# Effective action with composite fields and Clairaut-type equations

P. M. Lavrov

Tomsk, Russia

Summer School and Workshop on the Standard Model and Beyond, August 31-September 12, 2016, Corfu, Greece

Based on

P.M.L., B.S. Merzlikin, Phys. Rev. D 92 (2015) 085038
Phys. Lett. B 756 (2016) 188

# Contents

## Motivation

Effective action with composite fields - J. M. Cornwell, R. Jackiw, E. Tomboulis, Phys. Rev. D, 1974.

Extension to gauge theories - P.M.L., Theor. Math. Phys. 1990.
Gauge independence on-shell.

Functional renormalization group (FRG) approach, average effective action - C. Wetterich, Nucl. Phys. B, 1991; Phys. Lett. B, 1993.

Gauge dependence and new formulation of FRG - P.M.L., I. L. Shapiro, JHEP, 2013.

Explicit form of average effective action in Abelian gauge model - P.M.L., B. S. Merzlikin, Phys. Rev. D, 2015

# Clairaut-type equations

A Clairaut equation is a differential equation of the form

$$y - y'x = \psi(y'),$$

where $y = y(x), y' = dy/dx$ and $\psi = \psi(z)$ is a real function of $z$. It is well-known that the general solution of the Clairaut equation is the family of straight line functions given by

$$y(x) = Cx + \psi(C),$$

where $C$ is a real constant. The so-called singular solution is defined by the equation

$$\psi'(z) + x = 0, \quad z = y'$$

if a solution to this equation exists in the form $z = \varphi(x)$ with a real function $\varphi$ of $x$. Then

$$y(x) = x\varphi(x) + \psi(\varphi(x))$$

presents the singular solution.

# Clairaut-type equations

A first-order partial differential equation

$$y - y'_i x^i = \psi(y'),$$

which is also known as the Clairaut equation. Here $y = y(x)$ is the real function of variables $x \in \mathbf{R}^n$, $x = \{x^1, x^2, ..., x^n\}$, $\psi = \psi(z)$ is a real function of variables $z = \{z_1, z_2, ..., z_n\}$ and the notation

$$y'_i = \partial_i y(x) \equiv \frac{\partial y(x)}{\partial x^i},$$

is used. In terms of new function $z_i = z_i(x) = y'_i(x)$ this equation rewrites

$$y - z_i x^i = \psi(z).$$

Differentiation with respect to $x^i$ leads to a system of differential equations

$$\frac{\partial z_j}{\partial x^i}\left(\frac{\partial \psi}{\partial z_j} + x^j\right) = 0, \quad i = 1, 2, ..., n.$$

In the case of the Hessian matrix vanishing $H_{ij} = 0$ where

$$H_{ij} = \frac{\partial z_j}{\partial x^i} = \frac{\partial^2 y}{\partial x^i \partial x^j},$$

$z_i = C_i = \mathrm{const}.$ Therefore the solution to the Clairaut equation is the family of linear functions

$$y(x) = C_i x^i + \psi(C), \quad C = \{C_1, C_2, ..., C_n\}.$$

Consider the case when

$$\frac{\partial \psi}{\partial z_j} + x^j = 0, \quad j = 1, 2, ..., n.$$

If there are any real solutions to these equations

$$z_j = \varphi_j(x), \quad j = 1, 2, ..., n,$$

then the singular solution to the Clairaut equation is

$$y(x) = \varphi_i(x) x^i + \psi(\varphi(x)).$$

We have found two new types of $\psi(z)$ when the singular solutions to the Clairaut - type equation admits singular solutions:

$$a)\ \psi(z) = -\alpha(z_1 z_2 \cdots z_n)^{\beta}$$

$$b)\ \psi(z) = \alpha \ln(1 - a^i z_i)$$

where $\alpha, \beta$ and $a^i, i = 1, 2, ..., n$ are real constants.

In the case of a) it was known (E. Kamke, 1959) the singular solutions when

- $n = 2, \beta = 1, \alpha = -1$
- $n = 3$, $\beta = 1$, $\alpha = -1$
- $\beta = 1/(n+1)$, $\alpha = -(n+1)$

The case b) is most important for QFT with composite fields.

