Lattice Quantum Chromodynamics

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Outline

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- Lattice QCD
- Continuum limit
- Monte Carlo simulations
- Static quark potential
- Some results
- Conclusions

References

F. Knechtli, M. Günther and M. Peardon,
Lattice Quantum Chromodynamics – Practical Essentials,
SpringerBriefs in Physics

P. Weisz, Corfu Summer School 2014

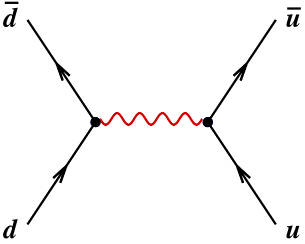
Lattice QCD

Quantumchromodynamics (QCD)

QCD is the theory of strong interactions with the Lagrangian

$$\mathcal{L}_{\mathrm{QCD}}(g_0, m_q) = -\frac{1}{2g_0^2}\mathrm{tr}\,\{F_{\mu\nu}F_{\mu\nu}\} + \sum_{q=u,d,s,c,b,t} \bar{q}\,(\gamma_\mu(\partial_\mu + A_\mu) + m_q)\,q$$

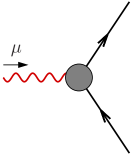
- bare parameters: gauge coupling g_0 and quark masses m_q
- Perturbation theory is an asymptotic expansion for small g_0

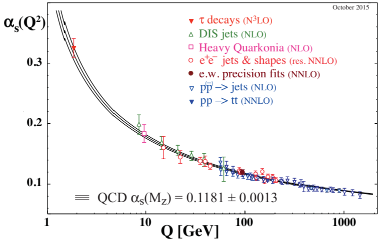


Asymptotic freedom

Renormalized coupling $\bar{g}$ and masses $\overline{m}$ depend on the renormalization scale μ and on the renormalization scheme

$\bar{g}(\mu) \quad =$





Asymptotic freedom for large μ ('t Hooft, 1972; Gross and Wilczek; Politzer, 1973):

$$\bar{g}^2(\mu) \overset{\mu\to\infty}{\sim} \frac{1}{2b_0 \log(\mu/\Lambda)}$$

[Particle Data Group (2015)]

$\overline{\mathrm{MS}}$ scheme, $\alpha_s = \bar{g}(\mu)^2/(4\pi)$

Confinement

Regimes of QCD

- Perturbation theory works at high energy $\gtrsim 2\,\mathrm{GeV}$ where α_{s} is small and quarks and gluons are (almost) free
- At lower energies, quarks and gluons are confined into hadrons $\Rightarrow$ we need a different tool to extract from $\mathcal{L}_{\mathrm{QCD}}$ the properties of hadrons: lattice formulation and computer simulations

Gauge theory on the lattice

Lattice regularization of path integral [Wilson; Smit; 1974]

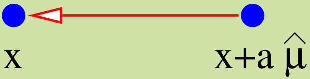
4d Euclidean hypercubic lattice with lattice spacing a:

$x = (n_0, n_1, n_2, n_3)a$, $n_\mu = 0, 1, \ldots$, $\mu = 0, 1, 2, 3$

For simulations consider finite lattice of size $T \times L^3$

$SU(N)$ Gauge field

$U_\mu(x) \in SU(N)$ associated with links



Local gauge transformation $\Omega(x) \in SU(N)$:

$$U_\mu^{(g)}(x) = \Omega(x)\, U_\mu(x)\, \Omega^{-1}(x + a\hat{\mu})$$

Group invariant (Haar) measure $\mathrm{d}U$:

$$I[f] = \int_{SU(N)} \mathrm{d}U\, f(U) = \int_{SU(N)} \mathrm{d}U\, f(VUW)\,,\ I[1] = 1$$

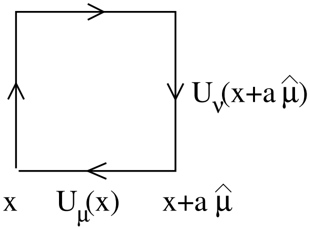
Wilson gauge action

Wilson plaquette action $S_{\rm w}$

Plaquette $P_{\mu,\nu}(x) = U_\mu(x)\,U_\nu(x+a\hat\mu)\,U_\mu^{-1}(x+a\hat\nu)\,U_\nu^{-1}(x)$,
$P_{\mu,\nu}^{-1}(x) = P_{\mu,\nu}^\dagger(x) = P_{\nu,\mu}(x)$
Under a gauge transformation $U'_{\mu,\nu}(x) = \Omega(x)\;P_{\mu,\nu}(x)\;\Omega^{-1}(x)$

$$S_{\rm w}[U] = \frac{\beta}{N}\sum_x\sum_{\mu<\nu}{\rm Re\,tr}\left[1 - P_{\mu,\nu}(x)\right]$$

