Holography, duality and integrability

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Recent developments in strings and gravity
The Corfu Summer Institute

Corfu – September 2016

Highlights

Applied holography

Macroscopic extension of AdS/CFT: gravity plus matter on $D+1$-dim asymptotically AdS background $\leftrightarrow$ phenomenological D-dim boundary CFT in a macroscopic state $\rightarrow$ potentially hydrodynamic

Quantum field theory: Compute correlation functions & transport coefficients of the CFT macroscopic states – AdS/QCD (quark-gluon plasma), AdS/CMT (superconductors, rotating cold atoms, ...)

Genuine exp. & th. puzzles: *measure* and *compute* the *Hall viscosity* in neutral rotating Bose–Einstein condensates

(Super)gravity:

- *Usual Holy Grail: string theory in non-perturbative regime*
- *Less usual: study the integrable sector of Einstein's equations*

Integrable sector of Einstein's equations

Reminder: solution generating techniques *[Ehlers '59; Ernst '68; Geroch '71]*

$$(\mathcal{M}, g, \xi) \to (\mathcal{S}, h, A, \phi) \underset{U}{\to} (\mathcal{S}, h', A', \phi') \to (\mathcal{M}, g', \xi')$$

- Examples with $\Lambda = 0$:
  - $4 \to 3$ $U \equiv SL(2, \mathbb{R})$ [Geroch '71]
  - $4 \to 2$ $U \equiv$ affine algebra $\to$ integrable [Belinskii, Zakharov '78;. . .]

  many others in supergravity up to 11 dim
- Caveat: less powerful when $\Lambda \neq 0$ [Leigh, Petkou, Petropoulos, Tripathy '14]

Question: does holography provide an alternative? *[Caldarelli, Gath, Leigh, Mukhopadhyay, Petkou, Petropoulos, Pozzoli, Siampos '11–'15]*

$$(\mathcal{M}, g_{bulk}) \underset{r\to\infty}{\longrightarrow} (\mathcal{B}, g_{bry.}, T) \underset{\text{“}U\text{”}}{\to} \overbrace{(\mathcal{B}, g'_{bry.}, T') \underset{\text{“}\infty\to r\text{”}}{\longrightarrow} (\mathcal{M}, g'_{bulk})}^{\textit{exact reconstruction}}$$

(Note: pure-gravity 4-dim bulk – simplest string low-energy approx.)

Here

The questions:

- *Can one appropriately design boundary data such that exact bulk Einstein solutions exist that reproduce them – reconstruction?*
- *Can one find "U" relating these distinguished boundary data – acting as bulk solution-generating transformations?*

The answer: *yes, boundary data can be tuned to produce exact bulk solutions and holographic Geroch-like transformations do exist*

The spirit: *set conditions that relate "initial momentum" and "initial position" – inspired from gravitational or Yang–Mills self-duality*

Bonus: *boundary data $\equiv$ macroscopic CFT state on a background $\rightarrow$ exact transport properties for the boundary state*

Highlights

Holography

Bulk: Einstein space with $\Lambda = -3k^2$, asymptotically locally AdS – emerges as an ADM-like radial Hamiltonian evolution from data on the time-like initial hypersurface at $r \to \infty$

- The "initial" hypersurface is the conformal boundary $\mathcal{B}$
- The "initial" data are
  - the boundary metric $\mathrm{d}s^2_{\text{bry.}} = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$
  - the boundary "second fundamental form" $T_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$

$T^{\mu\nu}$ symmetric, traceless with $\nabla_\mu T^{\mu\nu} = 0$: interpreted as holographic energy–momentum tensor vev on the macroscopic state of the boundary-CFT – possibly a fluid

Reminder: here pure gravitational bulk backgrounds $\to$ no boundary currents $J^\mu \to$ neutral boundary states

Boundary metric & boundary energy–momentum tensor

Packaged in the Fefferman–Graham on-shell expansion of the $D+1$-dim bulk metric for large r: *[Fefferman, Graham '85]*

- metric: "generalized coordinate" – leading term
- energy–momentum: "conjugate momentum" – subleading term

$$\mathrm{d}s^2_{\text{bulk}} = \frac{\mathrm{d}r^2}{k^2 r^2} + k^2 r^2 \mathrm{d}s^2_{\text{bry.}} + \cdots + \frac{16\pi G}{3k(kr)^{D-2}} T_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \cdots$$

- *This requires a specific r-coordinate gauge: no lapse/shift*
- *All other terms are either vanishing or expressed with $g_{\mu\nu}$ and $T_{\mu\nu}$ & derivatives (Schouten, Cotton, ...)*

A primer: Schwarzschild AdS$_4$ black hole ($d = 4$, $D = 3$)

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1+k^2r^2-\frac{2M}{r}} - \left(1 + k^2r^2 - \frac{2M}{r}\right)\mathrm{d}t^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right)$$

$$\mathrm{d}s^2_{\text{bry.}} = -\mathrm{d}t^2 + \frac{1}{k^2}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right) \quad \to \mathbb{R}\times S^2$$

$$T_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \frac{\varepsilon}{2}\left(2\mathrm{d}t^2 + \frac{1}{k^2}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right)\right) = \frac{\varepsilon}{2}\left(3\mathrm{u}^2 + \mathrm{d}s^2_{\text{bry.}}\right)$$

conformal perfect fluid with $\mathrm{u} = \partial_t$ and $\varepsilon = 2p = {}^{Mk^2}/_{4\pi G}$

- the holographic fluid *is not* perfect (${}^{\eta}/_{s} \gtrapprox {}^{\hbar}/_{4\pi k_{\mathrm{B}}}$)
- $T^{\mu\nu} = T^{\mu\nu}_{\text{perf}}$ because of the kinematic state (fluid at rest)

Note: no information on any transport coefficient – to get insight on transport coefficients we need more involved bulk Einstein spaces

Highlights

The question

Given a boundary geometry $ds^2_{bry.}$ can one determine

- *the conditions it should satisfy*
- *the stress tensor it should be accompanied with*

for the FG expansion to be exactly resummable? $\longrightarrow$ *answer encoded in integrability properties*

- Integrability deeply related with self-duality properties: in the '70s all integrable systems were thought to be SDYM reductions *... many (and perhaps all?) of the ordinary or partial differential equations that are regarded as being integrable or solvable may be obtained from the self-duality equations (or its generalizations) by reduction.* [Ward, '85]
- Self-duality appears in the ancestor of holography: LeBrun's filling-in problem [LeBrun, '82]
  - Problem: how to fill-in *analytically* the 3-dim Berger sphere?
  - Solution: with a quaternionic space (Einstein & Weyl-self-dual)

At the heart of duality: the Weyl tensor

Atiyah–Hitchin–Singer packaging of the 5 Ψ_as inside a symmetric and traceless 3×3 complex matrix

$W^{\pm}_{\mu\nu}$, $\mu, \nu \ldots = 0, 1, 2 \to$ (anti-)self-dual Weyl $\frac{W \pm i \star W}{2}$

