Holography, localization, black holes

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Recent Developments in Strings and Gravity

Outline

- **Introduction**
- **Part I: Field Theory Perspective**
- **Part II: AdS Black Holes, localization and entropy**

Introduction

Lot of recent activity in the study of supersymmetric and superconformal theories in various dimensions in closely related contexts

- A deeply interconnected web of supersymmetric theories arising from branes, most often strongly coupled, related by various types of dualities.
- Progresses in the evaluation of exact quantum observables. Related to localization and the study of supersymmetry in curved space.
- Many results on indices and counting problems that can be also related to BH physics.

Introduction

One of the success of string theory is the microscopic counting of micro-states for a class of asymptotically flat black holes [Vafa-Strominger'96]

- The black holes are realized by putting together D-branes, extended objects that have gauge theories on the world-volume
- The entropy is obtained by counting states in the corresponding gauge theory

Introduction

No similar result for AdS black holes in $d \geq 4$ was known until very recently. But AdS should be simpler and related to holography:

- A gravity theory in AdS_{d+1} is the dual description of a CFT_d

The entropy should be related to the counting of states in the dual CFT. People tried hard for AdS_5 black holes (states in N=4 SYM). Still an open problem.

Introduction

At the end of these two lectures I will have solved the problem for static AdS_4 BH, by relating two quantities

- the entropy of a supersymmetric AdS_4 black hole in M theory
- a field theory computation for a partition function in the dual CFT_3

The computation uses recent localization techniques that allow to evaluate exact quantities in supersymmetric gauge theories.

PART I : FIELD THEORY PERSPECTIVE

PART I : FIELD THEORY PERSPECTIVE

- FT and SCFT on curved space
- Localization
- The topologically twisted index

Chapter I : FT and SCFT on curved space

Euclidean FT in finite volume

To regularize IR physics we can put the theory on a Euclidean compact manifold. Supersymmetry restricts the manifold

Euclidean FT in finite volume

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Topological Field Theories [Witten, 1988]

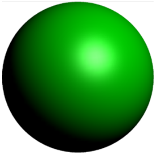
- $N = 2$ theories in 4d with $SU(2)_R$ R-symmetry can be consistently defined on any M_4 euclidean manifold
- The resulting theory is *topological*: independent of metric.

$$T_{\mu\nu} = \{Q, \cdots\}$$

Euclidean FT in finite volume

To regularize IR physics we can put the theory on a Euclidean compact manifold. Supersymmetry restricts the manifold

SCFT on Spheres [Pestun 2007]



Just use conformal map from $\mathbb{R}^n$ to S^n

Euclidean FT in finite volume

To regularize IR physics we can put the theory on a Euclidean compact manifold. Supersymmetry restricts the manifold

In between a pletora of possibilities.

- A general method is to couple to supergravity and find solution of [Festuccia-Seiberg]

$$\delta\psi_\mu(x) \quad = \quad \nabla_\mu \epsilon + \cdots \equiv 0$$

Backgrounds with twisted (conformal) Killing spinors have been classified in various dimensions and with various amount of supersymmetry.

[klare,tomasiello,A.Z.;dumitruescu,festuccia,seiberg;Closset,Dumitrescu,Festuccia,Komargodski; ...]

Superconformal theories on curved spaces

For SCFT, the variation of the gravitino can be written as a (twisted) conformal Killing equation

$$(\nabla_a - iA_a^R)\epsilon_+ + \gamma_a \epsilon_- = 0 \qquad \Longrightarrow \qquad \nabla_a^A \epsilon_+ = \frac{1}{d}\gamma_a \not{\nabla}^A \epsilon_+$$

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- for theories with more supercharges and in different dimensions, A_a^R is promoted to a non-abelian gauge field, there can be other backgrounds tensor fields and extra conditions.

Superconformal theories on curved spaces

A possible solution is to have covariantly constant spinors

$$\nabla^A_a \epsilon_+ = \frac{1}{d}\gamma_a \cancel{\nabla}^A \epsilon_+ \qquad \text{solved by} \qquad \nabla^A_a \epsilon_+ = 0$$

Topological Field Theories can be realized this way by canceling the spin connection with a background R-symmetry

$$\nabla^A_a \epsilon_+ = \partial_a \epsilon_+ + \frac{1}{4}\omega_a^{\alpha\beta}\Gamma^{\alpha\beta}\epsilon_+ + A^R_a \epsilon_+ = \partial_a \epsilon_+$$

which can be solved by $\epsilon_+ =$ constant on any manifold

Superconformal theories on curved spaces

The well know examples of Topological Field Theories back in the eighties

$$\nabla_a^A \epsilon_+ = \partial_a \epsilon_+ + \frac{1}{4}\omega_a^{\alpha\beta}\Gamma^{\alpha\beta}\epsilon_+ + A_a^R \epsilon_+ = \partial_a \epsilon_+$$

- $N = 2$ theories on any M_4: $SU(2)_R$ background gauge field

  ϵ_+ transform in the $(\mathbf{2}, \mathbf{0})$ representation of the local Euclidean group $SO(4) = SU(2) \times SU(2)$

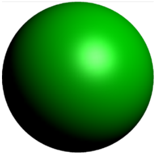
- A twist in 2d on any Riemann surface Σ_g: $U(1)_R$ background gauge field

Superconformal theories on curved spaces

A possible solution is to have Killing spinors

$$\nabla_a^A \epsilon_+ = \frac{1}{d}\gamma_a \cancel{\nabla}^A \epsilon_+ \qquad \text{solved by} \qquad \nabla_a \epsilon_+ = \gamma_a \epsilon_+$$

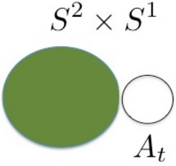
Realized for Theories on Spheres, with zero background field $A^R = 0$.



Superconformal theories on curved spaces

Or combinations: Killing spinors on S^d and R- and flavor holonomies on S^1

$$\nabla_a^A \epsilon_+ = \frac{1}{d}\gamma_a \not{\nabla}^A \epsilon_+ \qquad \text{solved by} \qquad \nabla_a \epsilon_+ = \gamma_a \epsilon_+ \, , A_t = i$$



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The path integral on $S^d \times S^1$ can be written as a trace over a Hilbert space

$$Z_{S^d \times S^1} = \mathrm{Tr}(-1)^F e^{-\beta\{Q,S\}+\sum \Delta_a J_a}$$

This is the superconformal index: it counts BPS states graded by dimension and charges

[Romelsberger; Kinney,Maldacena,Minwalla,Rahu]

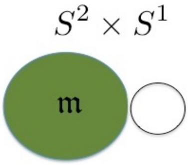
computed for large class of 4d SCFT [Gadde,Rastelli,Razamat,Yan]

Superconformal theories on curved spaces

Or combinations: a topological twist on Σ_g and flavor holonomies on S^1 (or T^2)

$$\nabla_a^A \epsilon_+ = \frac{1}{d}\gamma_a \not{\nabla}^A \epsilon_+ \qquad \text{solved by} \qquad \nabla_a^A \epsilon_+ = 0\,, A_a^R = \omega_a$$

Example $S^2 \times S^1$:



The integral $\mathfrak{m} = \frac{1}{2\pi}\int_{S^2} F^R$ gives a magnetic charge for R-symmetry

Superconformal theories on curved spaces

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$$\nabla_a^A \epsilon_+ = \frac{1}{d}\gamma_a \not{\nabla}^A \epsilon_+ \qquad \text{solved by} \qquad \nabla_a^A \epsilon_+ = 0\,,\, A_a^R = \omega_a$$

Example $S^2 \times S^1$:

The path integral on $\Sigma_g \times S^1$ can be written as a Witten index (elliptic genus)

$$Z_{\Sigma_g \times S^1} = \mathrm{Tr}_{\mathcal{H}}\left((-1)^F e^{iJ_F A^F} e^{-\beta H} \right)$$

$Q^2 = H - \sigma^F J_F$

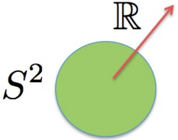
holomorphic in $A^F + i\sigma^F$

of the dimensional reduced theory on Σ_g and counts ground states graded by charges. This is the topologically twisted index.