positive, gauge invariant, parameter $\beta \geq 0$



$U_{\rm I} = U_\mu(x)\;U_\nu(x+a\hat\mu)$
$U_{\rm II} = U_\nu(x)\;U_\mu(x+a\hat\nu)$
curvature $M = U_{\rm I} - U_{\rm II}$

$$M\;M^\dagger = 2 - P_{\mu,\nu}(x) - P_{\mu,\nu}^{-1}(x)$$

$$\mathrm{tr}\,(M\;M^\dagger) \geq 0,\; = 0 \Leftrightarrow M = 0$$

Lattice path integral

Partition function

$$Z = \int \mathcal{D}[U] \;\; \mathrm{e}^{-S_{\mathrm{w}}[U]}$$

$\mathcal{D}[U] = \prod_{x,\mu} \mathrm{d}U_\mu(x)$, Haar measure $\mathrm{d}U$: compact, a non-perturbative definition of Euclidean path integral

Expectation values

Observable $=$ function of the gauge field $\mathcal{O}[U]$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}[U] \;\; \mathcal{O}[U] \, \mathrm{e}^{-S_{\mathrm{w}}[U]}$$

Gauge fixing is not required for gauge invariant observables

Wilson fermions

Quarks and antiquarks

Quark fields: $\psi_i^\alpha(x)$, Grassmann variables with indices $\alpha = 1,\ldots,4$ (Dirac) and $i = 1,\ldots,N$ (gauge; fundamental representation)
Antiquark fields: $\bar\psi_i^\alpha$: independent Grassmann field in the complex conjugate representation

$$\{\psi_i^\alpha(x),\psi_j^\beta(y)\} = 0\,,\quad \{\bar\psi_i^\alpha(x),\bar\psi_j^\beta(y)\} = 0\,,\quad \{\psi_i^\alpha(x),\bar\psi_j^\beta(y)\} = 0$$

Under local gauge transformation $(\psi^{(\mathrm{g})})_i^\alpha(x) = [\Omega(x)_{ij}]\psi_j^\alpha(x)$ and $(\bar\psi^{(\mathrm{g})})_i^\alpha(x) = \bar\psi_j^\alpha[\Omega(x)^{-1}]_{ji}$

Wilson fermion action

$$S_\mathrm{F} = a^4\sum_x \bar\psi_i^\alpha(x)[D_\mathrm{w}+m_0]_{ij}^{\alpha\beta}\psi_j^\beta(x)\,,\quad m_0 \text{ bare mass}$$

Wilson–Dirac operator

Wilson–Dirac operator

$$[D_{\rm w}]_{ij}^{\alpha\beta} = \frac{1}{2}\sum_\mu \left\{[\gamma_\mu]^{\alpha\beta}(\nabla_\mu^* + \nabla_\mu)_{ij} - a[\delta]^{\alpha\beta}(\nabla_\mu^*\nabla_\mu)_{ij}\right\}$$

Covariant difference operators

$$(\nabla_\mu\psi)_i^\alpha(x) = \{[U_\mu(x)]_{ij}\psi_j^\alpha(x+a\hat\mu) - \psi_i^\alpha(x)\}/a$$
$$(\nabla_\mu^*\psi)_i^\alpha(x) = \{\psi_i^\alpha(x) - [U_\mu^{-1}(x-a\hat\mu)]_{ij}\psi_j^\alpha(x-a\hat\mu)\}/a$$

Large sparse matrix of dimension $12\times(T/a)\times(L/a)^3$

What is the role of the Wilson term $-\frac{a}{2}\nabla_\mu^*\nabla_\mu$?

Chiral symmetry

Chiral transformation:

$$\psi(x) \to \psi'(x) = \mathrm{e}^{i\alpha\gamma_5}\psi(x)\,, \quad \bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x)\mathrm{e}^{i\alpha\gamma_5} \quad \alpha \in \mathbb{R}$$

Naive free Dirac operator: $\not{\partial}\psi(x) = \frac{1}{2}\gamma_\mu(\nabla^*_\mu + \nabla_\mu)\psi(x)$
Action $S = a^4 \sum_x \bar{\psi}(x)(\not{\partial} + m_0)\psi(x)$ is invariant under chiral transformation for $m_0 = 0 \Leftrightarrow \{\not{\partial}, \gamma_5\} = 0$
Problem: propagator in momentum space is

$$S(p) = \left[i\gamma_\mu\bar{p}_\mu + m_0\right]^{-1}\,, \quad \bar{p}_\mu = \frac{1}{a}\sin(ap_\mu)$$

When $ap_\mu \ll 1$, $\bar{p}_\mu \approx p_\mu$, $S(p)$ has continuum form. But also if $ap_\mu = \pi - aq_\mu$, $aq_\mu \ll 1$, then $\bar{p}_\mu \approx q_\mu \Rightarrow$ species doubling: $2^4 = 16$ fermion copies in the continuum