The existence of 4 principal null directions, potentially degenerate with higher multiplicity, translates into special algebraic relationships among the Ψs: Petrov type I, II, III, D, N, O

Goldberg–Sachs theorem and extensions: $\exists$ null, shearless, geodesic congruence $\leftrightarrow$ algebraically special space

The 3-dim origin of the Weyl

Large-r expansion of $W^{\pm}$ *[Mansi, Petkou, Tagliabue '08; de Haro '08; Miskovic, Olea '09]*

$$W^{\pm} = \frac{8\pi G}{k^2 r^3} P^{-1} T^{\pm} P + \cdots$$

$\mathrm{P} = \mathrm{diag}(\mp i, -1, 1)$ and $\mathrm{T}^{\pm}$ symmetric, traceless and conserved:

$$\boxed{T^{\pm}_{\mu\nu} = T_{\mu\nu} \pm \frac{i}{8\pi G k^2} C_{\mu\nu}}$$

Property: remarkably simple (i.e. with degenerate eigenvalues) if the space is Petrov algebraically special
Integrability/resummability ↔ setting conditions between "initial momentum" and "initial position" ↔ tuning $T^{\pm}$

Reminder: the Cotton tensor

In 3 dim the Weyl tensor vanishes – conformal properties are captured by the Cotton tensor

$$C_{\mu\nu} = \frac{\epsilon_{\mu\rho\sigma}}{\sqrt{|g|}} \nabla^{\rho} \left(R_{\nu}{}^{\sigma} - \frac{R}{4} \delta_{\nu}{}^{\sigma} \right)$$

- symmetric and traceless
- conformally covariant of weight 1
- identically conserved: $\nabla_{\mu} C^{\mu\nu} = 0$

The pattern for exact bulk reconstruction

The metric $ds^2_{bry.}$ must admit 2 symmetric, traceless and exactly conserved rank-2 tensors $T^{\pm}$ related by complex conjugation
Crucial observation: $T^{\mu\nu}$ and $C^{\mu\nu}$ can be arbitrarily complicated but $T^{\pm}_{\mu\nu}$ is expected to be simple for Petrov algebraically special bulks

For given $ds^2_{bry.}$

- choose simple (canonical) $T^{\pm}$ s.t.

$$\nabla^{\mu} T^{\pm}_{\mu\nu} = 0 \qquad \text{(h)}$$

- further impose on $ds^2_{bry.}$ the condition

$$C = 8\pi G k^2 \, \mathrm{Im} T^{+} \qquad \text{(C)}$$

- build the bulk with the resulting $ds^2_{\text{bry.}}$ and the stress tensor

$$\boxed{T = \text{Re}T^+} \qquad (T)$$

The exact bulk reconstruction works

- (h) and (C) are the boundary version of Einstein's Eqs. for algebraically special Petrov classes (which class $\leftrightarrow T^\pm$)
- (T) is a non-trivial boundary stress tensor allowing for probing transport (fluid plus possibly non-hydrodynamic modes)

How does reconstruction work?

Derivative expansion *[Bhattacharyya et al '08]*

Alternative to the Fefferman–Graham expansion – originally designed for the hydrodynamic regime & fluid/gravity correspondence

- *boundary metric* $ds^2_{bry.}$
- *boundary*
  - *energy–momentum tensor* T
  - *time-like hydrodynamic congruence* u

The congruence u is redundant and arbitrary

- $ds^2_{bry.}$ and T are *sufficient* to determine the bulk
- non-perfect relativistic fluids have *no* natural velocity

Purpose: organize the perturbative long-wave-length expansion, both on the boundary and in the bulk, and potentially resum it

On the boundary $(D = 3)$

Fluids in 3-dim gravitational backgrounds are described in terms of 4 *independent quantities* u, ε, p *all in* $T^{\mu\nu}$

$$T^{\mu\nu} = T^{\mu\nu}_{perf} + \Pi^{\mu\nu}$$

with $\nabla_\mu T^{\mu\nu} = 0$ *plus an Eq. of state*

- $T^{\mu\nu}_{perf} = \varepsilon u^\mu u^\nu + p h^{\mu\nu}$ *(*$h_{\mu\nu}$*: metric on* $\Sigma \perp u$*)*
- $\Pi^{\mu\nu}$ *as expansion in* $\nabla^n u \rightarrow$ *transport coefficients*

Remarks:

- $\Pi^{\mu\nu}$ may contain *non-hydrodynamic* modes – asymptotic series
- Freedom in choosing u – Landau–Lifshitz frame: $\Pi^{\mu\nu}_{\text{hydro}} u_\nu = 0$
  example $\Pi^{\mu\nu}_{(1)} = -2\eta\sigma^{\mu\nu} - \zeta\Theta h^{\mu\nu} - \frac{\zeta_{\mathrm{H}}}{\sqrt{-g}}\epsilon^{\rho\lambda(\mu} u_\rho \sigma_\lambda{}^{\nu)}$
- $\nabla \cdot T_{(n)} = \mathrm{O}\left(\nabla^{n+1}\mathrm{u}\right)$ for the expansion to make sense, and this restricts the choice for u

In the bulk [Bhattacharyya et al '08; Caldarelli et al '12]

Guideline for the bulk reconstruction along null tubes supporting the holographic coordinate r: Weyl covariance – the bulk metric should be invariant under boundary conformal transformations

Tool: Weyl connection $A = a - \frac{\Theta}{2} u$ and Weyl covariant derivative $\mathscr{D} = \nabla + wA$ (a is the acceleration and $\Theta = \nabla \cdot u$)

$$
\begin{aligned}
ds^2_{bulk} &= -2u(dr + rA) + r^2 k^2 ds^2_{bry.} + \frac{1}{k^2}\Sigma \\
&+ \frac{u^2}{r^2}\left(1 - \frac{1}{2k^4 r^2}\omega_{\alpha\beta}\omega^{\alpha\beta}\right)\left(\frac{3T_{\lambda\mu}u^\lambda u^\mu}{\kappa k} r + \frac{C_{\lambda\mu}u^\lambda \eta^{\mu\nu\sigma}\omega_{\nu\sigma}}{2k^6}\right) \\
&+ \text{terms with } \sigma, \sigma^2, \nabla\sigma, \ldots + O\left(\mathscr{D}^4 u\right)
\end{aligned}
$$

- $\Sigma = -2u\nabla_\nu \omega^\nu{}_\mu dx^\mu - \omega_\mu{}^\lambda \omega_{\lambda\nu} dx^\mu dx^\nu - \frac{1}{2}u^2 \mathscr{R}$
- $\omega = \frac{1}{2}(du + u \wedge a) \quad \eta^{\mu\nu\sigma} = \epsilon^{\mu\nu\sigma}/\sqrt{-g_{bry.}}$
- $\mathscr{R} = R + 4\nabla_\mu A^\mu - 2A_\mu A^\mu \quad \kappa = {}^{3k}/{}_{8\pi G}$

The resummation [Caldarelli et al '12; Mukhopadhyay et al '13; Gath et al '15]