Explicit form of singular solutions of Clairaut-type equations in these cases reads

$$a)\;\; y(x) = \alpha(n\beta - 1)\left(\prod_{i=1}^{n} \frac{x^i}{\alpha\beta}\right)^{\frac{\beta}{n\beta-1}}$$

$$b)\;\; y(x) = b_i x^i - \alpha\ln(b_i x^i) - \alpha + \alpha\ln\alpha,$$

where the constant vector $b = \{b_1, b_2, ..., b_n\}$ satisfies the condition $b_i a^i = 1$.

We can consider the functional Clairaut-type equations. Let $\Gamma = \Gamma[F]$ be a functional of fields $F^m = F^m(x)\,, m = 1, 2, ..., N$, which are real integrable functions of real variables $x \in \mathbf{R}^n$. We use the notion of *functional Clairaut-type equations* for the equations of the form

$$\Gamma - \frac{\delta\Gamma}{\delta F^m} F^m = \Psi\left[\frac{\delta\Gamma}{\delta F}\right],$$

where $\Psi = \Psi[Z]$ is a given real functional of real variables $Z_m = Z_m(x)\,, m = 1, 2, ..., N$. Here the following notation

$$\frac{\delta\Gamma}{\delta F^m} F^m = \int d^n x \frac{\delta\Gamma}{\delta F^m(x)} F^m(x)$$

is used. The functional derivatives are defined by the rule

$$\frac{\delta F^m(x)}{\delta F^k(y)} = \delta^m_k\, \delta(x - y)\,.$$

# Clairaut-type equations

In the available scientific literature we didn't find discussions concerning functional Clasiraut-type equations. Let $Z_m(x) = \delta\Gamma/\delta F^m(x)$ be new unknown functions of $x$. Then

$$\Gamma - Z_m F^m = \Psi[Z].$$

Varying this equation with respect to $F^n(y)$ we have

$$\int dx\, \frac{\delta Z_m(x)}{\delta F^n(y)} \left[\frac{\delta \Psi[Z]}{\delta Z_m(x)} + F^m(x)\right] = 0.$$

If $\delta Z_m(x)/\delta F^n(y) = 0$ then

$$\Gamma = C_m F^m + \Psi[C], \quad C_m = \text{const}, m = 1, 2, ..., N$$

are solutions for any set of $\{C_m\}$. Singular solutions (if exist) should be found from the functional system of equations

$$\frac{\delta \Psi[Z]}{\delta Z_m(x)} + F^m(x) = 0, \quad m = 1, 2, ..., N.$$

Let us consider a field model which is described by a non-degenerate action, $S[\phi]$, of the scalar field $\phi = \phi(x)$. The generating functional of the Green functions, $Z[J]$, and the generating functional of the connected Green functions, $W[J]$, are defined in the standard way

$$Z[J] = \int \mathcal{D}\phi \; e^{i(S[\phi]+J\phi)} = e^{iW[J]} \,,$$

where $J = J(x)$ is usual source to $\phi$ and $J\phi = \int dx J(x)\phi(x)$. The effective action, $\Gamma = \Gamma[\Phi]$, is introduced by the Legendre transformation of $W[J]$

$$\Gamma[\Phi] = W[J] - J\Phi \,,$$

$$\frac{\delta W[J]}{\delta J(x)} = \Phi(x) \,, \qquad \frac{\delta \Gamma[\Phi]}{\delta \Phi(x)} = -J(x) \,.$$

Eliminating the source $J$ one obtains the equation ($\Gamma = \Gamma[\Phi]$)

$$\Gamma - \frac{\delta\Gamma}{\delta\Phi}\Phi = W\left[-\frac{\delta\Gamma}{\delta\Phi}\right],$$

which has exactly the form of functional Clairaut-type equation. The practical application requires solving the equation for $W[J]$. The standard way to find a solution of this equation with the functional $W$ is related to using the perturbation theory. In the tree approximation, $\Gamma[\Phi] = \Gamma^{(0)}[\Phi]$, it follows $\Gamma^{(0)}[\Phi] = S[\Phi]$. In the one-loop approximation, $\Gamma[\Phi] = S[\Phi] + \Gamma^{(1)}[\Phi]$, one derives

$$\Gamma^{(1)}[\Phi] = \frac{i}{2}\,\mathsf{Tr}\,\ln S^{''}[\Phi], \quad S^{''}[\Phi](x,y) = \frac{\delta^2 S[\Phi]}{\delta\Phi(x)\delta\Phi(y)}$$

In fact, specific features of the Clairaut-type equation disappear within the perturbation expansion.

