Supersymmetric Flux Backgrounds and Generalised Special Holonomy

Charles Strickland-Constable

IHES and CEA/Saclay

Recent Developments in Strings and Gravity,
Corfu Summer Institute, 15th September 2016

1411.5721 with André Coimbra and Daniel Waldram

and 1504.02465 & 1606.09304 with André Coimbra

Introduction

Minkowski backgrounds : $M^{D-1,1} \times M_{\text{int}}$

- No fluxes: $\rightarrow \nabla\epsilon = 0$

  $\rightarrow$ Special holonomy! E.g. CY_3, G_2 holonomy, etc.

  [Candelas, Horowitz, Strominger & Witten '85]

- Fluxes $\rightarrow \nabla\epsilon = (\text{“Flux”}) \cdot \epsilon \neq 0$ [Strominger '86, Hull '86]

  $\rightarrow$ intrinsic torsion [Gauntlett, Martelli, Pakis & Waldram '02]

- Hitchin: Geometry on $T \oplus T^*$ includes H flux! (But not RR)

  $\rightarrow$ Generalised Calabi-Yau $\quad \mathrm{d}_H \Phi^{\pm} = 0$

  [Graña, Minasian, Petrini & Tomasiello '04]

- Can we geometrise all fluxes? Generalised special holonomy?

11D Supergravity [Cremmer, Julia & Scherk '78]

- Field content $\{g_{\mu\nu}, \mathcal{A}_{\mu\nu\rho}, \psi_\mu\}$ with $\mathcal{F}_4 = \mathrm{d}\mathcal{A}_3$
- Bosonic Action

$$S_{\mathsf{B}} \sim \int \left(\mathrm{vol}_g \mathcal{R} - \tfrac{1}{2}\mathcal{F} \wedge *\mathcal{F} - \tfrac{1}{6}\mathcal{A} \wedge \mathcal{F} \wedge \mathcal{F}\right)$$

- Supersymmetry

$$\delta\psi_\mu = \nabla_\mu \varepsilon + \tfrac{1}{288}({\Gamma_\mu}^{\nu_1 \dots \nu_4} - 8{\delta_\mu}^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4})\mathcal{F}_{\nu_1 \dots \nu_4}\varepsilon,$$

Restricting to 7 dimensions

- Warped metric ansatz $(\mu, \nu = 0, \ldots, 3)$ $(m, n = 1, \ldots, 7)$

$$\mathrm{d}s_{11}^2 = \mathrm{e}^{2\Delta(x)}\eta_{\mu\nu}\mathrm{d}y^\mu \mathrm{d}y^\nu + g_{mn}(x)\mathrm{d}x^m \mathrm{d}x^n$$

- Internal fields: $\{g_{mn}, A_{mnp}, \tilde{A}_{m_1 \ldots m_6}, \Delta; \psi_m, \rho\}$

- Field strengths

$$F_4 = \mathrm{d}A_3 \qquad \tilde{F}_7 = \mathrm{d}\tilde{A}_6 - \tfrac{1}{2}A_3 \wedge F_4,$$

- Gauge transformation: $(\Lambda \in \Lambda^2 T^* M,\ \tilde{\Lambda} \in \Lambda^5 T^* M)$

$$A' = A + \mathrm{d}\Lambda \qquad \tilde{A}' = \tilde{A} + \mathrm{d}\tilde{\Lambda} - \tfrac{1}{2}\mathrm{d}\Lambda \wedge A$$