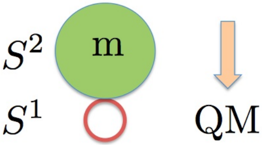
[Benini,AZ; Closset-Kim]

Superconformal theories on curved spaces

The two indices are quite different



- the superconformal index counts the BPS states on S^2/operators



- the topologically twisted index counts the ground states/Landau levels of the theory on S^2 with magnetic charges for R and flavor symmetries

Superconformal theories on curved spaces

In between a pletora of possibilities. Backgrounds with (twisted) conformal Killing spinors have been classified in various dimensions and with various amount of supersymmetry. For example,

- In $N = 1$ in 4d, the existence of a CKS is (locally) equivalent to the existence of a complex structure.
- In $N = 2$ in 3d, the existence of a CKS is (locally) equivalent to the existence of a holomorphic transverse foliation.

[klare,tomasiello,A.Z.;dumitruescu,festuccia,seiberg]

Chapter II : Localization

Localization

Exact quantities in supersymmetric theories with a charge $Q^2 = 0$ can be obtained by a saddle point approximation

$$Z = \int e^{-S} = \int e^{-S+t\{Q,V\}} \underset{t\gg 1}{=} e^{-\bar{S}|_{class}} \times \frac{\det_{fermions}}{\det_{bosons}}$$

$$\partial_t Z = \int \{Q, V\} e^{-S+t\{Q,V\}} = 0$$

Very old idea that has become very concrete recently, with the computation of partition functions on spheres and other manifolds supporting supersymmetry.

Localization

Localization ideas apply to path integral of Euclidean supersymmetric theories

- Compact space provides IR cut-off, making path integral well defined
- Localization reduces it to a finite dimensional integral, a matrix model

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- Compact space provides IR cut-off, making path integral well defined
- Localization reduces it to a finite dimensional integral, a matrix model

$$\int \prod_{i=1}^{N_1} du_i \prod_{j=1}^{N_2} dv_j \frac{\prod_{i<j} \sinh^2 \frac{u_i - u_j}{2} \sinh^2 \frac{v_i - v_j}{2}}{\prod_{i<j} \cosh^2 \frac{u_i - v_j}{2}} e^{\frac{ik}{4\pi}\left(\sum u_i^2 - \sum v_j^2\right)}$$

ABJM, 3d Chern-Simon theories, [Kapustin,Willet,Yakoov;Drukker,Marino,Putrov]

Localization

Carried out recently in many cases

- many papers on topological theories
- S^2, T^2
- S^3, $S^3/\mathbb{Z}_k$, $S^2 \times S^1$, Seifert manifolds
- S^4, $S^4/\mathbb{Z}_k$, $S^3 \times S^1$, ellipsoids
- S^5, $S^4 \times S^1$, Sasaki-Einstein manifolds

with addition of boundaries, codimension-2 operators, ...

Pestun 07; Kapustin,Willet,Yakoov; Kim; Jafferis; Hama,Hosomichi,Lee, too many to count them all $\cdots$

Localization

In all cases, it reduces to a finite-dimensional matrix model on gauge variables, possibly summed over different topological sectors

$$Z_M(y) = \sum_{\mathfrak{m}} \int_C dx\, Z_{\text{int}}(x, y; \mathfrak{m})$$

with different integrands and integration contours.

When backgrounds for flavor symmetries are introduced, $Z_M(y)$ becomes an interesting and complicated function of y which can be used to test dualities

- Sphere partition function, Kapustin-Willet-Yakoov; $\cdots$
- Superconformal index, Spironov-Vartanov; Gadde,Rastelli,Razamat,Yan; $\cdots$
- Topologically twisted index, Benini,AZ; Closset-KIm; $\cdots$

Chapter III : The topologically twisted index

The background

Consider an $\mathcal{N} = 2$ gauge theory in three dimensions. There are two types of multiplets, the vector multiplet

$$V = (A_\mu, \sigma, \lambda, \lambda^\dagger, D)$$

and the chiral multiplet

$$\Phi = (\phi, \psi, F)$$

Supersymmetry allows a $U(1)_R$ R-symmetry

$$\Phi(\theta) \to e^{iR_\Phi \alpha} \Phi(e^{-i\alpha}\theta)\,, \qquad \lambda \to e^{i\alpha}\lambda$$

We consider theories where the R-charges R_Φ of the chiral fields are integer.

The background

Consider an $\mathcal{N}=2$ gauge theory on $S^2 \times S^1$

$$ds^2 = R^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) + \beta^2 dt^2$$

with a background for the R-symmetry proportional to the spin connection:

$$A^R = -\frac{1}{2}\cos\theta\, d\varphi = -\frac{1}{2}\omega^{12}$$

so that the Killing spinor equation

$$D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon - iA_\mu^R \epsilon = 0 \qquad \Longrightarrow \qquad \epsilon = \mathrm{const}$$

The partition function

The path integral for an $\mathcal{N} = 2$ gauge theory on $S^2 \times S^1$ with gauge group G localizes on a set of BPS configurations specified by data in the vector multiplets $V = (A_\mu, \sigma, \lambda, \lambda^\dagger, D)$

$$\delta\lambda = (\frac{1}{2} F_{\mu\nu}\gamma^{\mu\nu} - D)\epsilon + i\epsilon\gamma^\mu D_\mu \sigma = 0$$

which, with $\gamma^3\epsilon = \epsilon$, is solved by

$$\sigma = \text{constant} \qquad A_t = \text{constant} \qquad F_{12} = D$$

$$\frac{1}{2\pi}\int_{S^2} F \neq 0$$

The partition function

The path integral for an $\mathcal{N} = 2$ gauge theory on $S^2 \times S^1$ with gauge group G then localizes on the BPS configurations specified by the following data in the vector multiplets

- A magnetic flux on S^2, $\mathfrak{m} = \frac{1}{2\pi}\int_{S^2} F$ in the co-root lattice
- A Wilson line A_t along S^1
- The vacuum expectation value σ of the real scalar

Up to gauge transformations, the BPS manifold is

$$(u = A_t + i\sigma, \mathfrak{m}) \in \mathcal{M}_{\text{BPS}} = \big(H \times \mathfrak{h} \times \Gamma_{\mathfrak{h}}\big)/W$$

The partition function

The path integral reduces to a the saddle point around the BPS configurations

$$\sum_{\mathfrak{m}\in\Gamma_{\mathfrak{h}}} \int du d\bar{u}\, \mathcal{Z}^{\text{cl }+\text{1-loop}}(u,\bar{u},\mathfrak{m})$$

- The integrand has various singularities where chiral fields become massless
- There are fermionic zero modes

The two things nicely combine and the path integral reduces to an r-dimensional contour integral of a meromorphic form

$$\boxed{\frac{1}{|W|}\sum_{\mathfrak{m}\in\Gamma_{\mathfrak{h}}} \oint_C Z_{\text{int}}(u,\mathfrak{m})}$$

[Benini-AZ; arXiv 1504.03698]