Nielsen–Ninomiya theorem

Nielsen-Ninomiya theorem [H.B. Nielsen and M. Ninomiya, 1981; D. Friedan, 1982]

Cannot construct local (free) lattice operator D with $\{D, \gamma_5\} = 0$ without doubling the quark species

Wilson–Dirac operator

$D_\mathrm{w} = \frac{1}{2}\left\{\gamma_\mu(\nabla_\mu^* + \nabla_\mu) - a\nabla_\mu^*\nabla_\mu\right\}$

Doublers are removed but chiral symmetry is violated: $\{D_\mathrm{w}, \gamma_5\} = -a\nabla_\mu^*\nabla_\mu \neq 0$, massive operator $D_\mathrm{w} + m_0$: m_0 has to be tuned to realize zero physical quark mass

Ginsparg–Wilson relation

Old suggestion [Ginsparg and Wilson, 1982]: avoid Nielsen–Ninomiya theorem by demanding only $\{D, \gamma_5\} = aD\gamma_5 D$

Revived 1997 [P. Hasenfratz, V. Laliena and F. Niedermayer, hep-lat/9801021; M. Lüscher, hep-lat/9802011; H. Neuberger, hep-lat/9707022]

Lattice QCD

Path integral of QCD

$U_\mu(x) \in SU(3)$, N_f quark fields ψ_f, $f = 1, \ldots, N_\mathrm{f}$ with bare masses m_{0f}

$$Z = \int \mathcal{D}[U] \prod_{f=1}^{N_\mathrm{f}} \mathcal{D}[\bar{\psi}_f]\,\mathcal{D}[\psi_f]\;\mathrm{e}^{-S_\mathrm{w}[U] + \sum_{f=1}^{N_\mathrm{f}} \bar{\psi}_f (D_\mathrm{w} + m_{0f}) \psi_f}$$

with $\mathcal{D}[\psi_f] = \prod_{x,i,\alpha} \mathrm{d}(\psi_f)_i^\alpha(x)$. Using the Matthews–Salam formula: $\int \mathcal{D}[\eta]\mathcal{D}[\bar{\eta}]\;\mathrm{e}^{\bar{\eta}_i M_{ij} \eta_j} = \det(M)$

$$Z = \int \mathcal{D}[U]\;\mathrm{e}^{-S_\mathrm{w}[U]} \prod_{f=1}^{N_\mathrm{f}} \det(D_\mathrm{w} + m_{0f})$$

Nucleon

How to extract the nucleon mass

Lattice operator for a nucleon at rest: $N \sim \sum_{\underline{x}} \epsilon_{ijk} q^i q^j C\gamma_5 \tau^2 q^k$
with $q = (\psi_1\ \psi_2)^T$

$$\langle N(t = na, \underline{p} = 0)\overline{N}(0)\rangle \propto \int \mathcal{D}[U]\,\mathcal{D}[\bar{q}]\,\mathcal{D}[q]\ \ \mathrm{e}^{-S[U,\bar{q},q]}\, N(t,\underline{0})\overline{N}(0)$$

$$\stackrel{n\ \text{large}}{\sim}\ \mathrm{e}^{-n/\xi_N(g_0,\ldots)}$$

assuming the theory has an (effective) transfer matrix

g_0 bare gauge coupling; ξ_N nucleon correlation length

$am_N = 1/\xi_N$ nucleon mass in lattice units

Continuum limit

Classical continuum limit

Parameter β in Wilson gauge action is related to continuum gauge coupling g_0. Use $U_\mu(x) = \exp(aA_\mu(x))$ * and expand in powers of a

$$S_{\mathrm{W}} = -\frac{\beta}{4N} \sum_x \sum_{\mu,\nu} a^4 \mathrm{tr}\left\{F_{\mu\nu}(x) F_{\mu\nu}(x)\right\} + \mathrm{O}(a^5)$$

yields the relation

$$\beta = \begin{cases} 2N/g_0^2 & \text{for gauge group } SU(N) \\ 1/g_0^2 & \text{for gauge group } U(1) \end{cases}$$

Note: in pure Yang–Mill's theory no gauge invariant scalar operator of dimension 5, $\mathrm{O}(a^5) \longrightarrow \mathrm{O}(a^6)$

*More precisely $U_\mu(x) = \mathcal{P} \exp a \int_0^1 \mathrm{d}t\, A_\mu(x + (1-t)a\hat{\mu})$, path ordering $\mathcal{P}$: fields at larger values of t come first.