Restricting to 7 dimensions

- Spinor decomposition

$$\varepsilon = \eta_+ \otimes \epsilon + \eta_- \otimes \epsilon^c$$

- Internal SUSY variations / Killing spinor equations

$$\delta\psi_m = \nabla_m \epsilon + \tfrac{1}{288}({\gamma_m}^{n_1 \dots n_4} - 8{\delta_m}^{n_1}\gamma^{n_2 n_3 n_4})F_{n_1 \dots n_4}\epsilon - \frac{1}{12}\frac{1}{6!}\tilde{F}_{m n_1 \dots n_6}\gamma^{n_1 \dots n_6}\epsilon = 0$$

$$\delta\rho = \nabla\!\!\!/\,\epsilon + (\partial\!\!\!/\Delta)\epsilon - \frac{1}{4}F\!\!\!/\,\epsilon - \frac{1}{4}\tilde{F}\!\!\!/\,\epsilon = 0$$

- $\mathcal{N} = N$ SUSY background iff

$\exists\, N$ independent non-vanishing complex ϵ satisfying $\uparrow$

G-structure analysis

Definition: G-structure $\equiv$ covering w/local frames related by G

(E.g. Orthonormal frames $g(\hat{e}_a, \hat{e}_b) = \delta_{ab}$ give $SO(d)$ structure)

- $\epsilon = \epsilon_1 + \mathrm{i}\epsilon_2$ stabilised by $\begin{cases} G_2 & \epsilon_1 \propto \epsilon_2 \\ SU(3) & \text{otherwise} \end{cases}$
- So ϵ defines local G_2 or $SU(3)$ structure
- Or: Form bilinears $\Phi_k \sim \epsilon\gamma_{m_1 \dots m_k}\epsilon$
- Find: intrinsic torsion $\sim$ Flux

$$\mathrm{d}\Phi_k \sim \text{"Flux"}$$

[Gauntlett, Martelli, Pakis & Waldram '02] [Cardoso, Curio, Dall'Agata, Lüst, Manousselis & Zoupanos '02] [Kaste, Minasian & Tomasiello '03] [Lukas & Saffin '04]

Intrinsic Torsion

- Given $\hat{\nabla}^{\mathfrak{g}}\epsilon = 0$, any G-connection can be written

$$\nabla^{\mathfrak{g}} = \hat{\nabla}^{\mathfrak{g}} + \Sigma \qquad \Sigma \in T^* \otimes \mathfrak{g}$$

- Let $W = T \otimes \Lambda^2 T^* \sim$ {Torsions}
- Torsion map on Σ:

$$\tau(\Sigma) = T(\nabla^{\mathfrak{g}}) - T(\hat{\nabla}^{\mathfrak{g}})$$

- G-structure defines section of $W/\mathrm{Im}(\tau)$ $\quad\rightarrow$ Intrinsic torsion!

$\exists G$-compatible torsion-free $\nabla^{\mathfrak{g}}$ $\Leftrightarrow$ Intrinsic torsion vanishes

$\Leftrightarrow$ Special holonomy G

Generalised Tangent Bundle [Hull '07; Pacheco & Waldram '08]

- Consider a bundle

$$E \simeq TM \oplus \Lambda^2 T^* M \oplus \Lambda^5 T^* M \oplus (T^* M \otimes \Lambda^7 T^* M)$$

- Carries action of $E_{7(7)} \times \mathbb{R}^+$ in $\mathbf{56}_{+1}$ representation
- Sections of $E \leftrightarrow$ infinitesimal diffeos + gauge transformations
- Dorfman derivative:

$$\begin{aligned} L_V &= \partial_V - (\partial \times_{\mathrm{ad}} V) \cdot \\ &\sim \mathcal{L}_v + (\mathrm{d}\Lambda_{(2)} \cdot) + (\mathrm{d}\tilde{\Lambda}_{(5)} \cdot) \end{aligned}$$

Generalised Connections and Torsion [CSW '11]

- Connection: For $\Omega_M \in \mathrm{ad}(E_{d(d)} \times \mathbb{R}^+)$

$$D : E \to E^* \otimes E$$

$$D_M V^A = \partial_M V^A + \Omega_M{}^A{}_B V^B$$

- Torsion: Define $T(V) \in \mathrm{ad}(E_{d(d)} \times \mathbb{R}^+)$

$$T(V) \cdot = L_V^{(\partial \to D)} - L_V$$

- For a torsion-free connection D

$$L_V = \partial_V - (\partial \times_{\mathrm{ad}} V) \cdot = D_V - (D \times_{\mathrm{ad}} V) \cdot$$