The partition function

The classical and 1-loop contribution give a meromorphic form

$$Z_{\text{int}}(u, \mathfrak{m}) = Z_{\text{class}} Z_{\text{1-loop}}$$

in each sector with gauge flux $\mathfrak{m}$, where

$$\boxed{Z_{\text{class}}^{\text{CS}} = x^{k\mathfrak{m}}} \qquad x = e^{iu}$$

$$\boxed{Z_{\text{1-loop}}^{\text{chiral}} = \prod_{\rho \in \mathfrak{R}} \left[\frac{x^{\rho/2}}{1 - x^{\rho}} \right]^{\rho(\mathfrak{m}) - q + 1}} \qquad q = \text{R charge}$$

$$\boxed{Z_{\text{1-loop}}^{\text{gauge}} = \prod_{\alpha \in G} \left(1 - x^{\alpha}\right) (i\, du)^r}$$

The partition function

Recall that the path integral can be re-interpreted as a twisted index: a trace over the Hilbert space $\mathcal{H}$ of states on a sphere in the presence of a magnetic background for the R symmetry

$$\mathrm{Tr}_{\mathcal{H}}\left((-1)^F e^{-\beta H}\right)$$

$$Q^2 = H$$

The partition function

The magnetic flux on S^2 generates Landau levels. Massive bosons and fermions cancel in pairs, while zero modes give

$$|\rho(\mathfrak{m}) - q + 1| \text{ Fermi multiplets on } S^1 \qquad \rho(\mathfrak{m}) - q + 1 < 0$$
$$|\rho(\mathfrak{m}) - q + 1| \text{ Chiral multiplets on } S^1 \qquad \rho(\mathfrak{m}) - q + 1 > 0$$

reduces to Witten index of (0,2) Quantum Mechanics

$$\mathrm{Tr}_{\mathcal{H}} \left((-1)^F e^{-\beta H} \right) \Big|_{\mathfrak{R}} = \prod_{\rho \in \mathfrak{R}} \left[\frac{x^{\rho/2}}{1 - x^{\rho}} \right] \quad , \quad \prod_{\rho \in \mathfrak{R}} \left[\frac{1 - x^{\rho}}{x^{\rho/2}} \right]$$

Chiral   Fermi

[compare with Hori,Kim,Yi '14]

The partition function

The magnetic flux on S^2 generates Landau levels. Massive bosons and fermions cancel in pairs, while zero modes

$|\rho(\mathfrak{m}) - q + 1|$ Fermi multiplets on S^1 $\qquad \rho(\mathfrak{m}) - q + 1 < 0$

$|\rho(\mathfrak{m}) - q + 1|$ Chiral multiplets on S^1 $\qquad \rho(\mathfrak{m}) - q + 1 > 0$

$$\boxed{Z^{\text{chiral}}_{\text{1-loop}} = \prod_{\rho \in \mathfrak{R}} \left[\frac{x^{\rho/2}}{1 - x^{\rho}} \right]^{\rho(\mathfrak{m}) - q + 1}}$$

Due to magnetic flux for R-symmetry, R charges of the fields must be integer.

The contour

$Z_{\text{int}}(u, \mathfrak{m})$ has pole singularities at

- along the hyperplanes $x^\rho = e^{i\rho(u)} = \mathbb{1}$ determined by the chiral fields
- at the boundaries of $H \times \mathfrak{h}$ $\qquad (\mathrm{Im}(u) = \pm\infty, \;\; x = e^{iu} = 0, \infty)$

Supersymmetric localization selects a particular contour of integration C and picks some of the residues of the form $Z_{\text{int}}(u, \mathfrak{m})$.

The contour

Consider a $U(1)$ theory with chiral fields with charges Q_i. We can use the prescription: sum the residues

- at the poles of fields with positive charge, at $x = 0$ if $k < 0$ and at $x = \infty$ if $k > 0$

In a generic theory one should use the effective Chern-Simons coupling

$$k_{eff}(\sigma) = k + \frac{1}{2} \sum_i Q_i^2 \mathrm{sign}(Q_i \sigma)$$

The contour

The prescription can be written in a compact form by using the so-called Jeffrey-Kirwan residue

$$\underset{y=0}{\text{JK-Res}}(Q,\eta)\frac{dy}{y}=\theta(Q\eta)\,\mathrm{sign}(Q)$$

as

$$\boxed{\frac{1}{|W|}\sum_{\mathfrak{m}\in\Gamma_{\mathfrak{h}}}\Bigg[\sum_{x_*\in\mathfrak{M}_{\text{sing}}}\underset{x=x_*}{\text{JK-Res}}\big(Q(x_*),\eta\big)\;Z_{\text{int}}(x;\mathfrak{m})\;+\;\underset{x=0,\infty}{\text{JK-Res}}(Q_x,\eta)\;Z_{\text{int}}(x;\mathfrak{m})\Bigg]}$$

where

$$Q_{x=0}=-k_{\text{eff}}(+\infty)\,,\qquad\qquad Q_{x=\infty}=k_{\text{eff}}(-\infty)$$

Similar to the localization of the elliptic genus for 2d theories and of the Witten index in 1d [Benini,Eager,Hori,Tachikawa '13; Hori,Kim,Yi '14]

The partition function

We can introduce background fluxes $\mathfrak{n}$ and fugacities y for flavor symmetries

$$x^\rho \to x^\rho\, y^{\rho_f}\ , \qquad\qquad \rho(\mathfrak{m}) \to \rho(\mathfrak{m}) + \rho_f(\mathfrak{n})\ ,$$

where ρ_f is the weight under the flavor group, and

$$x = e^{iu}\,, \qquad y = e^{iu^F}\,, \qquad u = A_t + \sigma\,, \qquad u^F = A_t^F + \sigma^F$$

A $U(1)$ topological symmetry with background flux $\mathfrak{t}$ and fugacity ξ contributes

$$Z_{\text{class}}^{\text{top}} = x^{\mathfrak{t}}\, \xi^{\mathfrak{m}}\ .$$

The path integral becomes a function of a set of magnetic charges $\mathfrak{n}, \mathfrak{t}$ and chemical potentials y, ξ.

The partition function

The path integral is still a trace over the Hilbert space $\mathcal{H}$ of states on a sphere in the presence of a magnetic background for the R and the global symmetries, with the insertion of chemical potentials

$$\mathrm{Tr}_{\mathcal{H}}\left((-1)^F e^{iJ_F A^F} e^{-\beta H}\right)$$

$Q^2 = H - \sigma^F J_F$

holomorphic in u^F

where J_F is the generator of the global symmetry.