Scale setting and continuum limit

QCD-like theory with massless quarks, input parameter: $\beta = 6/g_0^2$. The value of the lattice spacing a is not an input!
Compute a mass, like the proton mass $am_N(g_0)$ Scale setting: declare

$$a(g_0) = \frac{am_N(g_0)}{946\,\mathrm{MeV}}$$ Note: value of a depends on quantity chosen

Continuum limit

Correlation length $\xi_N = 1/(am_N)$. $a \to 0$ if consider m_N fixed means $\xi_N \to \infty$, i.e. existence of critical point

$$g_0 \to g_{\mathrm{crit}} \quad \text{such that} \quad \xi_N \to \infty$$

Conventional wisdom (plausible but not yet rigorously proven)

- $g_{\mathrm{crit}} = 0$
- continuum limit of lattice QCD is asymptotically free

Symanzik's conjecture

Consider mass spectrum am_k, $k = 1, 2, \ldots$ near the continuum limit. Presently $\xi_k = 1/(am_k)$ attained not so large $\longrightarrow$ need extrapolations

$$\left[\frac{m_k}{m_1}\right](g_0) = \left[\frac{m_k}{m_1}\right](g_{\mathrm{crit}}) + c_k \mathrm{O}((am_1)^p)\,, k > 1$$

What is the value of p? integer? log corrections?

Symanzik effective theory

Symanzik conjecture [K. Symanzik, 1980]: as $a \to 0$ the lattice theory can be described by an effective continuum theory with the lattice spacing a as an expansion parameter. Only operators with the same symmetries as the lattice theory appear. See [P. Weisz, 1004.3462]

Lattice artifacts

Expectations based on Symanzik

a) generic O(a^2) artifacts in pure $SU(N)$ theory but O(a) effects with pure Wilson fermions

b) it is possible to construct O(a^2)-improved lattice action for $SU(N)$ theory and O(a)-improved Wilson fermions

$W_{a,b}$ rectangular $a \times b$ Wilson loop with $b = \gamma a$ and γ fixed

$$W_{a,b} = a^2 b^2 G_{2,2} + a^2 b^4 G_{2,4} + a^4 b^2 G_{4,2} + \mathrm{O}(a^8)$$

where $G_{2,2} = \int \mathrm{d}^4 x \, \mathrm{tr}\,\{F_{\mu\nu} F_{\mu\nu}\}$. Then

$$\frac{5}{3} W_{a,a} - \frac{1}{12}\{W_{2a,a} + W_{a,2a}\} = a^4 G_{2,2} + \mathrm{O}(a^8)$$

O(a^2) improved action at tree level [M. Lüscher and P. Weisz, 1985]

Monte Carlo simulations

Monte Carlo

Restricting first to pure $SU(3)$ gauge theory. Task is to compute expectation values

$$\langle O \rangle = \int \mathcal{D}[U]\, O(U)\, \Pi(U)\,, \quad \Pi(U) = \frac{\mathrm{e}^{-S(U)}}{Z}, Z \;\leftrightarrow\; \langle 1 \rangle = 1$$

Dimension of the integral $= 8 \times 4 \times (T/a) \times (L/a)^3$ is huge!

Monte Carlo

Importance sampling: approximate

$$\langle O \rangle = \frac{1}{n} \sum_{i=1}^{n} O(U_i) + \mathrm{O}(n^{-1/2})$$

from sequence $\{U_1, U_2, \ldots, U_n\}$ drawn at random from $\Pi(U)$

Markov chain

U_k is obtained from U_{k-1} by a stochastic process. The chain depends on U_1 and a transition probability function $T(U,U')$ with properties

1. $T(U,U') \geq 0 \; \forall U \,, U'$, $\sum_{U'} T(U,U') = 1$
2. stability: $\int \mathcal{D}[U] \Pi(U) T(U,U') = \Pi(U')$
3. ergodicity: $\forall U, U'$ there is a finite n s.t. $T^n(U,U') > 0$.
   Aperiodicity: $T(U,U) > 0 \; \forall U$

Note: detailed balance $\Pi(U)T(U,U') = \Pi(U')T(U',U) \Rightarrow$ 2.

Link algorithms: e.g. Metropolis

1. choose a link at random
2. choose $X \in \mathfrak{su}(3)$ randomly in a ball with uniform distribution
3. generate random number r uniformly in the interval $[0,1]$, accept $U' = \mathrm{e}^X U$ if $\Pi(\mathrm{e}^X U) > r\Pi(U)$, otherwise set $U' = U$

Algorithms for QCD

For large-scale simulations need excellent random number generators, e.g. RANLUX [M. Lüscher, hep-lat/9309020]

$SU(3)$ gauge theory

heatbath [N. Cabibbo and E. Marinari, 1982], overrelaxation [M. Creutz; F. R. Brown and T. J. Woch, 1987]

$SU(3)$ + fermions

Hybrid Monte Carlo (HMC) [Duane, Kennedy, Pendleton and Roweth, 1987]: rewrite fermion determinant using pseudofermion ϕ, define Hamiltonian: $H(U,P) = \langle P,P\rangle/2 + S_{\mathrm{w}}(U) + \langle \phi, (DD^{\dagger})^{-1}\phi\rangle$ $(N_{\mathrm{f}}=2)$, solve molecular dynamics equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}P_{\mu}(x) = -F_{\mu}(x)\,, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}U_{\mu}(x) = P_{\mu}(x)U_{\mu}(x)$$

Fermionic force $F = \partial H/\partial U$ requires the inversion of D !