Supergravity Fields and the Generalised Metric

- Can build generalised metric

$$G \sim \{g_{mn}, A_{mnp}, \tilde{A}_{m_1 \dots m_6}, \Delta\} \in \frac{E_{7(7)} \times \mathbb{R}^+}{SU(8)}$$

$\rightarrow$ Defines $SU(8)$ structure on E

- Supergravity fermions $\rightarrow SU(8)$ representations

$$\epsilon^\alpha \in \mathbf{8} \qquad \rho_\alpha \in \bar{\mathbf{8}} \qquad \psi^{[\alpha\beta\gamma]} \in \mathbf{56}$$

- Generalised vector in $SU(8)$ indices

$$\mathbf{56} \rightarrow \mathbf{28} + \bar{\mathbf{28}} \qquad V = (V^{[\alpha\beta]}, \bar{V}_{[\alpha\beta]})$$

Generalised Levi-Civita & SUSY variations [CSW '11]

- $SU(8)$ (metric) compatible connection defined by

$$DG = 0$$

- $\exists$ family of D torsion-free & compatible (Not unique!!!)
- $T = 0 \Rightarrow$ Some cpts fixed to be $\nabla, F, \tilde{F}, \mathrm{d}\Delta$
- SUSY variations are:

$$\delta\rho_\alpha = \bar{D}_{\alpha\beta}\epsilon^\beta \qquad \delta\psi^{[\alpha\beta\gamma]} = D^{[\alpha\beta}\epsilon^{\gamma]}$$

- These operators are unique!
- Write them collectively as $(\delta\psi, \delta\rho) = D \times_{\text{SUSY}} \epsilon$

SUSY backgrounds revisited [Pacheco & Waldram '08; Grana & Orsi '10]

- ϵ defines (global) $SU(7)$ structure on E

Key result:

Killing spinor eqns $(D \times_{\text{SUSY}} \epsilon) \equiv SU(7)$ Intrinsic torsion

- I.e SUSY $\Rightarrow SU(7)$ structure has vanishing intrinsic torsion
- Analogue of spaces with special holonomy
- $SU(7)$ "generalised holonomy"

Generalised special holonomy!

$\mathcal{N} = 1$ Minkowski background $\Leftrightarrow SU(7)$ generalised holonomy

Comment:

- $D^{[\alpha\beta}\epsilon^{\gamma]} = 0$ and $\bar{D}_{\alpha\beta}\epsilon^{\beta} = 0$ (SUSY)

  $\Rightarrow$ Generalised Ricci flat (Eqns of motion)

Extension : $\mathcal{N} = 1$ AdS backrounds [Coimbra, CSC '15]

- $\mathcal{N} = 1$ AdS backgrounds

$$D^{[\alpha\beta}\epsilon^{\gamma]} = 0 \qquad\qquad D_{\alpha\beta}\epsilon^{\beta} = \Lambda\bar{\epsilon}_{\alpha}$$

$\Lambda \rightarrow$ (generalised) singlet torsion

- "Weak generalised holonomy"

[c.f. Sasaki-Einstein, weak G_2 etc.]

- SUSY $\Rightarrow$ Generalised Einstein

$$R_{MN} \sim \Lambda^2 G_{MN}$$

Extension : Higher $\mathcal{N}$ backgrounds?

Would like that:

$\mathcal{N} = N$ Minkowski background $\overset{?}{\Leftrightarrow} SU(8-N)$ generalised holonomy

- $\mathcal{N} = 2 \to$ torsion-free $SU(6)$ structure

  (same proof: $(D \times_{\text{SUSY}} \epsilon_{1,2}) \equiv$ Intrinsic torsion)

- $\mathcal{N} \geq 3$ more difficult: $(D \times_{\text{SUSY}} \epsilon_i) \subsetneq$ Intrinsic torsion

  $\to$ What happens then?...