A Simple Example: SQED

The theory has gauge group $U(1)$ and two chiral Q and $\tilde{Q}$

$$Z = \sum_{\mathfrak{m}\in\mathbb{Z}} \int \frac{dx}{2\pi i\, x} \left(\frac{x^{\frac{1}{2}} y^{\frac{1}{2}}}{1 - xy}\right)^{\mathfrak{m}+\mathfrak{n}} \left(\frac{x^{-\frac{1}{2}} y^{\frac{1}{2}}}{1 - x^{-1} y}\right)^{-\mathfrak{m}+\mathfrak{n}}$$

	$U(1)_g$	$U(1)_A$	$U(1)_R$
Q	1	1	1
$\tilde{Q}$	-1	1	1

Consistent with duality with three chirals with superpotential XYZ

$$Z = \left(\frac{y}{1 - y^2}\right)^{2\mathfrak{n}-1} \left(\frac{y^{-\frac{1}{2}}}{1 - y^{-1}}\right)^{-\mathfrak{n}+1} \left(\frac{y^{-\frac{1}{2}}}{1 - y^{-1}}\right)^{-\mathfrak{n}+1}$$

A Simple Example: $U(1)_{1/2}$ with one chiral

The theory has just a topological $U(1)_T$ symmetry: $J_\mu = \epsilon_{\mu\nu\tau} F_{\nu\tau}$. With background flux $\mathfrak{t}$ and fugacity ξ

$$Z = \sum_{\mathfrak{m}\in\mathbb{Z}} \int \frac{dx}{2\pi i x} x^{\mathfrak{t}} (-\xi)^{\mathfrak{m}} x^{\mathfrak{m}/2} \left(\frac{x^{1/2}}{1-x}\right)^{\mathfrak{m}} = \frac{\xi}{(1-\xi)^{\mathfrak{t}+1}}$$

$$k_{\text{eff}}(\sigma) = \frac{1}{2} + \frac{1}{2}\text{sign}(\sigma) \qquad \rightarrow \qquad Q_{x=0} = -1, \; Q_{x=\infty} = 0$$

pick just the residues at $x = 1$

Consistent with duality with a free chiral.

	$U(1)_g$	$U(1)_T$	$U(1)_R$
X	1	0	1
T	0	1	0
$\tilde{T}$	-1	-1	0

Aharony and Giveon-Kutasov dualities

The twisted index can be used to check dualities: for example, $U(N_c)$ with $N_f = N_c$ flavors is dual to a theory of chiral fields M_{ab}, T and $\tilde{T}$, coupled through the superpotential $W = T\tilde{T}\det M$

$$Z_{N_f=N_c} = \left(\frac{y}{1-y^2}\right)^{(2\mathfrak{n}-1)N_c^2}\left(\frac{\xi^{\frac{1}{2}}y^{-\frac{N_c}{2}}}{1-\xi y^{-N_c}}\right)^{N_c(1-\mathfrak{n})+\mathfrak{t}}\left(\frac{\xi^{-\frac{1}{2}}y^{-\frac{N_c}{2}}}{1-\xi^{-1}y^{-N_c}}\right)^{N_c(1-\mathfrak{n})-\mathfrak{t}}$$

Aharony and Giveon-Kutasov dual pairs for generic (N_c, N_f) have the same partition function.

Dualities and generalizations

Many generalizations

- We can add refinement for angular momentum on S^2 [Benini,AZ '15].
- We can consider higher genus $S^2 \to \Sigma$ [Benini,AZ '16; Closset-Kim '16]

Dualities and generalizations

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We can go up and down in dimension and compute

- amplitudes in gauged linear sigma models for $(2,2)$ theories in 2d on S^2 [Benini,AZ '15; Cremonesi-Closset-Park '15]
- an elliptically generalized twisted index for $\mathcal{N}=1$ theory on $S^2 \times T^2$ [Benini,AZ; see also Closset-Shamir '13;Nishioka-Yaakov '14;Yoshida-Honda '15]

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The index adds to and complete the list of existing tools (superconformal indices, sphere partition functions) for testing dualities.

PART II : AdS_4 black holes

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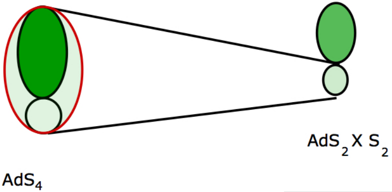
- Magnetically charged AdS_4 black holes
- Localization for ABJM
- Comments and discussions

Chapter I : Magnetically charged AdS_4 black holes

AdS$_4$ black holes

A nice arena where many of the previous ingredients meet is the counting of microstates of asymptotically AdS$_4$ BPS black holes [Benini,Hristov, AZ]



Entropy of black holes

Counting of microstates

Partition function of twisted

3d CFT on S$_2$ X S$_1$

QM fixed point

AdS_4 black holes

I'm talking about BPS asymptotically AdS_4 static black holes

$$\mathrm{d}s^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}\left(dr^2 + V(r)^2 \mathrm{d}s^2_{S^2}\right)$$

- vacua of $N = 2$ gauged supergravities arising from M theory truncations
- supported by magnetic charges on Σ_g: $\mathfrak{n} = \frac{1}{2\pi}\int_{S^2} F$
- preserving supersymmetry via an R-symmetry twist

$$(\nabla_\mu - iA_\mu)\epsilon = \partial_\mu \epsilon \qquad \Longrightarrow \qquad \epsilon = \text{cost}$$

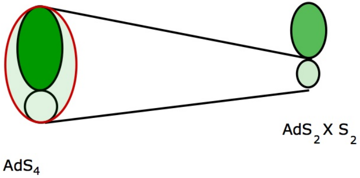
[Cacciatori,Klemm; Gnecchi,Dall'agata; Hristov,Vandoren;Halmagyi;Katmadas]

AdS_4 black holes

These static black holes

$$\mathrm{d}s^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}\left(dr^2 + V(r)^2\mathrm{d}s^2_{S^2}\right)$$

are asymptotic to AdS_4 for $r \gg 1$ and with horizon $AdS_2 \times S^2$ at some $r = r_h$



$$ds^2 \sim \frac{dr^2}{r^2} + r^2(-dt^2 + d\theta^2 + \sin\theta^2 d\phi^2)) \qquad ds^2 \sim -(r-r_h)^2dt^2 + \frac{dr^2}{(r-r_h)^2} + (d\theta^2 + \sin\theta^2 d\phi^2)$$

AdS_4 black holes

They have been found in $\mathcal{N} = 2$ gauged supergravity with n_V vector multiplets. The bosonic field content is

$$g_{\mu\nu}\,, \qquad n_V + 1 \;\; \text{vectors } A_\mu^\Lambda\,, \qquad n_V \;\; \text{complex scalars } z^i$$

It is convenient to write the scalars in terms of $n_V + 1$ homogeneous coordinates X^Λ. The Lagrangian is then uniquely specified by a prepotential

$$\mathcal{F}(X^\Lambda)$$

and it is covariant under symplectic transformations $Sp(2n_V + 1)$, that correspond to electric-magnetic dualities, and a set of electric and magnetic gaugings (FI)

$$(g_\Lambda, g^\Lambda)$$

AdS_4 black holes

A static dyonic black hole is characterized by electric and magnetic charges

$$\int_{S^2} F^\Lambda = \mathrm{Vol}(S^2)\, p^\Lambda \,, \qquad \int_{S^2} G_\Lambda = \mathrm{Vol}(S^2)\, q_\Lambda \,,$$

where $G_\Lambda = 8\pi G_{\rm N}\, \delta(\mathscr{L}\, d\mathrm{vol}_4)/\delta F^\Lambda$ and $G_{\rm N}$ is the Newton constant.

Supersymmetry impose two constraints on the charges, leaving n_V electric charges and n_V magnetic ones. One constraint is linear

$$g_\Lambda p^\Lambda - g^\Lambda q_\Lambda = -1$$

and can be understood as the fact the background for A^Λ cancels the spin connection

$$(\nabla_\mu - iA_\mu)\epsilon = \partial_\mu \epsilon \qquad \Longrightarrow \qquad \epsilon = \text{cost}$$

AdS$_4$ black holes

The entropy of the black hole can be obtained from the value of the scalar fields at the horizon. This is the attractor mechanism and it follows from the BPS equations at the horizon. It states that we have to extremize the quantity

$$\tilde{\mathcal{I}} = -i \frac{q_\Lambda X^\Lambda - p^\Lambda F_\Lambda}{g_\Lambda X^\Lambda - g^\Lambda F_\Lambda}\,, \qquad F_\Lambda = \frac{d\mathcal{F}}{dX^\Lambda}$$

with respect to the values of X^Λ at the horizon, and the entropy is just

$$S = \tilde{\mathcal{I}}(X^\lambda)|_{\mathrm{extremum}}$$

Notice that this also implies that $\tilde{\mathcal{I}}(X^\Lambda)|_{\mathrm{extremum}}$ is real. This condition gives the non-linear constraint on the charges.