Statistical error analysis

Autocorrelations

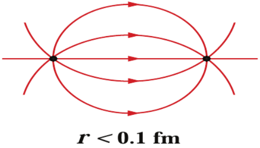
In a Markov chain consecutive configurations are not statistically independent. These autocorrelations reduce the number of independent measurement of an observable O by $n \longrightarrow n/(2\tau_{\rm int}(O))$, see e.g. [U. Wolff, hep-lat/0306017]

Critical slowing down: as $a \to 0$, $\tau_{\rm int}(O) \propto (1/a)^z$. z can be as large as $z \approx 5$ [S. Schaefer, R. Sommer and F. Virotta, 1009.5228]. Open boundary conditions in time achieve $z = 2$ [M. Lüscher and S. Schaefer, 1105.4749]

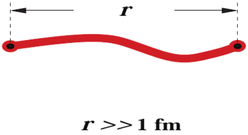
Static quark potential

Static quark potential

$V_n(r)$, $n = 0, 1, \ldots$ are the energy levels of a static quark and anti-quark pair at distance r. $V_0(r) \equiv V(r)$: static potential
Pure $SU(N)$ gauge theory:



Coulomb potential



Confining potential

Short distance

Small distance

$SU(N)$ + fermions: as $r \to 0$ perturbation theory can be applied (asymptotic freedom)

$$V(r) \overset{r \to 0}{\sim} -C_{\mathrm{F}} \frac{g_0^2}{4\pi r}\,, \quad C_{\mathrm{F}} = (N^2-1)/(2N)$$

Expansion of $V(r)$ in the $\overline{\mathrm{MS}}$ coupling is known to 3 loops [N. Brambilla, A. Pineda, J. Soto, A. Vairo, hep-ph/9907240, hep-ph/9903355; A. V. Smirnov, V. A. Smirnov and M. Steinhauser, 0911.4742; C. Anzai, Y. Kiyo and Y. Sumino, 0911.4335]

Bosonic string

Large distance: pure $SU(N)$

As $r \to \infty$ the potential can be computed using an effective bosonic string theory [Y. Nambu, 1979; M. Lüscher, K. Symanzik, P. Weisz, 1980; M. Lüscher, 1981; M. Lüscher and P. Weisz, hep-th/0406205]: the string describes a flux tube joining the static sources and fluctuating in $d-2$ transverse directions

$$V(r) = \sigma r + \mu + \frac{\gamma}{r} + O(1/r^3)$$

σ is the string tension, μ a mass parameter; $\gamma = -\pi(d-2)/24$ depends only on the number of dimensions d

Broadening of the string: its width increases logarithmically in the distance r [M. Lüscher, G. Münster and P. Weisz, 1981]

String breaking

Large distance: QCD

When sea quarks are present, the string breaks at a distance $r_{\mathrm{b}} \approx 1.5\,\mathrm{fm}$ when the energy of the flux tube is sufficient to form two static-light mesons

- fundamental property of QCD with dynamical (sea) quarks
- experimental indications for hadrons with exotic quark content in systems with two charm or bottom quarks
- important input for potential models to describe such systems

Static force and strong coupling

Static force F

The potential contains an additive constant which diverges when the ultra-violet cut-off $(1/a)$ goes to infinity. The divergence originates from the self-energy of the static quarks.
A renormalized quantity is the static force $F(r) = \mathrm{d}V(r)/\mathrm{d}r$

Runnning coupling

$$\alpha_{\mathrm{qq}}(\mu) \equiv \frac{\bar{g}^2_{\mathrm{qq}}(\mu)}{4\pi} = \frac{1}{C_{\mathrm{F}}} r^2 F(r)$$

coupling $\bar{g}_{\mathrm{qq}}(\mu)$ runs with the energy scale $\mu = 1/r$

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \bar{g}_{\mathrm{qq}}(\mu) = \beta_{\mathrm{qq}}(\bar{g}_{\mathrm{qq}}(\mu))$$

β_{qq} function is known up to the 4-loop term, cf. [M. Donnellan, F. Knechtli, B. Leder and R. Sommer, 1012.3037]

Renormalization group equation

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \bar{g}_{\mathrm{qq}}(\mu) = \beta_{\mathrm{qq}}(\bar{g}_{\mathrm{qq}}(\mu))$$

$$\beta_{\mathrm{qq}}(\bar{g}_{\mathrm{qq}}) = -\bar{g}_{\mathrm{qq}}^3 [\sum_{n=0}^{3} b_n^{(\mathrm{qq})} \bar{g}_{\mathrm{qq}}^{2n} + b_{3,l}^{(\mathrm{qq})} \bar{g}_{\mathrm{qq}}^6 \log(3\bar{g}_{\mathrm{qq}}^2/(8\pi)) + \mathrm{O}(\bar{g}_{\mathrm{qq}}^8)]$$

