3}} + \ldots\,, \quad k \gg 1\,,$$

we have the pseudodual model

$$S_{\text{pseudo-dual}} = \frac{1}{8\pi}\int \mathrm{d}^2\sigma\, \partial_+ v^a \left(\frac{\delta_{ab}}{b^{2/3}} + \frac{1}{3} f_{ab}\right) \partial_- v^b$$

Nappi 1980

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Constraints on β function

The β function at one-loop in $1/k$ expansion takes the form:

$$\beta = \frac{d\lambda}{d\ln\mu^2} = -\frac{1}{k} f(\lambda)\,.$$

- From CFT perturbations we expect that:

$$f(\lambda) \simeq \frac{1}{2} c_G \lambda^2 + \mathcal{O}(\lambda^3)$$

- Due to the weak–strong duality we have the constraint:

$$\lambda f(\lambda^{-1})\,\lambda = f(\lambda)\,.$$

Let us compute $f(\lambda)$

General approach

Consider a 1+1-dimensional non-linear σ-model with action

$$S = \frac{1}{2\pi\alpha'} \int \mathrm{d}^2\sigma \, E_{\mu\nu} \, \partial_+ X^\mu \partial_- X^\nu \,, \quad E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$$

The one-loop β-functions for $G_{\mu\nu}$ and $B_{\mu\nu}$ read:
Ecker–Honerkamp 71, Friedan 80, Braaten–Curtright–Zachos 85

$$\frac{\mathrm{d}\, E_{\mu\nu}}{\mathrm{d} \ln \mu^2} = R^-_{\mu\nu} + \nabla^-_\nu \xi_\mu \,,$$

where the last term corresponds to field redefinitions (diffeomorphisms).

Generalities:

- The Ricci tensor and the covariant derivative includes torsion, i.e. $H = \mathrm{d}B$
- The σ-model is renormalizable within the zoo of metrics and 2-forms
- Not given that the RG flows will retain the form at hand of $G_{\mu\nu}$ and $B_{\mu\nu}$

Isotropic case

The RG flow at one-loop in $^1/_k$ expansion retains the form of the σ-model

Itsios–Sfetsos–KS (2014) $\beta = \frac{d\lambda}{d\ln\mu^2} = -\frac{c_G\,\lambda^2}{2k(1+\lambda)^2}\,, \quad 0 \leqslant \lambda \leqslant 1\,, \quad k$ does not flow

Properties of the flow

1. In agreement with the all-loop isotropic Thirring model Kutasov 89
2. Invariance under the weak–strong duality, i.e. $\lambda \mapsto \lambda^{-1}\,, \quad k \mapsto -k \quad$ for $\quad k \gg 1$
3. It behaves according to CFT expectations around $\lambda \ll 1 \Longrightarrow \beta \simeq -\frac{c_G\,\lambda^2}{2k} + \mathcal{O}(\lambda^3)$
4. The β-function can be solved explicitly:

$$\lambda - \lambda^{-1} + 2\ln\lambda = -\frac{c_G}{2k}\,(t - t_0)\,,$$

   where UV at $\lambda \to 0^+$ and towards the IR at $\lambda \to 1^-$.

$$S = S_{WZW,k} + k\,\frac{\lambda}{2\pi}\int J_+^a J_-^a \Leftrightarrow S_{k,\lambda}(g) = S_{WZW,k} + \frac{k}{2\pi}\int J_+\left(\lambda^{-1} - D^T\right)^{-1} J_-$$

Plan of the talk

Regularization method

Starting point is the non-Abelian Thrirring model

$$S = S_{\mathrm{WZW},k} + k\,\frac{\lambda}{2\pi}\int \mathrm{d}^2\sigma\, J_+^a\, J_-^a\,.$$

To compute the current correlators we expand around the WZW model.