AdS$_4$ black holes

Some black holes arise in truncation of M theory on AdS$_4 \times S^7$

- four abelian vectors $U(1)^4 \subset SO(8)$ that come from the reduction on S^7.
- vacua of a $N = 2$ gauged supergravity with 3 vector multiplets; one vector is the graviphoton.

$$F = -2i\sqrt{X^0 X^1 X^2 X^3}$$

$$g_\Lambda \equiv g\,, \qquad g^\Lambda = 0$$

The linear constraint on charges just becomes $\sum_{\Lambda=0}^{3} p^\Lambda = -1/g$.

AdS$_4$ black holes

The explicit expression for the entropy of the AdS$_4 \times S^7$ black hole is quite complicated. In the case of purely magnetical black holes with just

$$p^1 = p^2 = p^3$$

is given by

$$S = \sqrt{-1 + 6p^1 - 6(p^1)^2 + (-1 + 2p^1)^{3/2}\sqrt{-1 + 6p^1}}$$

Dual Field Theory Perspective

General vacua of a bulk effective action

$$\mathcal{L} = -\frac{1}{2}\mathcal{R} + F_{\mu\nu}F^{\mu\nu} + V \ldots$$

with a metric

$$ds^2_{d+1} = \frac{dr^2}{r^2} + (r^2 ds^2_{M_d} + O(r)) \qquad A = A_{M_d} + O(1/r)$$

and a gauge fields profile, correspond to CFTs on a d-manifold M_d and a non trivial background field for the R- or global symmetry

$$L_{CFT} + J^\mu A_\mu$$

Dual Field Theory Perspective

The boundary is $S^2 \times \mathcal{R}$ or $S^2 \times S^1$ in the Euclidean, with a non vanishing background gauge field for the global symmetries on S^2

$$A^\Lambda = -\frac{p^\Lambda}{2} \cos\theta d\phi$$

- The magnetic charges p^Λ corresponds to a deformation of the boundary theory, which is topologically twisted, with a magnetic charge for the R-symmetry and for the global symmetries of the theory.
- The electric charges q_Λ gives sub-leading contributions at the boundary. They are a VEV in the boundary theory, meaning the the average electric charge of the CFT states is non zero.

Dual Field Theory Perspective

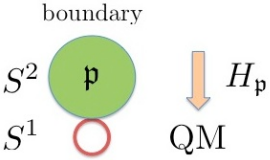
All what previously said and to be said can be easily generalized to the case of a horizon

$$S^2 \to \Sigma_g$$

with small modifications. We will mostly consider S^2 for simplicity.

Dual Field Theory Perspective

It is then natural to evaluate the topologically twisted index with magnetic charges $\mathfrak{p}$ for the R-symmetry and for the global symmetries of the theory

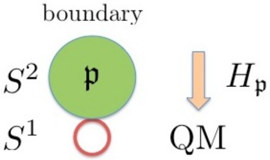


$$Z^{\text{twisted}}_{\Sigma_g^2 \times S^1}(\mathfrak{p}, \Delta) = \mathrm{Tr}_{\mathcal{H}} \left((-1)^F e^{iJ\Delta} e^{-\beta H_{\mathfrak{p}}} \right)$$

$$\Delta = A_t^F + i\sigma^F$$

Dual Field Theory Perspective

It is then natural to evaluate the topologically twisted index with magnetic charges $\mathfrak{p}$ for the R-symmetry and for the global symmetries of the theory



$$Z^{\text{twisted}}_{\Sigma_g^2\times S^1}(\mathfrak{p},\Delta) = \mathrm{Tr}_{\mathcal{H}}\left((-1)^F e^{iJ\Delta} e^{-\beta H_\mathfrak{p}}\right)$$

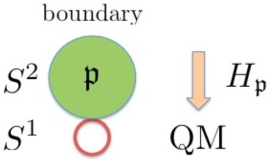
$$\Delta = A_t^F + i\sigma^F$$

This is the Witten index of the QM obtained by reducing $S^2 \times S^1 \to S^1$.

- magnetic charges $\mathfrak{p}$ are not vanishing at the boundary and appear in the Hamiltonian
- electric charges $\mathfrak{q}$ can be introduced using chemical potentials Δ

Dual Field Theory Perspective

It is then natural to evaluate the topologically twisted index with magnetic charges $\mathfrak{p}$ for the R-symmetry and for the global symmetries of the theory



$$Z^{\text{twisted}}_{\Sigma_g^2 \times S^1}(\mathfrak{p}, \Delta) = \mathrm{Tr}_{\mathcal{H}} \left((-1)^F e^{iJ\Delta} e^{-\beta H_\mathfrak{p}} \right)$$

$$\Delta = A_t^F + i\sigma^F$$

The BH entropy is related to a Legendre Transform of the index [Benini-Hristov-AZ]

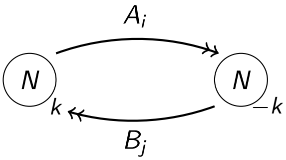
$$S_{BH}(\mathfrak{q}, \mathfrak{n}) \equiv \mathbb{R}\mathrm{e}\, \mathcal{I}(\Delta) = \mathbb{R}\mathrm{e}(\log Z(\mathfrak{p}, \Delta) - i\Delta\mathfrak{q}) \,, \qquad \frac{d\mathcal{I}}{d\Delta} = 0$$

[similar to Sen's formalism, OSV, etc]

Chapter II : Localization for ABJM

The ABJM theory

The dual field theory to $\mathrm{AdS}_4 \times S^7$ is known: is the ABJM theory with gauge group $U(N) \times U(N)$

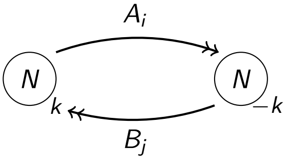


with quartic superpotential

$$W = A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1$$

The ABJM theory

Black hole supported by magnetic charges: is the ABJM theory with gauge group $U(N) \times U(N)$



with quartic superpotential

$$W = A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1$$

defined on twisted $S^2 \times \mathbb{R}$ with magnetic fluxes $\mathfrak{n}_i$ for the R/global symmetries

$$U(1)^4 \subset SU(2)_A \times SU(2)_B \times U(1)_B \times U(1)_R \subset SO(8)$$

.