$b_0^{(\mathrm{qq})} = b_0 = \frac{1}{(4\pi)^2}\left(11 - \frac{2}{3}N_{\mathrm{f}}\right)$, $b_1^{(\mathrm{qq})} = b_1 = \frac{1}{(4\pi)^4}\left(102 - \frac{38}{3}N_{\mathrm{f}}\right)$

Integration of renormalization group equation

$$\Lambda_{\mathrm{qq}} = \mu \left(b_0 \bar{g}_{\mathrm{qq}}^2\right)^{-b_1/(2b_0^2)} \mathrm{e}^{-1/(2b_0\bar{g}_{\mathrm{qq}}^2)} \times \exp\left\{-\int_0^{\bar{g}_{\mathrm{qq}}} \mathrm{d}x \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}$$

Λ_{qq} is the Λ parameter in the qq scheme

n-loop solution: truncate β_{qq} after term $b_{n-1}^{(\mathrm{qq})}$ and solve for $\bar{g}_{\mathrm{qq}}$

Static force and scale r_0

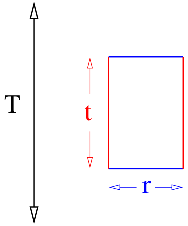
Setting the scale from the static force

$$r^2 F(r)|_{r=r(c)} = c$$

$c = 1.65$ leads to the scale $r_0 = r(1.65)$ which has a value of about $0.5\,\mathrm{fm}$ in QCD [R. Sommer, hep-lat/9310022]

Other choices are $r_1 = r(1.0)$ [C. W. Bernard, T. Burch, K. Orginos, D. Toussaint, T. A. DeGrand, C. E. DeTar, S. A. Gottlieb, U. M. Heller, J. E. Hetrick and B. Sugar, hep-lat/0002028] and $r_c = r(0.65)$ [S. Necco and R. Sommer, hep-lat/0108008]

Wilson loops



T is the temporal size of the lattice
Planar Wilson loop:

$$\langle W(r,t)\rangle = \Big\langle \mathrm{tr}\,\Big\{ P(0,\vec{0};0,r\hat{k})P(0,r\hat{k};t,r\hat{k}) \\ P^{\dagger}(t,\vec{0};t,r\hat{k})P^{\dagger}(0,\vec{0};t,\vec{0}) \Big\} \Big\rangle$$

$P(x_0,\vec{0};x_0,r\hat{k})$ is the product of spatial links joining the static sources at time x_0; it represents a string-like state
$P^{\dagger}(0,\vec{x};t,\vec{x})$ is the product of temporal links at position $\vec{x}$; it represents the propagator of the static quark

$$\langle W(r,t)\rangle \overset{T\to\infty}{\sim} \sum_n c_n c_n^* \mathrm{e}^{-V_n(r)t}$$

assuming the theory has an (effective) transfer matrix

Generalized eigenvalue problem

Smearing

Gauge link smearing replaces spatial lines in the Wilson loops by $P_i(x_0,\vec{0};x_0,r\hat{k})$, i is the smearing index $\longrightarrow$ matrix $\langle W_{ij}(r,t)\rangle$

GEVP

For fixed r correlation matrix $C_{ij}(t)=\langle W_{ij}(r,t)\rangle$

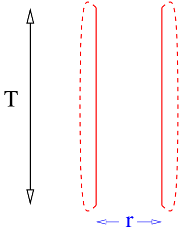
$$C(t)v_n(t,t_0)=\lambda_n(t,t_0)C(t_0)v_n(t,t_0)\,,\quad \lambda_n>\lambda_{n+1}$$

Energy levels can be computed as [M. Lüscher and U. Wolff, 1990]

$$V_n(r)=-\frac{1}{a}\ln\left\{\frac{\lambda_n(t+a,t_0)}{\lambda_n(t,t_0)}\right\}+\epsilon_n(t,t_0)$$

with exponential corrections $\epsilon_n = \mathrm{O}(\mathrm{e}^{-\Delta E_n t})$ [B. Blossier, M. Della Morte, G. von Hippel, T. Mendes and R. Sommer, arXiv:0902.1265]

Polyakov loops



T is the temporal size of the lattice
Correlator of Polyakov loops:

$$\langle P(x)^* P(y) \rangle \,|_{y=x+r\hat{k}}$$

Polyakov loop
$P(x) = \mathrm{tr}\,\{U_\mu(x)U_\mu(x+a\hat{\mu})\ldots U_\mu(x+(T-a)\hat{\mu})\}|_{\mu=0}$:
trace of path through x which winds once around the time axis

$$V(r) \stackrel{T\to\infty}{\sim} -\frac{1}{T}\ln\langle P(x)^* P(y)\rangle\,|_{y=x+r\hat{k}}$$