Schematically we compute the $\mathcal{O}(\lambda^n)$ correction of the correlation function

$$\begin{aligned}&\langle F_1(x_1,\bar{x}_1)F_2(x_2,\bar{x}_2)\dots\rangle^{(n)}_{k,\lambda}\\ &= \frac{1}{n!}\left(-\frac{\lambda}{\pi}\right)^n \int \mathrm{d}^2 z_1 \dots \mathrm{d}^2 z_n\, \langle J^{a_1}(z_1)\dots \bar{J}^{a_1}(\bar{z}_1)\dots\; F_1(x_1,\bar{x}_1)F_2(x_2,\bar{x}_2)\dots\rangle\end{aligned}$$

Regularization scheme

The internal points cannot coincide with external ones:

$$\boxed{D_n = \{(z_1,z_2,\dots,z_n)\in\mathbb{C} : |z_i - x_j| > \varepsilon,\, \varepsilon > 0\}\,,\quad \forall i \neq j\,,}$$

and some basic integrals

$$\int_{D_1}\frac{\mathrm{d}^2 z}{(z-x_1)(\bar{x}_2-\bar{z})} = \pi \ln|x_{12}|^2\,,\quad \int_{D_1}\frac{\mathrm{d}^2 z}{(x_1-z)^2(\bar{z}-\bar{x}_2)^2} = \pi^2\,\delta^{(2)}(x_{12})\,.$$

Results

Perturbative results, well defined behaviour at $\lambda = \pm 1$ and weak-strong duality:

1. β-function and anomalous dimension of the current operator

$$\boxed{\beta_\lambda = -\frac{c_G\,\lambda^2}{2k\,(1+\lambda)^2} \leqslant 0\,, \qquad \gamma^{(J)} = \frac{c_G\,\lambda^2}{k\,(1-\lambda)(1+\lambda)^3} \geqslant 0}$$

In agreement with the results derived from the effective action.
Itsios, Sfetsos, KS (2014), Appadu, Hollowood (2015), Georgiou, Sfetsos, KS (2015)

2. All-loop two and three-point functions – leading in $1/k$ expansion

$$\langle J^a(x_1)J^b(x_2)\rangle_{k,\lambda} = \frac{\delta_{ab}}{x_{12}^{2+\gamma^{(J)}}\bar{x}_{12}^{\gamma^{(J)}}}\,, \quad \langle J^a(x_1)\bar{J}^b(\bar{x}_2)\rangle_{k,\lambda} = -\gamma^{(J)}\frac{\delta_{ab}}{|x_{12}|^2}\,.$$

$$\langle J^a(x_1)J^b(x_2)J^c(x_3)\rangle_{k,\lambda} = \frac{1+\lambda+\lambda^2}{\sqrt{k(1-\lambda)(1+\lambda)^3}}\frac{f_{abc}}{x_{12}x_{13}x_{23}}\,,$$

$$\langle J^a(x_1)J^b(x_2)\bar{J}^c(\bar{x}_3)\rangle_{k,\lambda} = \frac{\lambda}{\sqrt{k(1-\lambda)(1+\lambda)^3}}\frac{f_{abc}\bar{x}_{12}}{x_{12}^2\bar{x}_{13}\bar{x}_{23}}$$

Georgiou, Sfetsos, KS (2016)
The expressions match (under a rescaling) with Konechny–Quella (2011) for supergroups.

OPEs and equal-time commutators

Using the above we find the OPE algebra

$$J^a(x_1)J^b(x_2) = \frac{\delta_{ab}}{x_{12}^{2+\gamma^{(J)}}\bar{x}_{12}^{\gamma^{(J)}}} + c(\lambda)\frac{f_{abc}J^c(x_2)}{x_{12}} + d(\lambda)\frac{f_{abc}\bar{J}^c(\bar{x}_2)\bar{x}_{12}}{x_{12}^2} + \dots,$$

$$J^a(x_1)\bar{J}^b(\bar{x}_2) = -\gamma^{(J)}\frac{\delta_{ab}}{|x_{12}|^2} + d(\lambda)\frac{f_{abc}\bar{J}^c(\bar{x}_2)}{x_{12}} + d(\lambda)\frac{f_{abc}J^c(x_2)}{\bar{x}_{12}} + \dots,$$

where $x_{12} := x_1 - x_2$ and

$$d(\lambda) = \frac{1}{\sqrt{k(1-\lambda^2)}}\,\frac{\lambda}{1+\lambda}, \quad \frac{c(\lambda)}{d(\lambda)} = 1 + 2\chi, \quad \chi = \frac{1+\lambda^2}{2\lambda}.$$