The ABJM theory

It is useful to introduce a basis of four R -symmetries R_a, $a = 1, 2, 3, 4$

	R_1	R_2	R_3	R_4
A_1	2	0	0	0
A_2	0	2	0	0
B_1	0	0	2	0
B_2	0	0	0	2

A basis for the three flavor symmetries is given by $J_a = \frac{1}{2}(R_a - R_4)$. Magnetic fluxes $\mathfrak{n}_a$ and complex fugacity y_a for the symmetries can be introduced. They satisfy

$$\sum_{a=1}^{4} \mathfrak{p}_a = 2\,, \qquad \text{supersymmetry}$$

$$\prod_{a=1}^{4} y_a = 1\,, \qquad \text{invariance } of\ W$$

ABJM twisted index

The ABJM twisted index is

$$Z = \frac{1}{(N!)^2} \sum_{\mathfrak{m},\widetilde{\mathfrak{m}}\in\mathbb{Z}^N} \int \prod_{i=1}^{N} \frac{dx_i}{2\pi i x_i} \frac{d\tilde{x}_i}{2\pi i \tilde{x}_i} x_i^{k\mathfrak{m}_i} \tilde{x}_i^{-k\widetilde{\mathfrak{m}}_i} \times \prod_{i\neq j}^{N} \left(1 - \frac{x_i}{x_j}\right)\left(1 - \frac{\tilde{x}_i}{\tilde{x}_j}\right) \times$$

$$\times \prod_{i,j=1}^{N} \left(\frac{\sqrt{\frac{x_i}{\tilde{x}_j} y_1}}{1 - \frac{x_i}{\tilde{x}_j} y_1}\right)^{\mathfrak{m}_i - \widetilde{\mathfrak{m}}_j - \mathfrak{p}_1 + 1} \left(\frac{\sqrt{\frac{x_i}{\tilde{x}_j} y_2}}{1 - \frac{x_i}{\tilde{x}_j} y_2}\right)^{\mathfrak{m}_i - \widetilde{\mathfrak{m}}_j - \mathfrak{p}_2 + 1}$$

$$\left(\frac{\sqrt{\frac{\tilde{x}_j}{x_i} y_3}}{1 - \frac{\tilde{x}_j}{x_i} y_3}\right)^{\widetilde{\mathfrak{m}}_j - \mathfrak{m}_i - \mathfrak{p}_3 + 1} \left(\frac{\sqrt{\frac{\tilde{x}_j}{x_i} y_4}}{1 - \frac{\tilde{x}_j}{x_i} y_4}\right)^{\widetilde{\mathfrak{m}}_j - \mathfrak{m}_i - \mathfrak{p}_4 + 1}$$

$$\prod_i y_i = 1\,, \qquad \sum \mathfrak{p}_i = 2$$

where $\mathfrak{m}, \widetilde{\mathfrak{m}}$ are the gauge magnetic fluxes, $y_i = e^{i\Delta_i}$ are fugacities and $\mathfrak{n}_i$ the magnetic fluxes for the three independent $U(1)$ global symmetries

ABJM twisted index

We need to evaluate it in the large N limit. Strategy:

- Re-sum geometric series in $\mathfrak{m}, \widetilde{\mathfrak{m}}$.

$$Z = \int \frac{dx_i}{2\pi i x_i} \frac{d\tilde{x}_i}{2\pi i \tilde{x}_i} \frac{f(x_i, \tilde{x}_i)}{\prod_{j=1}^N (e^{iB_i} - 1) \prod_{j=1}^N (e^{i\widetilde{B}_j} - 1)}$$

- Step 1: find the zeros of denominator $e^{iB_i} = e^{i\tilde{B}_j} = 1$ at large N
- Step 2: evaluate the residues at large N

$$Z \sim \sum_I \frac{f(x_i^{(0)}, \tilde{x}_i^{(0)})}{\det \mathbb{B}}$$

[Benini-Hristov-AZ]

[extended to other models Hosseini-AZ; Hosseini-Mekareeya]

The large N limit

Step 1: solve the large N Limit of the algebraic equations $e^{iB_i} = e^{i\tilde{B}_i} = 1$ giving the positions of poles

$$1 = x_i^k \prod_{j=1}^{N} \frac{\left(1 - y_3 \frac{\tilde{x}_j}{x_i}\right)\left(1 - y_4 \frac{\tilde{x}_j}{x_i}\right)}{\left(1 - y_1^{-1} \frac{\tilde{x}_j}{x_i}\right)\left(1 - y_2^{-1} \frac{\tilde{x}_j}{x_i}\right)} = \tilde{x}_j^k \prod_{i=1}^{N} \frac{\left(1 - y_3 \frac{\tilde{x}_j}{x_i}\right)\left(1 - y_4 \frac{\tilde{x}_j}{x_i}\right)}{\left(1 - y_1^{-1} \frac{\tilde{x}_j}{x_i}\right)\left(1 - y_2^{-1} \frac{\tilde{x}_j}{x_i}\right)}$$

- We dubbed this set of equations *Bethe Ansatz Equations* in analogy with same expressions appearing in the integrability approach to 3d theories [Nekrasov-Shatashvili]
- They can be derived by a BA potential $\mathcal{V}_{BA}$

$$e^{iB_i} = e^{i\tilde{B}_i} = 1 \qquad \Longrightarrow \qquad \frac{d\mathcal{V}_{BA}}{dx_i} = \frac{\mathcal{V}_{BA}}{d\tilde{x}_i} = 0$$

The large N limit

Step 1: the Bethe Ansatz equations can be solved with the ansatz

$$u_i = i\sqrt{N}t_i + v_i\,, \qquad \log \tilde{u}_i = i\sqrt{N}t_i + \tilde{v}_i \qquad (x_i = e^{iu_i}\,, \tilde{x}_i = e^{i\tilde{u}_i})$$

which has the property of selecting contributions from $i \sim j$ and makes the problem local.

$$\rho(t) = \frac{1}{N}\frac{di}{dt}\,, \qquad \delta v(t) = v_i - \tilde{v}_i$$

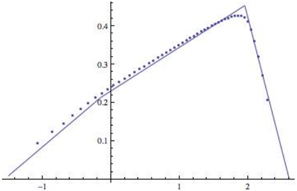
$$\frac{\mathcal{V}_{BA}}{iN^{\frac{3}{2}}} = \int dt \left[t\,\rho(t)\,\delta v(t) + \rho(t)^2 \left(\sum_{a=3,4} g_+\left(\delta v(t) + \Delta_a\right) - \sum_{a=1,2} g_-\left(\delta v(t) - \Delta_a\right) \right) \right]$$

where $g_\pm(u) = \frac{u^3}{6} \mp \frac{\pi}{2}u^2 + \frac{\pi^2}{3}u$.

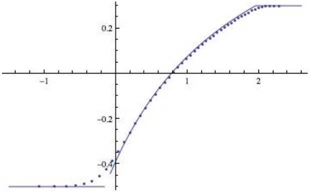
The large N limit

Step 1: the equations can be then explicitly solved

$$u_i = i\sqrt{N}t_i + v_i\,, \qquad \log \tilde{u}_i = i\sqrt{N}t_i + \tilde{v}_i$$



$\rho(t)$



$\delta v(t)$

and

$$\mathcal{V}_{BA} \sim N^{3/2}\sqrt{\Delta_1\Delta_2\Delta_3\Delta_4}$$

The large N limit

Step 1: it is curious that

- In the large N limit, these *auxiliary* BAE are the same appearing in a different localization problem: the path integral on S^3 [Hosseini-AZ; arXiv 1604.03122]

$$\mathcal{V}_{BA}(\Delta) = Z_{S^3}(\Delta) \qquad\qquad y_i = e^{i\Delta_i}$$

The same holds for other 3d quivers dual to M theory backgrounds $\mathrm{AdS}_4 \times Y_7$ ($N^{3/2}$) and massive type IIA ones ($N^{5/3}$).