[M. Lüscher and P. Weisz, hep-lat/0207003]

Signal-to-noise problem

Due to confinement

$$\langle W(r,t)\rangle \text{ “}= \langle \pm \rangle\text{ “} \approx \exp(-\sigma r T)$$

but

$$\left\langle W(r,t)^2\right\rangle \text{ “}= \langle + \rangle\text{ “} \approx \text{const}$$

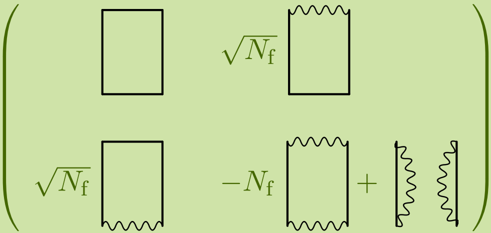
$\Rightarrow$ signal-to-noise ratio *decays exponentially* with the area of the loop

- in $SU(N)$: exponential reduction of the error with t using one-link integral [G. Parisi, R. Petronzio and F. Rapuano, 1983] and with the area using multilevel algorithm [M. Lüscher and P. Weisz, hep-lat/0108014]. But these methods are not applicable with fermions.
- in $SU(N)$+fermions: HYP smearing [A. Hasenfratz and F. Knechtli, hep-lat/0103029] of the gauge links in the Wilson loops [M. Donnellan, F. Knechtli, B. Leder and R. Sommer, 1012.3037]

String breaking as mixing

Mixing

Previous studies in $SU(2)$ + scalar matter fields [F. Knechtli and R. Sommer, hep-lat/9807022, hep-lat/0005021; O. Philipsen and H. Wittig, hep-lat/9807020]: mixing of "string-like" states with states made of two static-light mesons $\longrightarrow$ matrix $C(r,t) =$

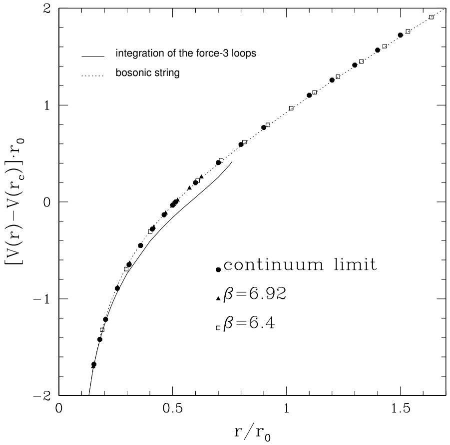


$$\sim \begin{pmatrix} \sigma r & g \\ g & 2E_\mathrm{stat} \end{pmatrix}$$

If $g \neq 0$, energy levels $2(E_\mathrm{stat} \pm g)$ where string breaks $\sigma r \approx 2E_\mathrm{stat}$ (E_stat is the energy of a static-light meson)

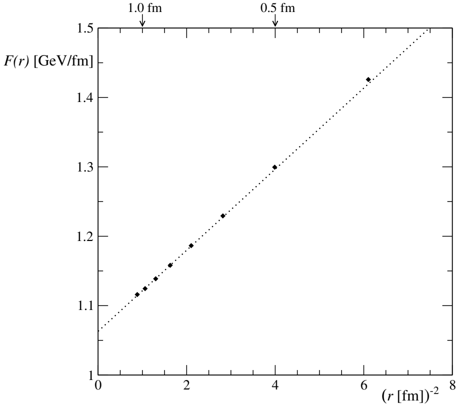
Some results

Static potential in pure $SU(3)$



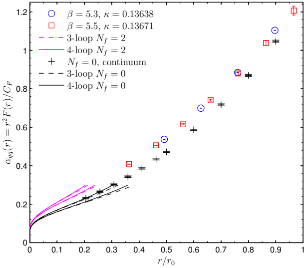
[S. Necco and R. Sommer, hep-lat/0108008] from Wilson loops, continuum limit

Static force in pure $SU(3)$



[M. Lüscher and P. Weisz, hep-lat/0207003], from Polyakov loops, $\beta = 6.0$

Strong coupling

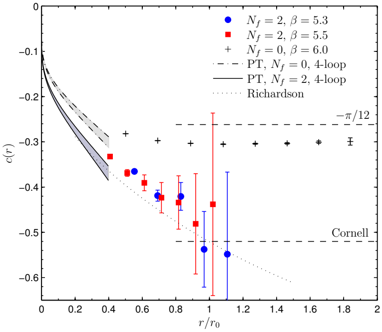


$\alpha_{\mathrm{qq}}(1/r)$ from the $N_{\mathrm{f}} = 2$ theory [F. Knechtli and B. Leder, arXiv:1112.1246] measured on CLS (Coordinated Lattice Simulations consortium) ensembles, Wilson gauge action and $N_{\mathrm{f}} = 2$ flavors of O(a) improved Wilson quarks with $a = 0.066\,\mathrm{fm}$ and $a = 0.049\,\mathrm{fm}$ at $m_\pi = 270\,\mathrm{MeV}$