Consider now an approach to QFT based on concept of composite fields. The starting point of such the approach is the generating functional of Green functions $Z[J,K]$,

$$Z[J,K] = \int \mathcal{D}\phi\, e^{i(S[\phi]+J\phi+KL(\phi))} = e^{iW[J,K]}\,,$$

where $W = W[J,K]$ is the generating functional of connected Green functions. Here $J = J(x)$ and $K = K(x,y)$ are sources to field $\phi = \phi(x)$ and composite field $L(\phi) = L(\phi)(x,y)$, respectively, and the notation $KL(\phi) = \int dxdy\, K(x,y)L(\phi)(x,y)$ is used. Let $L(\phi)(x,y)$ depend quadratically on the field $\phi$

$$L(\phi)(x,y) = \frac{1}{2}\phi(x)\phi(y)\,.$$

# Effective action with composite fields

The effective action with composite field, $\Gamma = \Gamma[\Phi, F]$, is defined by using the double Legendre transformation

$$\Gamma[\Phi, F] = W[J, K] - J\Phi - K\big(L(\Phi) + \frac{1}{2}F\big)\,,$$

$$\frac{\delta W[J,K]}{\delta J(x)} = \Phi(x)\,, \qquad \frac{\delta W[J,K]}{\delta K(x,y)} = L(\Phi)(x,y) + \frac{1}{2}F(x,y)\,,$$

$$\frac{\delta \Gamma[\Phi,F]}{\delta \Phi(x)} = -J(x) - \int dy K(x,y)\Phi(y)\,, \qquad \frac{\delta \Gamma[\Phi,F]}{\delta F(x,y)} = -\frac{1}{2}K(x,y)\,,$$

# Effective action with composite fields

Eliminating the sources $J$ and $K$ one obtains the equation

$$\Gamma - \frac{\delta\Gamma}{\delta\Phi}\Phi - \frac{\delta\Gamma}{\delta F}F = W\left[-\frac{\delta\Gamma}{\delta\Phi} + 2\frac{\delta\Gamma}{\delta F}\Phi, -2\frac{\delta\Gamma}{\delta F}\right] - 2\frac{\delta\Gamma}{\delta F}L(\Phi)\,.$$

Since the right-hand side of the last equation depends on the fields $\Phi$ not only through derivatives of functional $\Gamma = \Gamma[\Phi, F]$, the equation does not belong to the Clairaut-type equation.

But, the one-loop approximation for the effective action with composite field, $\Gamma^{(1)} = \Gamma^{(1)}[\Phi, F]$ by itself satisfies the equation

$$\Gamma^{(1)} - \frac{\delta\Gamma^{(1)}}{\delta F} F = \frac{i}{2}\,\mathsf{Tr}\,\ln\left(S^{''}[\Phi] - 2\frac{\delta\Gamma^{(1)}}{\delta F}\right),$$

being exactly the Clairaut-type with respect to field $F$ wherein the variable $\Phi$ should be considered as parameter.

Solution to this equation has been proposed by J. M. Cornwell, R. Jackiw, E. Tomboulis, Phys. Rev. D, 1974

$$\Gamma^{(1)}[\Phi, F] = \frac{1}{2}\,\mathsf{Tr}\,\big(FS''[\Phi]\big) - \frac{i}{2}\,\mathsf{Tr}\,\ln\big(iF\big) - \frac{i}{2}\delta(0)\,.$$