Mulitple Killing spinors [c.f. Gabella, Martelli, Sparks & Passias '12]

- Basis ϵ_i for $i = 1, \dots, N$ of $\mathbb{C}$-vector space of Killing spinors
- SUSY $\Rightarrow$ rescaled $\hat{\epsilon}_i = \mathrm{e}^{-\Delta/2}\epsilon_i$ satisfies

$$\tilde{\nabla}_m \hat{\epsilon} = \nabla_m \hat{\epsilon} - \frac{1}{4}\frac{1}{3!} F_{mnpq}\gamma^{npq}\hat{\epsilon} - \frac{1}{4}\frac{1}{6!}\tilde{F}_{mn_1 \dots n_6}\gamma^{n_1 \dots n_6}\hat{\epsilon} = 0$$

- $\{\gamma^{(2)}, \gamma^{(3)}, \gamma^{(6)}\}$ generate $SU(8)$ so $SU(8)$ connection!
- $\Rightarrow$ preserves inner products $\hat{\epsilon}_i^\dagger \hat{\epsilon}_j$
- So $\exists$ unitary basis $\boxed{\hat{\epsilon}_i^\dagger \hat{\epsilon}_j = \delta_{ij}}$

$SU(8-N)$ Intrinsic torsion mismatch

- Split $SU(8)$ index : $\alpha = (i, a)$, where $i \leftrightarrow$ Killing spinors.
- Let $D = \hat{D} + \Sigma$ where
  - D an $SU(8-N)$ connection $D\epsilon_i = 0$
  - $\hat{D}$ a torsion-free $SU(8)$ connection
  - $\Sigma = (\Sigma_{[\alpha\beta]}{}^{\gamma}{}_{\delta}, \Sigma^{[\alpha\beta]\gamma}{}_{\delta}) \in E^* \otimes \mathrm{ad}\, SU(8)$ $\leftarrow$ torsion is here!
- $SU(8-N)$ intrinsic torsion

$$\hat{\tau}_{\text{int}} = (\Sigma_{a\gamma}{}^{\gamma}{}_{i}, \Sigma_{i\gamma}{}^{\gamma}{}_{j}, \Sigma_{[ab}{}^{i}{}_{c]}, \Sigma_{[ab}{}^{i}{}_{j]}, \Sigma_{[ai}{}^{j}{}_{k]}, \Sigma_{[ij}{}^{k}{}_{l]}, \Sigma_{[ij}{}^{a}{}_{k]}) + (\text{c.c.})$$

- Projections $\hat{D}_{[\alpha\beta}\bar{\epsilon}_{\gamma]}$ and $\hat{D}_{\alpha\beta}\epsilon^{\beta}$ don't see $\Sigma_{[ij}{}^{a}{}_{k]}$!!!
- Looks impossible that SUSY could $\Rightarrow \hat{\tau}_{\text{int}} = 0$

Killing vectors

Killing vectors have

$$\mathcal{L}_v g = 0 \quad \Leftrightarrow \quad \nabla_{(m} v_{n)} = 0 \quad \Leftrightarrow \quad \nabla_m v_n = \nabla_{[m} v_{n]}$$

- Basically $(\nabla v) \in \mathrm{ad}(SO(d)) \subset \mathrm{ad}(GL(d,\mathbb{R}))$
- So ∇ and (∇v) can act on arbitrary $Spin(d)$ objects
- $\rightarrow$ Kosmann's spinorial Lie derivative

$$\mathcal{L}^{\mathsf{K}}_v \epsilon = \nabla_v \epsilon + \tfrac{1}{4}(\nabla_{[a} v_{b]})\gamma^{ab}\epsilon$$