Λ parameter is know from ALPHA Collaboration, $N_{\mathrm{f}} = 0$ [S. Capitani, M. Lüscher, R. Sommer and H. Wittig, hep-lat/9810063], $N_{\mathrm{f}} = 2$ [P. Fritzsch, F. Knechtli, B. Leder, M. Marinkovic, S. Schaefer, R. Sommer and F. Virotta, 1205.5380]

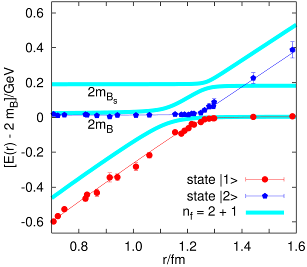
Slope $c(r)$

Slope of the static force $c(r) = \frac{1}{2} r^3 F'(r)$



- running coupling at small r: $\bar{g}_c^2(\mu) = -\frac{4\pi}{C_F} c(r)$, $\mu = 1/r$
- in pure $SU(N)$ theory $c(r)$ approaches the asymptotic value $c(\infty) = -\pi/12$
- c is an interesting but difficult quantity for holographic QCD models [Giataganas and Irges, arXiv:1104.1623]

String breaking in QCD



Result from [G. S. Bali, H. Neff, T. Düssel, T. Lippert and K. Schilling, hep-lat/0505012] with $N_{\mathrm{f}} = 2$ at $a = 0.083\,\mathrm{fm}$ and $m_\pi = 400\,\mathrm{MeV}$; qualitative sketch of expected behavior with $N_{\mathrm{f}} = 2 + 1$

Continuum limit and quark mass dependence with $N_{\mathrm{f}} = 2 + 1$ are still open issues

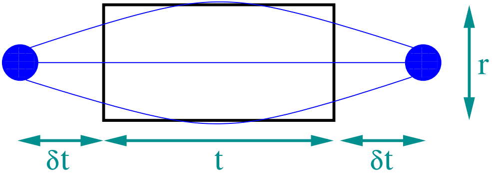
Hadro-quarkonium

Pentaquark

Candidates for a penta-quark state ($c\bar{c}uud$) $P_c^+(4380)$ ($J^P = \frac{3}{2}^-$) and $P_c^+(4450)$ ($J^P = \frac{5}{2}^+$) from $\Lambda_b \to J/\psi\ p\ K$ [LHCb: R. Aaij et al, 1507.03414, 1604.05708]

Hadro-quarkonia: quarkonia ($c\bar{c}$) bound "within" ordinary hadrons [S. Dubynskiy and M. Voloshin, 0803.2224]

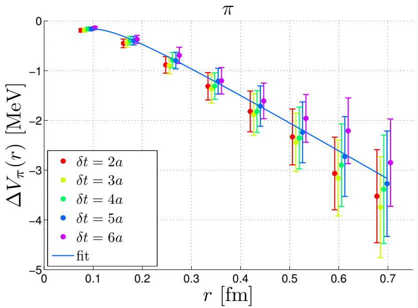
Lattice study [M. Alberti, G. S. Bali, S. Collins, F. Knechtli, G. Moir and W. Söldner, 1608.06537]



$$C_H(r,\delta t,t) = \frac{\langle W(r,t) C_{H,\mathrm{2pt}}(t+2\delta t)\rangle}{\langle W(r,t)\rangle\langle C_{H,\mathrm{2pt}}(t+2\delta t)\rangle}$$

Modification of the static quark potential

Shift of the static quark potential in the presence of a hadron
$\Delta V_H(r,\delta t) \equiv V_H(r,\delta t) - V_0(r) = -\lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} \ln[C_H(r,\delta t,t)]$
CLS $N_\mathrm{f} = 2+1$ lattice, $a = 0.0854(15)\,\mathrm{fm}$, $m_\pi \approx 223\,\mathrm{MeV}$



$\Delta V_H(r) < 0$ for mesons (π, K, ρ, $K^\star$ and ϕ) as well as for baryons (N, Σ, Λ, Ξ, Δ, Σ^*, Ξ^* and Ω of both parities)
$\Rightarrow$ binding energies of charmonium of few MeV like deuterium

Conclusions

Conclusions

- this lecture is a compact introduction to the formulation of QCD on the lattice discussing its continuum limit and Monte Carlo simulations
- the static quark potential is a quantity which shows the properties of QCD from small to large distances and is therefore taken as an example
- more than 40 years after its invention, lattice field theory has developed into an active field of research connecting physics, mathematics and informatics