Assuming u shear-free a partial resummation is performed with

$$\frac{1}{r^2}\left(1-\frac{1}{2k^4r^2}\omega_{\alpha\beta}\omega^{\alpha\beta}\right)\to\frac{1}{r^2+\frac{1}{2k^4}\omega_{\alpha\beta}\omega^{\alpha\beta}}\equiv\frac{1}{\rho^2}$$

$$\begin{aligned} ds^2_{res.} &= -2u(dr+rA)+r^2k^2ds^2_{bry.}+\frac{1}{k^2}\Sigma \\ &+\frac{u^2}{\rho^2}\left(\frac{8\pi G T_{\lambda\mu}u^\lambda u^\mu}{k^2}r+\frac{C_{\lambda\mu}u^\lambda\eta^{\mu\nu\sigma}\omega_{\nu\sigma}}{2k^6}\right) \end{aligned} \qquad (R)$$

Why shear-free: convenient and unique in Lorentzian $D=3$

Theorem: when

- *u is the boundary time-like shearless congruence of* $ds^2_{bry.}$
- $ds^2_{bry.}$ *obeys Eqs. (h) and (C) for some canonical* $T^{\pm}$
- *T given in Eq. (T)*

(R) exact Einstein & Petrov-algebraic (∂_r: bulk null, geodesic, shearless → Goldberg–Sachs)

This resummation method

- gives access to *all* Petrov-algebraic Einstein spaces: Kundt, Robinson–Trautman, Plebański–Demiański, . . .
- is based on a non-perfect-fluid stress–energy tensor: $\mathrm{T} = \mathrm{T}^{\text{perf}} + \Pi$ with *finite-derivative corrections*
  - rich information on transport properties
  - not in Landau–Lifshitz frame: $\Pi_{\mu\nu} u^{\nu} \neq 0 \quad \Pi_{\mu\nu} u^{\mu} u^{\nu} = 0$

We are controlling the integrable algebraically special sector of Einstein's equations from non-trivial boundary fluid dynamics

Highlights

Boundary metric & tensors $T^{\pm}$

A simple boundary metric without isometries

$$\boxed{ds^2_{bry.} = -dt^2 + \frac{2}{k^2 P^2} d\zeta d\bar{\zeta}} \qquad (nv)$$

with a unique shearless congruence $u = -dt$

- $P(t, \zeta, \bar{\zeta})$ real & *a priori* arbitrary
- Cotton-tensor components $C_{\mu\nu}$: 3rd-derivative of P

The $T^{\pm}$s: Petrov–Segre type D_t & type III & N

$$\begin{cases} T^{\pm}_{pf} = \frac{M_{\pm} k^2}{8\pi G} \left(3 \left(u^{\pm}\right)^2 + ds^2_{bry.}\right), \quad u^{+} = -dt + \frac{\alpha^{+}}{P^2} d\zeta \quad \& \quad c.c. \\ T^{+}_{mr} = \frac{1}{4\pi G} d\zeta \left(\beta dt + \frac{\gamma}{k^2} d\zeta\right) \end{cases}$$

Integrability conditions

P, $M_{\pm}$, $\alpha^{\pm}$, β, γ and c.c. functions of $t, \zeta, \bar{\zeta}$

Obey

- partial differential equations following $\nabla \cdot T^{\pm} = 0$ (h)
- the integrability constraints $\mathrm{C} = 8\pi G k^2 \, \mathrm{Im} \mathrm{T}^{+}$ (C)

Combined $\rightarrow$ *Robinson–Trautman equation*

$$\boxed{\Delta K - 12 M \partial_t \log P + 4 \partial_t M = 0}$$

without any reference to a 4-dim bulk

Energy–momentum tensor and resummation

Using $ds^2_{bry.}$, $T = ReT^+$ *and* u *in Eq. (R)*

$$\boxed{ds^2_{res.} = 2dt\,dr - 2Hdt^2 + 2\frac{r^2}{P^2}d\zeta d\bar{\zeta}}$$

where

$$2H = k^2r^2 + K + 2r\partial_t \log P - \frac{2M}{r}$$

This is *Robinson–Trautman* space–time with t retarded time

- Einstein thanks to Robinson–Trautman
- Petrov type determined by the choice of $\mathrm{T}^{\pm}$
    - generically type II
    - $M = 0$: type III ($\Psi_2 = 0$)
    - $M = \beta = 0$: type N ($\Psi_2 = \Psi_3 = 0$)
    - $\beta = \gamma = 0$: type D

Robinson–Trautman physics

The bulk: generically time-dependent and singular at $r = 0$ (sources)

- Type II ($M \neq 0$): black hole with horizon radiating away gravitational waves $\underset{t\to\infty}{\longrightarrow}$ Schwarzschild AdS_4
- Type III & N ($M = 0$): pure gravitational waves $\underset{t\to\infty}{\longrightarrow}$ AdS_4
- Type D: Schwarzschild AdS_4 and C-metric (stationary)

The boundary: generically non-global-equilibrium

- Type II: claim for hydrodynamic regime at late times
- Type III & N: "pure radiation" at late times
- Type D: genuine stationary hydrodynamics at any time

Probing correlations and transport: under investigation [de Freitas, Reall '14; Bakas, Skenderis '14 & '15; Mukhopadhyay et al work in progress]

Highlights

Foreword

From the bulk to the boundary: the boundary data

From the boundary to the bulk: integrability and resummation

A concrete example

Transforming the boundary data

Outlook

Invariance transformation

Eqs. (h) $\nabla \cdot T^{\pm} = 0$ *invariant under*

$$T^{+} \to z\,T^{+}, \quad T^{-} \to \bar{z}\,T^{-}, \quad z \in \mathbb{C}$$

- *Drastic modification of boundary data*

Eqs. (T) & (C) $\Rightarrow$

$$\mathsf{T} \to \frac{1}{2}\left(z\,\mathsf{T}^{+} + \bar{z}\,\mathsf{T}^{-}\right), \quad \mathsf{C} \to \frac{3k^3}{2i\kappa}\left(z\,\mathsf{T}^{+} - \bar{z}\,\mathsf{T}^{-}\right)$$

local for T but *non-local* for $\mathrm{d}s^2_{\text{bry.}}$

- *Non-local modification of ds^2_{bulk}*

Setting $T^{\pm}_{\mu\nu}u^{\mu}u^{\nu} = \varepsilon(x) \pm i\frac{\kappa}{3k^3}c(x)$ and $z = |z| \exp i\psi$

- on the boundary: mixing of $\varepsilon(x)$ and $c(x)$

$$\begin{pmatrix} \varepsilon \\ \frac{\kappa}{3k^3}c \end{pmatrix} \to |z| \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \varepsilon \\ \frac{\kappa}{3k^3}c \end{pmatrix}$$

- in the bulk: mixing of W^+ and W^- – non-local in ds^2_{bulk}

$$\Psi_2 \approx -\frac{3}{2k\kappa r^3}\left(\varepsilon + \frac{i\kappa}{3k^3}c\right) \to -\frac{3|z|e^{i\psi}}{2k\kappa r^3}\left(\varepsilon + \frac{i\kappa}{3k^3}c\right)$$