- "Agrees" with Lie derivative if v a Killing vector!

$$\mathcal{L}_v = \nabla_v - (\nabla \times_{\mathrm{ad}\, GL(d,\mathbb{R})} v)\cdot = \nabla_v - (\nabla \times_{\mathrm{ad}\, SO(d)} v)\cdot$$

Generalised Killing vectors (GKVs) [Grana, Minasian, Petrini & Waldram '08]

Generalised Killing vectors (GKVs) have

$$L_V G = 0 \quad \Leftrightarrow \quad D \times_{\mathrm{ad}(E_{7(7)} \times \mathbb{R}^+)} V = D \times_{\mathrm{ad}(SU(8))} V$$

$\Rightarrow$ Dorfman derivative can naturally act on $SU(8)$ objects!

$$L_V = D_V - (D \times_{\mathrm{ad}(E_{7(7)} \times \mathbb{R}^+)} V) \cdot = D_V - (D \times_{\mathrm{ad}(SU(8))} V) \cdot$$

(for D a torsion-free generalised $SU(8)$ connection)

Kosmann-Dorfman (KD) derivative

Define

$$L_V = D_V - (D \times_{\mathrm{ad}(SU(8))} V)\cdot$$

For GKVs:

- "Agrees" with Dorfman derivative
- Algebra closes: $[L_V, L_{V'}] = L_{L_V V'}$
- Commutes with SUSY operators

$$L_V(D \times_{\mathsf{SUSY}} \epsilon) = D \times_{\mathsf{SUSY}} (L_V \epsilon)$$

(Kosmann)-Dorfman and GKVs

•) Vector part of GKV is flux preserving Killing vector.

$$\mathcal{L}_v g = \mathcal{L}_v \Delta = \mathcal{L}_v F = \mathcal{L}_v \tilde{F} = 0 \qquad \text{(Isometry)}$$

→ Preserves Killing spinor equations $\mathcal{L}_v(\mathcal{D}_{\text{SUSY}}\epsilon) = \mathcal{D}_{\text{SUSY}}(\mathcal{L}_v\epsilon)$

⇒ The Killing spinors form a representation of the isometry group!

•) In an appropriate "untwisted frame" on E have:

$$\boxed{L_V \equiv \mathcal{L}_v}$$

•) ⇒ If V a GKV then ∃ constant matrix $X_i{}^j$ s.t.

$$\boxed{L_V \epsilon_i = X_i{}^j \epsilon_j}$$

Spinor bilinears and trilinears

Complex generalised vectors V_{ij} and $W^{ij} = (V_{ij})^*$

$$(V_{ij})^{\alpha\beta} = \epsilon_i^{[\alpha} \epsilon_j^{\beta]} \qquad (V_{ij})_{\alpha\beta} = 0$$

Find (using SUSY and $\epsilon_i^\dagger \epsilon_j = \delta_{ij}$):

- Automatically GKVs
- $$\boxed{L_{V_{ij}} \epsilon_k \sim D_{V_{[ij}} \epsilon_{k]}} \qquad L_{W^{ij}} \epsilon_k = 0$$
- Then representation $\Rightarrow L_{V_{ij}} \epsilon_k = X_{ijk}{}^l \epsilon_l$

$$\Rightarrow \boxed{L_{V_{ij}} \epsilon_k = 0 \qquad L_{W^{ij}} \epsilon_k = 0}$$

GKVs preserving the Killing spinors!

$SU(8-N)$ Intrinsic torsion revisited

Return to setup $D^{SU(8-N)} = \hat{D}^{SU(8)}_{T=0} + \Sigma^{SU(8)}$ from before

We have that:

$$(L_{V_{ij}}\epsilon_k)^a \sim \epsilon^{\gamma}_{[i}\epsilon^{\gamma'}_{j}(\bar{\hat{D}}_{|\gamma\gamma'|}\epsilon^a_{k]}) \sim \Sigma_{[ij}{}^{a}{}_{k]}$$

This was the missing component of the intrinsic torsion!!!