The large N limit

Step 2: plug into the partition function. It is crucial to keep into account exponentially small corrections in tail regions where $y_i x_i/\tilde{x}_i = 1$

$$\log Z = N^{3/2}(\text{finite}) + \underbrace{\sum_{i=1}^{N} \log(1 - y_i x_i/\tilde{x}_i)}_{O(N)} \qquad y_i x_i/\tilde{x}_i = 1 + e^{-N^{1/2} Y_i}$$

The large N limit

Step 2: plug into the partition function. It is crucial to keep into account exponentially small corrections in tail regions where $y_i x_i / \tilde{x}_i = 1$

$$\log Z = \underbrace{N^{3/2}(\text{finite}) + \sum_{i=1}^{N} \log(1 - y_i x_i/\tilde{x}_i)}_{O(N)} \qquad\qquad y_i x_i/\tilde{x}_i = 1 + e^{-N^{1/2} Y_i}$$

One can by-pass it by using a general simple formula [Hosseini-AZ; arXiv 1604.03122]

$$\log Z = -\sum_a \mathfrak{p}_a \frac{\partial \mathcal{V}_{BA}}{\partial \Delta_a}$$

The final result

The Legendre transform of the index is obtained from $\mathcal{V}_{BA} \sim \sqrt{\Delta_1\Delta_2\Delta_3\Delta_4}$:

$$\mathcal{I}(\Delta) = \frac{1}{3}N^{3/2}\sum\nolimits_a\left(-\mathfrak{p}_a\frac{d\mathcal{V}_{BA}}{d\Delta_a} - i\Delta_a\mathfrak{q}_a\right) \qquad\qquad y_a = e^{i\Delta_a}$$

$\log Z$

This function can be extremized with respect to the Δ_a and

$$\boxed{\mathcal{I}|_{crit} = \mathrm{BH\,Entropy}(\mathfrak{p}_a, \mathfrak{q}_a)}$$

$$\boxed{\Delta_a|_{crit} \sim X^a(r_h)}$$

[Benini-Hristov-AZ]

The final result

Comparing the field theory result with the attractor mechanism:

$$\mathcal{F}(X) = -2i\sqrt{X^0X^1X^2X^3} \sim \mathcal{V}_{BA}(\Delta) = \sqrt{\Delta_1\Delta_2\Delta_3\Delta_4}$$

$$\tilde{\mathcal{I}}(X) = i\frac{\sum_\Lambda \left(p^\Lambda \frac{d\mathcal{F}}{dX^\Lambda} - q_\Lambda X^\Lambda\right)}{\sum X^\Lambda} \sim \sum_a \left(-\mathfrak{p}_a \frac{d\mathcal{V}}{d\Delta_a} - i\Delta_a \mathfrak{q}_a\right) = \mathcal{I}(\Delta)$$

perfect agreement under

$$\frac{X^\Lambda}{\sum_\Lambda X^\Lambda} \to \Delta_a, \qquad (p^\Lambda, q_\Lambda) \to (\mathfrak{p}_a, \mathfrak{q}_a)$$

Chapter III : Comments and discussions

A. Statistical ensemble

Δ_a can be seen as chemical potential in a macro-canonical ensemble defined by the supersymmetric index

$$Z = \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{i\Delta_a J_a} e^{-\beta H}$$

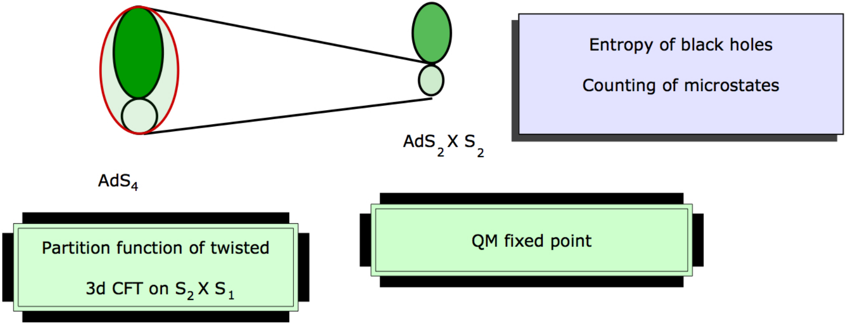
so that the extremization can be rephrased as the statement that the black hole has average electric charge

$$\frac{\partial}{\partial \Delta} logZ \sim < J >$$

- Similarities with Sen's entropy formalism based on AdS_2.
- Similarly to asymptotically flat BH, $(-1)^F$ does not cause cancellations at large N. What's about finite N?

B. R-symmetry extremization

Recall the cartoon



B. R-symmetry extremization

The extremization reflects exactly what's going on in the bulk. Consider no electric charge, for simplicity. The graviphoton field strength depends on r

$$T_{\mu\nu} = e^{K/2} X^\Lambda F_{\Lambda,\mu\nu}$$

suggesting that the R-symmetry is different in the IR and indeed

$$\boxed{\Delta_i|_{crit} \sim X^i(r_h)}$$

B. R-symmetry extremization

The twisted index depends on Δ_i because we are computing the trace

$$Z(\Delta) = \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{i\Delta_i J_i} \equiv \mathrm{Tr}_{\mathcal{H}}(-1)^{R(\Delta)}$$

where $R(\Delta) = F + \Delta_i J_i$ is a possible R-symmetry of the system.

For zero electric charges, the entropy is obtained by extremizing $\log Z(\Delta)$.

Some QFT extremization is at work?

B. R-symmetry extremization

The extremum $\log Z(\hat{\Delta})$ is the entropy.

- symmetry enhancement at the horizon AdS_2:

$$\mathrm{QM}_1 \to \mathrm{CFT}_1$$

- $R(\hat{\Delta})$ is the exact R-symmetry at the superconformal point
- all the BH ground states have $R(\hat{\Delta}) = 0$ because of superconformal invariance (AdS_2)

$$Z(\hat{\Delta}) = \mathrm{Tr}_{\mathcal{H}}(-1)^{R(\hat{\Delta})} = \sum 1 = e^{\mathrm{entropy}}$$

  and the extremum is obtained when all states have the same phase $(-1)^R$

- Z is the natural thing to extremize: in even dimensions central charges are extremized, in odd partition functions...

Conclusions

The main message of these lectures is that you can related the entropy of a class of AdS_4 black holes to a microscopic counting of states.

- first time for AdS black holes in four dimensions

Conclusions

The main message of these lectures is that you can related the entropy of a class of AdS_4 black holes to a microscopic counting of states.

- first time for AdS black holes in four dimensions

But don't forget that we also gave a general formula for the topologically twisted path integral of 2d (2,2), 3d $\mathcal{N} = 2$ and 4d $\mathcal{N} = 1$ theories.

- Efficient quantum field theory tools for testing dualities.
- With many field theory questions/generalizations

Appendix

Supersymmetric theories

Supersymmetric theories are usually formulated on Minkowski space-time $\mathbb{R}^{3,1}$. At the classical level, we have an action for bosonic and fermionic fields

$$S_{\rm SUSY}(\phi(x), \psi(x), A_\mu(x), ...)$$

invariant under transformations that send bosons into fermions and viceversa

$$\delta\phi(x) = \epsilon\psi(x)\,, \qquad \delta\psi = \partial_\mu\phi\gamma^\mu\epsilon + \cdots$$

where ϵ is a constant spinor.

The symmetry group of the theory contains translations, Lorentz transformations $SO(3,1)$ and the fermonic symmetries with the corresponding fermionic Noether charges Q. The theory can be also formulated on Euclidean space $\mathbb{R}^4$.

Can we define the theory on a general manifold M preserving supersymmetry?

Supersymmetric theories on curved spaces

The general strategy is to promote the metric to a dynamical field [Festuccia,Seiberg] .