This is a non-trivial bulk solution-generating transformation

Back to duality

Generically $\varepsilon \sim m$ *and* $c \sim n$ – *for the bulk parameters the mapping reads*

$$\begin{pmatrix} m \\ n \end{pmatrix} \to |z| \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}$$

- For $\psi = {}^{-\pi}/_{2}$ and $|z| = 1$: gravitational duality map

$$(m, n) \to (-n, m)$$

- For $|z| = 1$ and arbitrary ψ: missing $U(1)$ of the $SL(2, \mathbb{R})$ Ehlers–Geroch group for $\Lambda \neq 0$ [Leigh, Petkou, Petropoulos, Tripathy '14]

Highlights

The original question: is holography an alternative to Geroch-like solution-generating techniques?

$$\left(\mathcal{M}, g_{bulk}\right) \underset{r\to\infty}{\longrightarrow} \left(\mathcal{B}, g_{bry.}, T\right) \underset{\text{“}U\text{”}}{\rightarrow} \overbrace{\left(\mathcal{B}, g'_{bry.}, T'\right) \underset{\text{“}\infty\to r\text{”}}{\longrightarrow} \left(\mathcal{M}, g'_{bulk}\right)}^{\text{exact reconstruction}}$$

- *We handled the exact reconstruction: how to tune and combine boundary data for resummable ascendent – for given $ds^2_{bry.}$*
  - *design $T^{\pm}$ s.t. $\nabla \cdot T^{\pm} = 0$ (h)*
  - *impose $C = 8\pi G k^2\, \mathrm{Im} T^{+}$ (C) & $T = \mathrm{Re} T^{+}$ (T)*

  (C) & (h) $\equiv$ Einstein's Eqs. from the bry. in some integrable sector
- *We found "U": inside (C), (T) & (h) with gravitational duality as a discrete subgroup*

What can we recover? Algebraically special spaces

Next

In the framework of integrability:

- *State appropriately the invariance of the bulk under the choice of u: include shear on the boundary*
- *Beyond algebraically special spaces: integrability & resummation*
- *Euclidean holography: Przanowski–Tod & Calderbank–Pedersen spaces*
- *Higher dimensions: $8 \to 7$, $Spin_7$, G_2, ... as a generalization of $4 \to 3$, $SO(4)$, $SO(3)$, ...*

More on the physics of boundary CFTs: probe of transport properties and the remarkable relationships among coefficients, thermal correlators, way to equilibrium, ...

Highlights

LeBrun's filling-in – 1982

The problem (Euclidean)

- A round S^3 can be "filled-in" by H_4

$$ds^2_{H_4} = \frac{dr^2}{1+r^2} + r^2 d\Omega^2_{S^3} \to r^2 d\Omega^2_{S^3}$$

- How to fill-in *analytically* a Berger sphere?

$$d\Omega^2_{\text{Berger}} = (\sigma^1)^2 + (\sigma^2)^2 + \gamma(\sigma^3)^2$$

(σ^i: Maurer–Cartan forms of $SU(2)$)

The answer: Einstein space with self-dual Weyl tensor – quaternionic space *[LeBrun '82; Pedersen '86; Pedersen, Poon '90; Tod '94; Hitchin '95]*

A classic example

Bianchi IX AdS Schwarzschild–Taub–NUT

- Einstein space with $\Lambda = -3k^2$, mass M, nut charge n

$$\begin{aligned} \mathrm{d}s^2 &= \frac{\mathrm{d}r^2}{V(r)} + \left(r^2 - n^2\right)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right) \\ &\quad + V(r)\left(\mathrm{d}\tau + 4n\sin^2\frac{\vartheta}{2}\mathrm{d}\varphi\right)^2 \end{aligned}$$

$$V(r) = \tfrac{1}{r^2 - n^2}\left[r^2 + n^2 - 2Mr + k^2\left(r^4 - 6n^2r^2 - 3n^4\right)\right]$$

- Weyl (anti-)self-dual (*i.e.* quaternionic) iff

$$M = \pm n(1 - 4k^2n^2)$$

$\Longleftrightarrow$ no conical singularity at $r = n$

The boundary geometry: $ds^2 \underset{r\to\infty}{\longrightarrow} \frac{dr^2}{k^2 r^2} + k^2 r^2 ds^2_{bry.}$

$$
\begin{aligned}
\mathrm{d}s^2_{\mathrm{bry.}} &= \left(\mathrm{d}\tau + 4n\sin^2\frac{\vartheta}{2}\mathrm{d}\varphi\right)^2 + \frac{1}{k^2}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) \\
&= \frac{1}{k^2}\left(\left(\sigma^1\right)^2 + \left(\sigma^2\right)^2\right) + 4n^2\left(\sigma^3\right)^2
\end{aligned}
$$

with $\tau = -2n(\psi + \varphi)$ and $0 \leq \vartheta \leq \pi, 0 \leq \varphi \leq 2\pi, 0 \leq \psi \leq 4\pi$

$$
\begin{cases}
\sigma^1 = \sin\vartheta\sin\psi\,\mathrm{d}\varphi + \cos\psi\,\mathrm{d}\vartheta \\
\sigma^2 = \sin\vartheta\cos\psi\,\mathrm{d}\varphi - \sin\psi\,\mathrm{d}\vartheta \\
\sigma^3 = \cos\vartheta\,\mathrm{d}\varphi + \mathrm{d}\psi.
\end{cases}
$$

Conclusion: $\mathrm{d}s^2_{\mathrm{bry.}}$ is a Berger sphere

Highlights

- The ancestor of holography
- Gravitational duality
- Fluids dynamics: a primer
- Fluids and gravity
- Gravity, holography and the Fefferman–Graham expansion
- The Robinson–Trautman spacetimes
- The Plebański–Demiański type D class
- More on probing transport
- Higher dimensions

Gravitational duality

Similar to electric–magnetic duality in general relativity – Euclidean regime

- Solve Einstein's Eqs. – self-dual gravitational instantons
  [Newman, Tamburino, Unti '63; Eguchi, Hanson '78]
- Provide another handle for understanding the theory
  - linear regime [works by Bunster, Julia, Henneaux...]
  - mass and nut as electric and magnetic charges [Dowker '74]

Self-duality deeply related with integrability – in the '70 all integrable systems were thought to be SDYM reductions [Ward, '85]

Curvature decomposition

Metric $ds^2 = \delta_{AB}\theta^A\theta^B$*, connection one-form* ω_{AB} *and curvature two-form* $\mathcal{R}_{AB} \in \mathbf{6}$ *of* $SO(4) \cong SO(3)_{sd} \otimes SO(3)_{asd}$