In real physical situation the number of fields is more then one. It needs to generalize the results obtained by J. M. Cornwell, R. Jackiw, E. Tomboulis, to the case of a field model described by a set of scalar bosonic fields $\phi^A(x)\,, A=1,..,N$, with a classical non-degenerate action $S[\phi]$. Let $L^i(\phi)=L^i(\phi)(x,y), i=1,2,...,M$, be composite non-local fields,

$$L^i(\phi)(x,y)=\frac{1}{2}\,\mathcal{A}^i_{AB}\;\phi^A(x)\phi^B(y)\;,$$

where $\mathcal{A}^i_{AB}=\mathcal{A}^i_{BA}$ are constants. The generating functional of Green functions, $Z[J,K]$, is given by the following path integral

$$Z[J,K]=\int\mathcal{D}\phi e^{i\left(S[\phi]+J_A\phi^A+K_iL^i(\phi)\right)}=e^{iW[J,K]}\,,$$

where $K_i=K_i(x,y)$, $i=1,2,...,M$, are sources to composite fields $L^i(\phi)(x,y)$.

$$J_A\phi^A=\int dx\,J_A(x)\phi^A(x),\quad K_iL^i(\phi)=\int dxdy\,K_i(x,y)L^i(\phi)(x,y).$$

From the definition of $Z[J, K]$ the following relations hold

$$\frac{1}{2}\mathcal{A}^j_{AB}\frac{\delta^2 Z[J,K]}{\delta J_A(x)\delta J_B(y)} = i\frac{\delta Z[J,K]}{\delta K_j(x,y)}\,, \quad j = 1, 2, ..., N,$$

or, in terms of the functional $W[J, K]$,

$$\frac{1}{2}\mathcal{A}^j_{AB}\left[-i\frac{\delta^2 W[J,K]}{\delta J_A(x)\delta J_B(y)} + \frac{\delta W[J,K]}{\delta J_A(x)}\frac{\delta W[J,K]}{\delta J_B(y)}\right] = \frac{\delta W[J,K]}{\delta K_j(x,y)}\,.$$

We define the average fields $\Phi^A(x)$ and composite fields $F^i(x,y)$ as follows

$$\frac{\delta W[J,K]}{\delta J_A(x)} = \Phi^A(x), \quad \frac{\delta W[J,K]}{\delta K_i(x,y)} \;=\; L^i(\Phi)(x,y) + \frac{1}{2}F^i(x,y)\,.$$

The effective action with composite fields, $\Gamma = \Gamma[\Phi, F]$, is defined by using the double Legendre transformation of $W[J,K]$,

$$\Gamma[\Phi,F] \;=\; W[J,K] - J_A\Phi^A - K_i\big(L^i(\Phi) + \frac{1}{2}F^i\big)\,.$$

One can eliminate the sources using

$$\frac{\delta\Gamma[\Phi,F]}{\delta\Phi^A(x)} = -J_A(x) - \int dy\, K_i(x,y)\mathcal{A}^i_{AB}\Phi^B(y)\,,$$

$$\frac{\delta\Gamma[\Phi,F]}{\delta F^i(x,y)} = -\frac{1}{2}K_i(x,y)\,.$$

The basic relations rewritten in terms of $\Gamma[\Phi, F]$ read

$$F^j(x,y) - i(G^{-1})^{AB}(x,y)\mathcal{A}^j_{AB} = 0\,,$$

where $(G^{-1})$ is the matrix inverse to $G$,

$$G = \{G_{AB}(x,y)\}, \quad G_{AB}(x,y) = \Gamma^{''}_{AB}[\Phi, F](x,y) - 2\frac{\delta\Gamma[\Phi, F]}{\delta F^i(x,y)}\mathcal{A}^i_{AB}\,,$$

and we have used the notation

$$\Gamma^{''}_{AB}[\Phi, F](x,y) = \frac{\delta^2\Gamma[\Phi, F]}{\delta\Phi^A(x)\delta\Phi^B(y)}\,.$$

In the one-loop approximation, $\Gamma[\Phi,F]=S[\Phi]+\Gamma^{(1)}[\Phi,F]$, the equation for one-loop contribution, $\Gamma^{(1)}$, to the effective action can be found in the form

$$\Gamma^{(1)}-\frac{\delta\Gamma^{(1)}}{\delta F^i}F^i=\frac{i}{2}\,\mathrm{Tr}\,\ln\left(S''_{AB}[\Phi]-2\frac{\delta\Gamma^{(1)}}{\delta F^i}\mathcal{A}^i_{AB}\right),$$

being the exact functional Clairaut-type equation with respect to field $F$ wherein the variable $\Phi$ should be considered as parameter.