Having the OPEs, we can compute the equal-time commutators:

$$[S^a(\sigma_1), S^b(\sigma_2)] = \frac{ik}{2\pi}\,\delta_{ab}\delta'(\sigma_{12}) + f_{abc}\,S^c(\sigma_2)\,\delta(\sigma_{12}),$$

$$[\bar{S}^a(\sigma_1), \bar{S}^b(\sigma_2)] = -\frac{ik}{2\pi}\,\delta_{ab}\delta'(\sigma_{12}) + f_{abc}\,\bar{S}^c(\sigma_2)\,\delta(\sigma_{12}),$$

$$[S^a(\sigma_1), \bar{S}^b(\sigma_2)] = 0,$$

Rajeev's (1989)

with $x = \sigma + i\tau$ and $S^a = \frac{1}{2\pi}\sqrt{\frac{k}{1-\lambda^2}}\,\left(J^a - \lambda\bar{J}^a\right)$, $\quad \bar{S}^a = \frac{1}{2\pi}\sqrt{\frac{k}{1-\lambda^2}}\,\left(\bar{J}^a - \lambda J^a\right)$.

Deformation of the PB for the PCM, also realized by the resumed effective action.

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Our resummed action

$$S_{k,\lambda}(g) = S_{\mathrm{WZW},k} + \frac{k}{2\pi}\int \mathrm{d}^2\sigma\, J_+^a \left(\lambda^{-1}\mathbb{I} - D^T\right)^{-1}_{ab} J_-^b$$

enraptures the all-loop isotropic Thirring model at leading order in $1/k$ expansion

$$S = S_{\mathrm{WZW},k} + k\,\frac{\lambda}{2\pi}\int \mathrm{d}^2\sigma\, J_+^a\, J_-^a$$

The agreement is based upon:

1. Symmetries of the actions
2. Invariance under the weak–strong duality, i.e. $\lambda \mapsto \lambda^{-1}\,, \quad k \mapsto -k \quad \text{for} \quad k \gg 1$
3. β-functions and anomalous dimension $\gamma^{(J)}$
4. Current algebra – Rajeev's deformation of the PB of the isotropic PCM

Extensions:

- We can also include affine primary fields
  Georgiou, Sfetsos, KS (2016)
- Subleading in $1/k$ expansion, beyond the weak–strong duality:
  Kutasov (1989) $\lambda \mapsto \lambda^{-1}\,, \quad k \mapsto -k - c_G$
- Cases beyond isotropy $\lambda_{ab} \neq \lambda\,\delta_{ab}$

Fermionic model

Exactly solvable QFT describing self-interacting massless Dirac fields in 1+1 dimensions.

- An 1+1 dimensional action with fermions in the fundamental representation of $SU(N)$

  Dashen–Frishman (1973)&(1975) : $\mathcal{L}_{int} = -\frac{g_B}{2} J_\mu J^\mu - \frac{g_V}{2} J^a_\mu J^{a\mu}\,, \quad \mu = 0, 1\,,$

  where $J^a_\mu = \bar{\Psi} t^a \gamma_\mu \Psi$, with $a = 1, \ldots, N^2 - 1$, are the $SU(N)$ currents and J_μ the $U(1)$.

- For $N = 1$ we recover the Abelian case (prototype) Thirring (1958)

- It is invariant under $SU(N) \times U(1)$ (vector) and $U(1)_{\text{Axial}}$

- The non-Abelian term breaks $SU(N)_{\text{Axial}}$, i.e. $\partial^\mu J^{5a}_\mu = g_V f_{abc} J^{b\mu} J^{5c}_\mu$

- The theory is scale-invariant only for $g_V = 0$ and $g_V = \frac{4\pi}{n+1}$

- There is a current algebra at level one $J^a_\pm(z) J^b_\pm(0) = \frac{\delta_{ab}}{z^2} + \frac{f_{abc} J^c_\pm(0)}{z}$