- Reducible under $SO(3)_{\mathsf{sd}}$ and $SO(3)_{\mathsf{asd}}$: $\mathbf{6} = (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})$
- Curvature two-form $(\lambda, \mu \ldots = 0, 1, 2)$

  $$(\mathbf{3}, \mathbf{1}) \quad \mathcal{S}_\lambda = \frac{1}{2}\left(\mathcal{R}_{r\lambda} + \frac{1}{2}\epsilon_{\lambda\mu\nu}\mathcal{R}^{\mu\nu}\right)$$
  $$(\mathbf{1}, \mathbf{3}) \quad \mathcal{A}_\lambda = \frac{1}{2}\left(\mathcal{R}_{r\lambda} - \frac{1}{2}\epsilon_{\lambda\mu\nu}\mathcal{R}^{\mu\nu}\right)$$

  and similarly for the connection one-form
- Basis for the space of two-forms $\wedge^2$

  $$(\mathbf{3}, \mathbf{1}) \quad \phi^\lambda = \theta^r \wedge \theta^\lambda + \frac{1}{2}\epsilon^\lambda{}_{\mu\nu}\theta^\mu \wedge \theta^\nu$$
  $$(\mathbf{1}, \mathbf{3}) \quad \chi^\lambda = \theta^r \wedge \theta^\lambda - \frac{1}{2}\epsilon^\lambda{}_{\mu\nu}\theta^\mu \wedge \theta^\nu$$

Atiyah–Hitchin–Singer decomposition of $\mathcal{S}_\mu$, $\mathcal{A}_\mu$ *[Cahen, Debever, Defise '67; Atiyah, Hitchin, Singer '78]*

$$\begin{aligned} \mathcal{S}_\mu &= \tfrac{1}{2} W^+_{\mu\nu}\phi^\nu + \tfrac{1}{12} s\phi_\mu + \tfrac{1}{2} C^+_{\mu\nu}\chi^\nu \\ \mathcal{A}_\mu &= \tfrac{1}{2} W^-_{\mu\nu}\chi^\nu + \tfrac{1}{12} s\chi_\mu + \tfrac{1}{2} C^-_{\nu\mu}\phi^\nu \end{aligned}$$

with $W^\pm$ *and* $C^\pm$ 3×3 *matrices, and* s *a function encoding the 20 components of the Riemann*

- $s = {}^R/_2$ scalar curvature $\rightarrow 1$
- $C^\pm_{\mu\nu}$ traceless Ricci $\rightarrow 9$
- $W^+_{\mu\nu}$ self-dual Weyl tensor symmetric and traceless $\rightarrow 5$
- $W^-_{\mu\nu}$ anti-self-dual Weyl tensor symmetric and traceless $\rightarrow 5$

Quaternionic spaces: $C^\pm = 0 \quad s = 2\Lambda \quad W^- = 0$ *or* $W^+ = 0 \Leftrightarrow$ *Einstein & Weyl (anti-)self-dual*

Here the signature is Lorentzian : $(+-++)$

- W^+ and W^- are complex-conjugate
- The 10 independent components are captured in 5 complex functions $\Psi_a, \ a = 0, \ldots, 4$ projections of W onto a null tetrad

The existence of 4 principal null directions, potentially degenerate with higher multiplicity, translates into special algebraic relationships among the Ψ*s: Petrov type I, II, III, D, N, O*

Highlights

Vector field u with $u_\mu u^\mu = -1$ *and spacetime variation* $\nabla_\mu u_\nu$

$$\nabla_\mu u_\nu = -u_\mu a_\nu + \sigma_{\mu\nu} + \frac{1}{D-1}\Theta h_{\mu\nu} + \omega_{\mu\nu}$$

- $h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu}$: projector/metric on the orthogonal space
- $a_\mu = u^\nu \nabla_\nu u_\mu$: acceleration
- $\sigma_{\mu\nu}$: symmetric traceless part – shear
- $\Theta = \nabla_\mu u^\mu$: trace – expansion
- $\omega_{\mu\nu}$: antisymmetric part – vorticity

In $2+1$ *dimensions*

$$T^{\mu\nu}_{visc} = -\left(2\eta\sigma^{\mu\nu} + \zeta h^{\mu\nu}\Theta + \zeta_H \epsilon^{\rho\lambda(\mu} u_\rho \sigma_\lambda{}^{\nu)}\right) + O\left(\nabla^2 u\right)$$

Conformal fluids (tracelessness): $\varepsilon = 2p, \zeta = 0, \ldots$

On conformal perfect fluids with some time-like velocity field u

- $\mathsf{T}^{\text{perf}} = p\left(3\mathsf{u}^2 + \mathrm{d}s^2_{\text{bry.}}\right)$
- Euler equations $\begin{cases} 2\mathsf{u}(p) + 3p\,\Theta = 0 \\ \mathsf{u}(p)\,\mathsf{u} + \mathrm{d}p + 3p\,\mathsf{a} = 0 \end{cases}$
- Integrability criterion: $\mathrm{d}\mathsf{A} = 0$ with $\mathsf{A} = \mathsf{a} - \frac{\Theta}{2}\mathsf{u}$

$\Longrightarrow$ geodesic and expansionless u solve them with constant p

On the actual stress tensor $T = \mathrm{Re}\, T^{\mathrm{ref}+}$

- Not expected to be perfect: $\mathsf{T} = \mathsf{T}^{\text{perf}} + \Pi$
- The fluid congruence u is read off from the perfect piece
- T^{perf} and Π *are not* separately conserved

Highlights

The ancestor of holography

Gravitational duality

Fluids dynamics: a primer

Fluids and gravity

Gravity, holography and the Fefferman–Graham expansion

The Robinson–Trautman spacetimes

The Plebański–Demiański type D class

More on probing transport

Higher dimensions

Fluid/gravity correspondence

Profound relationship

- Originally in a classical framework: relationship between two sets of non-linear equations, Einstein's and Navier–Stokes [Damour 1979; also Eling, Lysov, Oz, Strominger, ... since 2009]
- More recently within the holographic correspondence with a quantum perspective [Bhattacharyya, Hubeny, Loganayagam, Minwalla, Rangamani, ... since 2007]

Connected via renormalization-group flow from the boundary (UV) to the horizon (IR) where the former correspondence takes place

[Kuperstein, Mukhopadhyay, ... since 2012]

Fluids and gravity

Originally: black-hole horizon responds to perturbations as a viscous fluid *[Damour 1979]*

- damped shear waves
- viscosity $\eta = {}^{1}/_{16\pi G}$
- Bekenstein–Hawking entropy $s = {}^{1}/_{4G} \Rightarrow {}^{\eta}/_{s} = {}^{1}/_{4\pi}$

Origin? Deeper and more general relationship between Einstein's Eqs. and fluid dynamics *[Eling, Lysov, Oz, Strominger, ...since 2009]*

Gravity in 4 dim: 10 Einstein's Eqs. involving G_{AB} ($\nabla_A G^{AB} = 0$)

- 6 evolution: $G_{\mu\nu}$ (2nd order)
- 4 constraint: G_{rr}, $G_{r\mu}$ (1st order)

Initial-value formulation (Cauchy problem): Σ_t, $g_{\mu\nu}$ (6), $K_{\mu\nu}$ (6) *with*