Further: showed that $L_{V_{ij}}\epsilon_k = 0$, so it vanishes!!!

Generalised special holonomy!!!

Result:

$\mathcal{N} = N$ Minkowski background $\Leftrightarrow SU(8 - N)$ generalised holonomy

In other dimensions:

d	$\tilde{H}_d$	Generalised Holonomy
7	$SU(8)$	$SU(8 - \mathcal{N})$
6	$Sp(8)$	$Sp(8 - 2\mathcal{N})$
5	$Sp(4) \times Sp(4)$	$Sp(4 - 2\mathcal{N}_+) \times Sp(4 - 2\mathcal{N}_-)$
4	$Sp(4)$	$Sp(4 - 2\mathcal{N})$

Also for type IIA and IIB

Internal Killing superalgebra

Reminiscent of "superalgebra"

$$
\begin{aligned}
[\epsilon_i, \epsilon_j] &= V_{ij} \\
[V_{ij}, \epsilon_k] &= L_{V_{ij}} \epsilon_k = 0 \\
[V_{ii'}, V_{jj'}] &= L_{V_{ii'}} V_{jj'} = 0
\end{aligned}
\qquad
\begin{aligned}
[Q, Q\} &= P \\
[P, Q\} &= 0 \\
[P, P\} &= 0
\end{aligned}
$$

Not a coincidence – "internal sector" of Killing superalgebra

11d Killing superalgebra (KSA) [Figueroa-O'Farrill, Meessen & Philip '04]

Lie Superalgebra on { Killing vectors } $\oplus$ { Killing spinors }

$$
\begin{aligned}
[\varepsilon_1, \varepsilon_2\} &= v(\varepsilon_1, \varepsilon_2) \\
[v, \varepsilon\} &= \mathcal{L}_v \varepsilon \\
[v_1, v_2\} &= \mathcal{L}_{v_1} v_2
\end{aligned}
$$

Algebra of "super-isometries"

11d Killing superalgebra (KSA)

Introduce basis of external Weyl spinors $\eta_1 = (1,0)$, $\eta_2 = (0,1)$

$$Q_{i,\alpha} = \eta_\alpha \otimes \epsilon_i, \qquad \bar{Q}_i{}^{\dot\alpha} = \bar\eta^{\dot\alpha} \otimes \epsilon_i^c,$$

and internal (complex) vectors $z_{ij} = (\epsilon_i^c \gamma^m \epsilon_j)\frac{\partial}{\partial x^m}$

$$\begin{aligned}
[Q_{i,\alpha}, \bar{Q}_{j,\dot\beta}\} &= \delta_{ij}(\sigma^\mu)_{\alpha\dot\beta}\frac{\partial}{\partial x^\mu}, \\
[Q_{i,\alpha}, Q_{j,\beta}\} &= \epsilon_{\alpha\beta} z_{ij}, \\
[\bar{Q}_{i,\dot\alpha}, \bar{Q}_{j,\dot\beta}\} &= \epsilon_{\dot\alpha\dot\beta}\bar{z}_{ij},
\end{aligned}$$

KSA $\cong$ Supertranslational part of super-Poincaré algebra

(Internal isometries always central $[z_{ij}, \cdot\} = 0$)

Further extensions?

- $\mathcal{N} = N$ AdS? [work in progress]
- Moduli? [Garcia-Fernandez, Rubio & Tipler '15; Ashmore & Waldram '15] [Ashmore, Petrini & Waldram '15; Ashmore, Gabella, Grana, Petrini & Waldram '16]
- Definition of "generalised holonomy"?
- Higher derivative corrections? [Garcia-Fernandez '13] [Coimbra, Minasian, Triendl & Waldram '14]
- Non-geometric backgrounds? [See talks by Hull, Lüst]

The End

- Thanks for your attention!