The difference between this equation and the equation appearing in the case of one field $\Phi$ is the same as between the ordinary differential Clairaut equation and the partial differential Clairaut equation.

# Solution to the functional Clairaut-type equations

To solve the basic equation we introduce new functions $Z_i(x,y)$

$$\frac{\delta\Gamma[\Phi,F]}{\delta F^i(x,y)} = Z_i(x,y)\,,$$

and then we have

$$\Gamma^{(1)} = \quad Z_i F^i + \tfrac{i}{2}\,\mathsf{Tr}\,\ln Q.$$

where the matrix $Q = \{Q_{AB}(x,y)\}$ is defined as

$$Q_{AB}(x,y) = S''_{AB}[\Phi](x,y) - 2Z_i(x,y)\mathcal{A}^i_{AB}.$$

Varying the functional $\Gamma^{(1)}$ with respect to $F^i$ we obtain

$$\delta\Gamma^{(1)} = \delta Z_i F^i + Z_i \delta F^i + \frac{i}{2}\,\mathsf{Tr}\,Q^{-1}\delta Q,$$

where $Q^{-1}$ is the inverse to $Q$

$$\int dz (Q^{-1})^{AC}(x,z) Q_{CB}(z,y) = \delta^A_B\,\delta(x-y).$$

## Solution to the functional Clairaut-type equations

Taking into account the explicit form of $Q_{AB}$ one obtains

$$\int dz dz' \frac{\delta Z_i(z,z')}{\delta F^j(x,y)} \left[F^i(z,z') - i(Q^{-1})^{AB}(z,z')\mathcal{A}^i_{BA}\right] = 0\,.$$

Thus the equation defining non-trivial functions $Z_i(x,y)$ reads

$$F^j(x,y) - i(Q^{-1})^{AB}(y,x)\mathcal{A}^j_{BA} = 0\,.$$

$$\Big(Q_{AB}(x,y) = S''_{AB}[\Phi](x,y) - 2Z_i(x,y)\mathcal{A}^i_{AB}\Big)$$

Now we introduce a set of matrices $\mathcal{B}_j = \{\mathcal{B}_j^{AB}\}$ by the relations

$$\mathcal{A}^j_{\ AB}\mathcal{B}_j^{CD} = \frac{1}{2}\left(\delta_A^C\delta_B^D + \delta_A^D\delta_B^C\right).$$

Then we have

$$F^j(x,y)\mathcal{B}_j^{AB} - i(Q^{-1})^{AB}(x,y) = 0.$$

The matrices introduced obey the properties

$$\mathcal{B}_j^{AB}\mathcal{A}^i_{BA} = \delta^i_j \quad \text{or} \quad \mathrm{tr}\,\mathcal{B}_j\mathcal{A}^i = \delta^i_j\,,$$

which lead to the restriction on parameters $N$ and $M$

$$\frac{1}{2}N(N+1) = M\,.$$

This condition has a simple sense: in a given theory with the set of $N$ fields $\phi^A$ there exists exactly the $(1/2)N(N+1)$ independent combinations of $\phi^A\phi^B$.

## Solution to the functional Clairaut-type equations

Then we can rewrite the equation for $Z_i(x,y)$ in the form

$$Z_i(x,y)\mathcal{A}^i_{AB} = \frac{1}{2}S''_{AB}[\Phi](x,y) - \frac{i}{2}(F^i\mathcal{B}_i)^{-1}_{AB}(x,y)\,,$$

where

$$\int dz (F^i\mathcal{B}_i)^{-1}_{AC}(x,z)(F^j(z,y)\mathcal{B}_j)^{CB} = \delta^B_A\delta(x-y)\,.$$