- Hamiltonian constraint (1): $R^{(3)} - 2\Lambda + K^2 - K_{\mu\nu}K^{\mu\nu} = 0$
- momentum constraint (3): $\nabla_\mu \left(K^{\mu\nu} - g^{\mu\nu}K\right) = 0$

constraints in 4-dim $\mathcal{M} \leftrightarrow$ dynamics for $K_{\mu\nu}$ on 3-dim Σ_t

Imposing algebraic Petrov in 4 dim: $K_{\mu\nu}$ (6) $\rightarrow$ (4) *as for a fluid on* Σ_t

- $K_{\mu\nu} \leftrightarrow \epsilon, p, \mathrm{u}$
- $R^{(3)} - 2\Lambda + K^2 - K_{\mu\nu}K^{\mu\nu} = 0$: Eq. of state
- $\nabla_\mu \left(K^{\mu\nu} - g^{\mu\nu}K\right) = 0$: energy-momentum conservation

incompressible Navier–Stokes appear e.g. on black-hole horizons and conformal fluids appear on the conformal boundary $\Sigma_\mathrm{t}|_{r\to\infty}$

Highlights

Gravity in $d = 4$

Palatini formulation and $3+1$ *split* *[Leigh, Petkou '07; Mansi, Petkou, Tagliabue '08]*

$$I_{\mathrm{EH}} = -\frac{1}{32\pi G}\int_{\mathcal{M}} \epsilon_{ABCD}\left(\mathcal{R}^{AB} + \frac{k^2}{2}\theta^A \wedge \theta^B\right) \wedge \theta^C \wedge \theta^D$$

θ^A an orthonormal frame $\mathrm{d}s^2 = \eta_{AB}\theta^A\theta^B$ $(\eta : +\varepsilon++)$
gauge: no lapse, no shift

- Coframe: $\theta^r = \frac{\mathrm{d}r}{kr}$ and θ^μ

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{k^2 r^2} + \eta_{\mu\nu}\theta^\mu\theta^\nu$$

- Connection: $\omega^{r\mu} = \mathcal{K}^\mu$ and $\omega^{\mu\nu} = -\epsilon^{\mu\nu\rho}\mathcal{B}_\rho$ or (a)sd combination $1/2(\mathcal{K}^\mu \pm \mathcal{B}^\mu)$ for $\varepsilon = +$

Hamiltonian evolution of $\theta^\mu, \mathcal{K}^\mu, \mathcal{B}_\rho$ from boundary data – what are the independent boundary data? Answer in asymptotically AdS: Fefferman–Graham expansion for large r *[Fefferman, Graham '85; subtleties: de Haro, Skenderis, Solodukhin, '00]*

$$
\begin{aligned}
\theta^\mu(r,x) &= kr\,E^\mu(x) + \frac{1}{kr}F^\mu_{[2]}(x) + \frac{1}{k^2r^2}F^\mu_{[3]}(x) + \cdots \\
\mathcal{K}^\mu(r,x) &= -k^2r\,E^\mu(x) + \frac{1}{r}F^\mu_{[2]}(x) + \frac{2}{kr^2}F^\mu_{[3]}(x) + \cdots \\
\mathcal{B}^\mu(r,x) &= B^\mu(x) + \frac{1}{k^2r^2}B^\mu_{[2]}(x) + \cdots
\end{aligned}
$$

Independent $2+1$ boundary data: E^μ and $F^\mu_{[3]}$

The holographic fluid

Interpretation of the boundary data

- E^μ: boundary orthonormal coframe – allows to determine

$$\mathrm{d}s^2_{\text{bry.}} = \eta_{\mu\nu} E^\mu E^\nu = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$$

  - $F^\mu_{[2]} = -{}^1\!/_{2k^2} S^{\mu\nu} e_\nu$: Schouten
  - $B^\mu_{[2]} = {}^1\!/_{2k^2} C^{\mu\nu} e_\nu$: Cotton
  - ...

- $F^\mu_{[3]}$: stress current one-form – allows to construct the vev of the boundary stress tensor

$$\mathrm{T} = \frac{3k}{8\pi G} F^\mu_{[3]} e_\mu = T^\mu{}_\nu E^\nu \otimes e_\mu$$

Macroscopic object carrying microscopic data from the bulk

Bulk Weyl self-duality and its boundary manifestation

Expanding $W^{\pm} = 0$ *leads to* $B_{[2]} = \pm(i)\frac{3k}{2}F_{[3]}$ *i.e.*

$$8\pi G k^2 T_{\mu\nu} \pm (i) C_{\mu\nu} = 0$$

[Leigh, Petkou '07; de Haro '08; Mansi, Petkou, Tagliabue '08; Miskovic, Olea '09]

Key property: C *and* T are

- traceless
- conserved

Away from the self-dual point, so is

$$\boxed{T^{\text{ref}\pm}_{\mu\nu} = T^{\mu\nu} \pm \frac{(i)}{8\pi G k^2} C^{\mu\nu}}$$

reflecting $W^{\pm}_{\mu\nu} = \frac{8\pi G}{k^2 r^3} T^{\text{ref}\pm}_{\mu\nu} + \cdots \neq 0$

The reference tensors $T^{ref\pm}$

Integrability in Einstein spaces is tight to Petrov special types

$\Longrightarrow$ $W^{\pm}$ are remarkably simple and so must be $\mathsf{T}^{\mathsf{ref}\pm}$

simpler to scan for $\mathsf{T}^{\mathsf{ref}\pm}$ than for C and T

Boundary geometries expected to lead to resummable series should have canonical $T^{ref\pm}$ i.e.

- *either possess complex-conjugate time-like geodesic congruences associated with perfect-fluid-form $T^{ref\pm}$*
- *or admit null congruences associated with pure-radiation $T^{ref\pm}$*
- *or a combination of both*

$\Longrightarrow$ $\mathsf{T}^{\mathsf{ref}\pm}$ follow the *Segre* classification of the 3-dim Cotton

the right integrability recipe

Highlights

Boundary metric

A simple boundary metric without isometries

$$\boxed{ds^2_{bry.} = -dt^2 + \frac{2}{k^2 P^2} d\zeta d\bar{\zeta}} \qquad (nv)$$

with a unique shearless congruence $u = -dt$

- $P(t, \zeta, \bar{\zeta})$ real & *a priori* arbitrary – $K = 2P^2 \partial_\zeta \partial_{\bar{\zeta}} \log P$
- Cotton-tensor components $C_{\mu\nu}$:

$$-i \begin{pmatrix} 0 & -\frac{k^2}{2}\partial_\zeta K & \frac{k^2}{2}\partial_{\bar{\zeta}} K \\ -\frac{k^2}{2}\partial_\zeta K & -\partial_t\left(\frac{\partial^2_\zeta P}{P}\right) & 0 \\ \frac{k^2}{2}\partial_{\bar{\zeta}} K & 0 & \partial_t\left(\frac{\partial^2_{\bar{\zeta}} P}{P}\right) \end{pmatrix}$$