This is done by coupling the rigid theory to the multiplet of supergravity $(g_{\mu\nu}, \psi_\mu, ...)$

$$S_{\rm SUGRA}(\phi(x), \psi(x), g_{\mu\nu}(x), \psi_\mu(x), ...)$$

which is invariant under local transformations

$$\delta\phi(x) = \epsilon(x)\psi(x)\,, \qquad \delta e^a_\mu(x) = \bar{\epsilon}(x)\gamma^a\psi_\mu(x) + \cdots$$

We are gauging the original symmetries of the theory. At linear level this is just the Noether coupling

$$-\frac{1}{2}g_{mn}T^{mn} + \bar{\psi}_m\mathcal{J}^m$$

Supersymmetric theories on curved spaces

The rigid theory is obtained by freezing the fields of the metric multiplet to background values

$$g_{\mu\nu} = g^M_{\mu\nu}\,, \quad \psi_\mu = 0$$

The resulting theory will be supersymmetric if the variation of supersymmetry vanish

$$\begin{aligned} \delta e^a_\mu(x) &= \bar{\epsilon}(x)\gamma^a\psi_\mu(x) + \cdots \equiv 0 \\ \delta\psi_\mu(x) &= \nabla_\mu\epsilon + \cdots \equiv 0 \end{aligned}$$

The graviton variation gives a differential equation for $\epsilon(x)$ which need to be solved in order to have supersymmetry and gives constraints on M.

Superconformal theories on curved spaces

The strategy here is to couple the CFT to conformal supergravity.
Consider for example, $N = 1$ SCFTs.

Superconformal theories on curved spaces

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Consider for example, $N = 1$ SCFTs.

The group of symmetries of a $N = 1$ SCFT is enlarged to the algebra $SU(2,2|1)$

- translations + Lorentz $SO(3,1) \rightarrow$ conformal group $SO(4,2)$
- supersymmetry Q is doubled: (Q, S)
- extra bosonic global symmetries rotating (Q, S) (R-symmetries)

$$U(1) : \qquad Q \rightarrow e^{i\alpha} Q$$

Superconformal theories on curved spaces

The strategy here is to couple the CFT to conformal supergravity. Consider for example, $N = 1$ SCFTs.

The $N = 1$ conformal supergravity multiplet $(g_{\mu\nu}, \psi_\mu, A_\mu)$ contains gauge fields for the superconformal symmetries

$$-\frac{1}{2} g_{mn} T^{mn} + A_m J^m + \bar{\psi}_m \mathcal{J}^m$$

We freeze $(g_{\mu\nu}, A_\mu)$ to background values and set $\psi_\mu = 0$. In order to preserve some supersymmetry, the gravitino variation must vanish.

$$(\nabla_a - iA_a)\epsilon_+ + \gamma_a \epsilon_- = 0$$

$\epsilon_\pm$ parameters for the supersymmetries and the superconformal transformations.

Refinement by angular momentum

Adding a fugacity $\zeta = e^{i\varsigma/2}$ for the angular momentum on S^2: the Landau zero-modes on S^2 form a representation of $SU(2)$.

$$\boxed{Z^{\text{chiral}}_{\text{1-loop}} = \prod_{\rho\in\mathfrak{R}} \prod_{j=-\frac{|B|-1}{2}}^{\frac{|B|-1}{2}} \left(\frac{x^{\rho/2}\zeta^j}{1-x^\rho\zeta^{2j}}\right)^{\text{sign}\, B}}\,, \qquad B = \rho(\mathfrak{m}) - q_\rho + 1$$

As noticed in other contexts: the refined partition function factorizes into the product of two vortex partition functions

$$Z = Z_{\text{1-loop}}\; Z_{\text{vortex}}(\zeta)\; Z_{\text{vortex}}(\zeta^{-1})$$

[Pasquetti '11;Beem-Dimofte-Pasquetti '12;Cecotti-Gaiotto-Vafa '13,$\cdots$]

Other dimensions

We can consider other dimensions too: $(2,2)$ theories in 2d on S^2

The BPS manifold is now $\mathfrak{M} = (\mathfrak{h} \times \mathfrak{h})/W$ and the 1-loop determinants depend on a complex scalar σ

$$\boxed{Z_{\text{1-loop}}^{\text{chiral}} = \prod_{\rho \in \mathfrak{R}} \left[\frac{1}{\rho(\sigma)}\right]^{\rho(\mathfrak{m}) - q + 1}}$$

$$\boxed{Z_{\text{1-loop}}^{\text{gauge}} = (-1)^{\sum_{\alpha>0} \alpha(\mathfrak{m})} \prod_{\alpha \in G} \alpha(\sigma)\,(d\sigma)^r}$$

Other dimensions

We are just repackaging results about the A-twist of gauged linear sigma models

For examples, for $U(1)$ with N flavors, 2d amplitudes compute the quantum cohomology of $\mathbb{P}^{N-1}$

$$\langle \sigma_1 \cdots \sigma_n \rangle = \sum_{\mathfrak{m}} \int \frac{dx}{2\pi i} \frac{1}{x^{(\mathfrak{m}+1)N}} q^{\mathfrak{m}} x^n = \sum_{\mathfrak{m}} q^{\mathfrak{m}} \delta_{N(\mathfrak{m}+1)-n-1,0}$$

$$\boxed{\sigma^N = q}$$

$$\boxed{\prod_{j=1}^{N} (\sigma - \mu_j) = q}$$

$\Omega-$background and non abelian G can be considered [related work by Cremonesi,Closset,Park '15]

Other dimensions

We can consider other dimensions too: $\mathcal{N}=1$ theories in 4d on $S^2\times T^2$, and obtain an elliptical generalization of our index.

The BPS manifold is now $\mathfrak{M}=(H\times H)/W$ and the 1-loop determinants

$$\boxed{Z^{\text{chiral}}_{\text{1-loop}}=\prod_{\rho\in\mathfrak{R}}\prod_{j=-\frac{|B|-1}{2}}^{\frac{|B|-1}{2}}\left(\frac{i\eta(q)}{\theta_1(q,x^\rho\zeta^{2j})}\right)^{\text{sign}(B)}}$$

$$\boxed{Z^{\text{gauge, off}}_{\text{1-loop}}=(-1)^{\sum_{\alpha>0}\alpha(\mathfrak{m})}\prod_{\alpha\in G}\frac{\theta_1\left(q,x^\alpha\zeta^{|\alpha(\mathfrak{m})|}\right)}{i\eta(q)}\,(du)^r}$$

[also Closset-Shamir '13;Nishioka-Yaakov '14]

[related work by Yoshida-Honda '15]

Other dimensions

The index on $S^2 \times T^2$ reduces to the elliptic genus of a flux dependent collection of $(0,2)$ multiplets on T^2

$\lvert\rho(\mathfrak{m}) - q + 1\rvert$	Fermi multiplets on T^2	$\rho(\mathfrak{m}) - q + 1 < 0$
$\lvert\rho(\mathfrak{m}) - q + 1\rvert$	Chiral multiplets on T^2	$\rho(\mathfrak{m}) - q + 1 > 0$

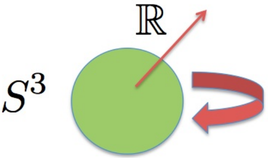
- Again R-charges should be integer.
- It can be tested against Seiberg's dualities.
- It adds to and complete the list of existing tools (superconformal indices, sphere partition functions) for testing 4d dualities.

One dimension more

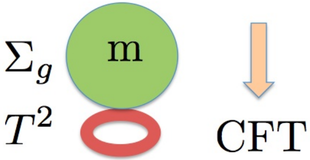
In AdS_5 there are two interesting objects

boundary

bulk



- AdS_5 rotating black hole; where the entropy comes from?



- AdS_5 black string; horizon $AdS_3 \times \Sigma_g$. 2d central charge of the CFT matched with gravity. c-extremization for R-symmetry

[Benini-Bobev]