It allows us to express the $Z_iF^i$ as a functional of $\Phi, F$ with the result

$$Z_jF^j = \frac{1}{2}\,\mathsf{Tr}\left((F^j\mathcal{B}_j)S''[\Phi]\right) - \frac{i}{2}\delta(0)N\,.$$

Finally we find the one-loop effective action, $\Gamma^{(1)}[\Phi, F]$, in the form

$$\Gamma^{(1)}[\Phi,F] = \frac{1}{2}\,\mathsf{Tr}\left((F^j\mathcal{B}_j)S''[\Phi]\right) - \frac{i}{2}\,\mathsf{Tr}\,\ln\left(i(F^j\mathcal{B}_j)\right) - \frac{i}{2}\delta(0)N\,.$$

We have studied the case with maximum number of composite fields, $M = \frac{1}{2}N(N+1)$, being quadratic in the given scalar fields $\phi^A\,, A = 1, ..., N$. In a similar manner one can consider the situation when the number of composite fields is less the maximum one, $L^i(\phi)(x,y) = \frac{1}{2}\mathcal{A}^i_{ab}\phi^a(x)\phi^b(y)\,, i = 1, ..., M < \frac{1}{2}N(N+1)\,, a = 1, ..., n < N$. Now the matrix of second derivatives of the classical action, $S''_{AB}[\Phi]$, should be presented in the block form

$$S''_{AB}[\Phi](x,y) = \left(\begin{array}{c|c} S''_{ab} & S''_{a\beta} \\ \hline S''_{\alpha b} & S''_{\alpha\beta} \end{array}\right)\,,$$

where $\quad a, b = 1, ..., n$ and $\alpha, \beta = n+1, ..., N\,.$

The equation for the one-loop contribution, $\Gamma^{(1)}[\Phi, F]$, to effective action takes the form

$$\Gamma^{(1)} - \frac{\delta\Gamma^{(1)}}{\delta F^i} F^i = \frac{i}{2}\,\mathsf{Tr}\,\ln\left(\tilde{S}''_{ab} - 2\frac{\delta\Gamma^{(1)}}{\delta F^i}\mathcal{A}^i_{ab}\right) + \frac{i}{2}\,\mathsf{Tr}\,\ln S''_{\alpha\beta}\,,$$

where $\tilde{S}''_{ab} = S''_{ab} - S''_{a\alpha}(S''^{-1})^{\alpha\beta} S''_{\beta b}$. The solution reads

$$\begin{aligned}\Gamma^{(1)}[\Phi, F] \;\; = \;\; & \frac{1}{2}\,\mathsf{Tr}\left((F^j\mathcal{B}_j)^{ab}\tilde{S}''_{bc}[\Phi]\right) - \frac{i}{2}\,\mathsf{Tr}\,\ln\left(i(F^j\mathcal{B}_j)^{ab}\right) + \\ & + \frac{i}{2}\,\mathsf{Tr}\,\ln S''_{\alpha\beta}[\Phi] - \frac{i}{2}\delta(0)n\,.\end{aligned}$$

Here the matrixes $\mathcal{B}^{ab}_i$ are introduced in the same manner as in previous case but for the $\mathcal{A}^i_{ab}$ ones.

## Conclusions

- We have studied relations existing between the Legendre transformations in quantum field theory and the functional differential equation for effective action which has the form of functional Clairaut-type equation. We have found that specific features of this equation do not hold within the perturbation theory in a quantum field theory without composite operators.

- Within the approach to the quantum field theory based on composite fields the perturbation expansion of the effective action leads exactly to a functional Clairaut-type equation with a special type of the right-hand side.

## Conclusions

- Partial first-order differential equations of Clairaut-type were our preliminary step in the study of solutions to the problem. It was shown that in case when the right-hand side of the equation has the form inspired by the real situation in quantum field theory with composite fields the solution to that functional Clairaut-type equation can be found with the help of algebraic manipulations only. In our knowledge the solutions found can be considered as a new result in the theory of partial first-order differential equations of Clairaut-type.
- We have found an explicit solution to the functional Clairaut-type equation appearing in the quantum field theory with composite fields to define one loop contribution to the corresponding effective action.

**Thank you**
**for attention!**