Reference tensors and integrability conditions

The perfect-fluid form – Petrov–Segre type D_t

$$T^{\pm}_{pf} = \frac{M_\pm k^2}{8\pi G}\left(3\left(u^\pm\right)^2 + ds^2_{bry.}\right)$$
$$u^+ = -dt + \frac{\alpha^+}{P^2}d\zeta \quad \& \quad c.c.$$

$M_\pm$, $\alpha^\pm$ and c.c. functions of $t, \zeta, \bar{\zeta}$

Conserved iff $\mathrm{A}^\pm = \mathrm{a}^\pm - \mathrm{u}^\pm \Theta_\pm/2$ is exact

- $M_\pm(t)$
- $M_\pm$ and $\alpha^\pm$ obey

$$\boxed{\frac{k^2 P}{2}\partial_\zeta \alpha^- + \frac{P}{3}\partial_t \ln M_\pm = k^2 \alpha^- \partial_\zeta P + \partial_t P \quad \& \quad \text{c.c.}} \quad \text{(h1)}$$

The matter–radiation form – Petrov–Segre type III & N

$$T^{+}_{mr} = \frac{1}{4\pi G} d\zeta \left(\beta dt + \frac{\gamma}{k^2} d\zeta\right)$$

β, γ and c.c. functions of $t, \zeta, \bar{\zeta}$

Conserved iff

- $\beta(t, \zeta)$ & $\bar{\beta}(t, \bar{\zeta})$
- β and γ obey

$$\boxed{\partial_t \beta - \beta \partial_t \ln P^2 - 2P^2 \partial_{\bar{\zeta}} \gamma = 0 \quad \& \quad \text{c.c.}} \qquad \text{(h2)}$$

Integrability conditions for $T^{\pm} = T^{\pm}_{pf} + T^{\pm}_{mr}$

- Requiring $\mathrm{C} = 8\pi G k^2 \, \mathrm{Im}\mathrm{T}^{+}$ sets 3 constraints:

$$\boxed{M_{+}(t) = M_{-}(t) = M(t)}$$

$$\boxed{\partial_t \left(\frac{\partial_\zeta^2 P}{P} \right) + \frac{3}{2} M k^4 \frac{(\alpha^{+})^2}{P^4} + \gamma = 0 \quad \& \quad \text{c.c.}} \qquad \text{(C1)}$$

and

$$\boxed{\partial_\zeta K + \beta = 3 M k^2 \frac{\alpha^{+}}{P^2} \quad \& \quad \text{c.c.}} \qquad \text{(C2)}$$

- (C2) combined with (h1) gives the *Robinson–Trautman Eq.*

$$\boxed{\Delta K - 12 M \partial_t \log P + 4 \partial_t M = 0} \qquad \text{(E)}$$

Energy–momentum tensor and resummation

The energy–momentum tensor T

- Using $\mathrm{T} = \mathrm{Re}\mathrm{T}^{+}$ one finds the *non-perfect* $^{8\pi G}/_{k^2}\mathrm{T}$

$$\begin{pmatrix} 2M(t) & -\frac{1}{2k^2}\partial_\zeta K & -\frac{1}{2k^2}\partial_{\bar\zeta} K \\ -\frac{1}{2k^2}\partial_\zeta K & -\frac{1}{k^4}\partial_t\left(\frac{\partial^2_\zeta P}{P}\right) & \frac{M}{k^2P^2} \\ -\frac{1}{2k^2}\partial_{\bar\zeta} K & \frac{M}{k^2P^2} & -\frac{1}{k^4}\partial_t\left(\frac{\partial^2_{\bar\zeta} P}{P}\right) \end{pmatrix} \text{ with } K = 2P^2\partial_\zeta\partial_{\bar\zeta}\log P$$

- The *perfect-fluid part* is $\mathrm{T}^{\mathrm{perf}} = \frac{Mk^2}{8\pi G}\left(3\mathrm{u}^2 + \mathrm{d}s^2_{\mathrm{bry.}}\right)$
- $\Pi = \mathrm{T} - \mathrm{T}^{\mathrm{perf}}$ contains hydrodynamic non-transverse modes ($\Pi(\mathrm{u}) \neq 0$, $\Pi(\mathrm{u},\mathrm{u}) = 0$) up to $\nabla^3\mathrm{u} \to$ rich information on transport in the non-Landau–Lifshitz frame

Resummation: using $ds^2_{bry.}$, C, T and u in Eq. (R)

$$\boxed{ds^2_{res.} = 2dt\,dr - 2Hdt^2 + 2\frac{r^2}{P^2}d\zeta d\bar{\zeta}} \qquad \text{(RT)}$$

with

$$2H = k^2r^2 + K + 2r\partial_t \log P - \frac{2M}{r}$$

This is *Robinson–Trautman* space–time with t retarded time

- Einstein thanks to Robinson–Trautman Eq. (E) $\equiv$ (h1) & (C2)
- Petrov type determined by Eq. (C1)
  - generically type II
  - $M = 0$: type III ($\Psi_2 = 0$)
  - $M = \beta = 0$: type N ($\Psi_2 = \Psi_3 = 0$)
  - $\beta = \gamma = 0$: type D

Robinson–Trautman physics

The bulk: generically time-dependent and singular at $r = 0$ (sources)

- Type II ($M \neq 0$): black hole with horizon radiating away gravitational waves $\underset{t\to\infty}{\longrightarrow}$ Schwarzschild AdS_4
- Type III & N ($M = 0$): pure gravitational waves $\underset{t\to\infty}{\longrightarrow}$ AdS_4
- Type D: Schwarzschild AdS_4 and C-metric (stationary)

The boundary: generically non-global-equilibrium

- Type II: claim for hydrodynamic regime at late times
- Type III & N: "pure radiation" at late times
- Type D: genuine stationary hydrodynamics at any time

Probing correlations and transport: under investigation [de Freitas, Reall '14; Bakas, Skenderis '14 & '15; Mukhopadhyay et al work in progress]

Highlights

The Plebański–Demiański type D class

$$\boxed{ds^2_{bry.} = -Q^2\left(dt - b\right)^2 + \frac{2}{k^2P^2}d\zeta d\bar{\zeta}} \qquad (nv)$$

P, Q real fcts and $\mathrm{b} = b_\zeta \mathrm{d}\zeta + b_{\bar{\zeta}}\mathrm{d}\bar{\zeta}$ a real form – *a priori* arbitrary

- Impose $\exists$ 1 Killing $\Rightarrow$ 2nd one [Mukhopadhyay *et al* '13]
- Impose $\exists$ 2 c.c. accelerating non-expanding congruences $\mathrm{u}_\pm \Rightarrow$ perfect-fluid conserved $\mathrm{T}^\pm$ (non-constant pressure)
- Impose $\mathrm{C} = 8\pi G k^2\, \mathrm{Im}\mathrm{T}^+ \Rightarrow$ solve for P, Q and b $\Rightarrow \mathrm{d}s^2_{\text{bry.}}$
- Extract $\mathrm{T} = \mathrm{Re}\mathrm{T}^+ = \mathrm{T}^{\text{perf}} + \Pi$
- T^{perf} generally non-conserved – aligned with $\mathrm{u} = -\mathrm{d}t + \mathrm{b}$ shearless, expanding accelerating congruence with vorticity
- Resum – Eq. (R): exact Petrov type D Plebański–Demiański familly (mass, rotation, nut, “twist”, acceleration)

Note: a suggestive expression

Generic comoving boundary frame adapted to a shearless congruence

$$ds^2_{bry.} = -Q^2(dt - b)^2 + \frac{2}{k^2 P^2} d\zeta d\bar{\zeta}$$
$$u = -Q(dt - b)$$

Resummed bulk derivative expansion (R) in null tetrad

$$ds^2_{res.} = -2\mathbf{lk} + 2\mathbf{m\bar{m}}$$

$$\mathbf{l} = dr + rA + \frac{u}{2}\left[r^2k^2 + \frac{q^2}{k^2} + \frac{\mathscr{R}}{2k^2} - \frac{3}{\rho^2 k}\left(\frac{r\varepsilon}{\kappa} + \frac{qc}{6k^5}\right)\right] - \frac{1}{2k^2} * (u \wedge dq)$$

$\mathbf{k} = u \quad \mathbf{m} = \frac{\rho}{P} d\zeta \quad q^2 = 2\omega_{\alpha\beta}\omega^{\alpha\beta} \quad c = C_{\lambda\mu}u^\lambda u^\mu \quad \varepsilon = T_{\lambda\mu}u^\lambda u^\mu$

The null congruence $\mathbf{k}$ *is a principal null direction of multiplicity* ≥ 2

Highlights

Summary of the reconstruction

Using $ds^2_{bry.}$ with shear-free u and C, T based on arbitrary exactly conserved canonical $T^{\pm}$ the partly resummed derivative expansion

- *is exactly Einstein $\rightarrow$ all Petrov-algebraic Einstein spaces: Kundt, Robinson–Trautman, Plebański–Demiański, ...*
- *gives access to transport properties beyond the derivative expansion $\leftarrow$ the stress–energy tensor is not perfect-fluid or even fluid: $T = T^{perf} + \Pi$*
- *allows for scanning new solutions by acting with $U_{hol.}$: $T^+ \rightarrow zT^+, T^- \rightarrow \bar{z}T^-$*

Output:

- *Integration achieved: limited derivative expansion is exact Einstein (Plebański–Demiański, Robinson–Trautman, Kundt...)*
- *Remarkable form of $T^{\text{ref}\pm} \Rightarrow$ special form of $W^{\pm}$: algebraic Petrov type (Kerr, Taub–NUT, C-metric, pp-waves...)*

Consequence for holographic fluids: *transport properties*

- *Status: exact solutions provide rich information on transport coefficients (in particular when T is non-perfect)* [Mukhopadhyay et al '13; de Freitas, Reall '14; Bakas, Skenderis '14]
- *Next: perturbation of exact Einstein spaces as a deeper probe for transport can be made more systematic – captured in the known h.d. terms of the ds^2_{bulk} expansion*

Boundary geometry and reference tensor

Boundary geometry:

$$\boxed{ds^2_{bry.} = -\left(dt - b\right)^2 + \frac{2}{k^2 P^2} d\zeta d\bar{\zeta}}$$

- $P(\zeta, \bar{\zeta})$ real fct and $\mathrm{b} = b_\zeta \mathrm{d}\zeta + b_{\bar{\zeta}} \mathrm{d}\bar{\zeta}$ a real form
- $\mathrm{u} = \partial_t$ Killing (geodesic, shear-, expansion-free, with vorticity)

Boundary reference tensor: special request

$$T^\pm \propto T = \varepsilon \left(\left(dt - b\right)^2 + \frac{1}{k^2 P^2} d\zeta d\bar{\zeta} \right) \propto C$$

(in Euclidean: $C = \pm 8\pi G k^2 T$ is the (a)sd condition)
conserved with constant $\varepsilon = {}^{Mk^2}/_{4\pi G}$

Full classification

Four Killings

- Boundary: homogeneous spaces (Bianchi XI, II, VII) solving

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \lambda g_{\mu\nu} = \frac{1}{\mu} C_{\mu\nu}$$

- No probe on fluid transport properties (too much symmetry)
- Bulk: AdS_4 Taub–NUT black holes

Two commuting Killings

- Boundary: "monopole plus dipole"
- Infinite sequence of non-vanishing tensors coupled to transport coefficients
- Bulk: AdS_4 Kerr–Taub–NUT black holes with regular horizons

Highlights

Higher dimensions

Playground for duality: 8 Euclidean dimensions – octonions

- $\mathcal{R} \in \mathbf{28}$ *of* $SO(8)$ *reducible wrt to* $Spin(7) \subset SO(8)$
- *Duality map:* $\tilde{\mathcal{R}}^{AB} = \Psi^{ABCD}\mathcal{R}_{CD}$ *with* $\mathbf{28} \to \mathbf{21} \oplus \mathbf{7}$

$$
\begin{aligned}
\mathcal{S}_{\mathbf{21}} &= W^{\mathbf{168}}\phi_{\mathbf{21}} + s^{\mathbf{1}}\phi_{\mathbf{21}} + W^{\mathbf{105}}\chi_{\mathbf{7}} \\
\mathcal{A}_{\mathbf{7}} &= W^{\mathbf{27}}\chi_{\mathbf{7}} + s^{\mathbf{1}}\chi_{\mathbf{7}} + S^{\mathbf{35}}\phi_{\mathbf{21}}
\end{aligned}
$$

- *"Quaternionic" spaces:*

$$
\begin{cases}
S^{\mathbf{35}} \to 0 & \textit{Einstein} \\
W^{\mathbf{27}} \to 0 & \textit{Weyl self-dual}
\end{cases}
$$

7-dimensional boundary

- $\mathcal{R} \in \mathbf{21}$ *of* $SO(7)$ *reducible wrt to* $G_2 \subset SO(7)$
- *Duality map: using octonions* $\mathbf{21} \to \mathbf{14} \oplus \mathbf{7}$

$$\begin{aligned} \mathcal{S}_{\mathbf{14}} &= W^{\mathbf{77}}\phi_{\mathbf{14}} + s^{\mathbf{1}}\phi_{\mathbf{14}} + W^{\mathbf{64}}\chi_{\mathbf{7}} \\ \mathcal{A}_{\mathbf{7}} &= W^{\mathbf{27}}\chi_{\mathbf{7}} + s^{\mathbf{1}}\chi_{\mathbf{7}} + S^{\mathbf{27}}\phi_{\mathbf{14}} \end{aligned}$$

- *7-dim boundary: energy–momentum* $T \in \mathbf{27}$
- $W^{\mathbf{27}}_{bulk} \to {}^{1}/_{r^7}\left(\kappa T + W^{\mathbf{27}}_{bry}\right) \to$ *generalized filling-in problem, Lorentzian extension and integrability*