Universal Gauge Theory in Two Dimensions

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Based on:

1608.03250 with **A. Deser** - **L. Jonke** - **T. Strobl**

1607.00342 (JHEP) with **A. Deser** - **L. Jonke** - **T. Strobl** $\leftarrow$ mainly this one

1604.03739 (PoS)

1509.01829 (JHEP) with **A. Deser** - **L. Jonke**

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General Motivations

- When does a theory have a local symmetry?
  cf. Weinstein's math/9602220 for the math viewpoint
- Strings in quotient spaces beyond group actions
  cf. G/H WZW models (Gawedzki, Kupiainen)
- Target space dualities without global symmetry
  cf. Poisson-Lie (Klimcik, Severa; Sfetsos), but w/o group action; Hull '06
- "Non-geometric" string backgrounds from a worldsheet/worldvolume perspective
  Hull '04; Halmagyi '09; Mylonas, Schupp, Szabo '12; 1311.4878 and 1505.05457 with L. Jonke and O. Lechtenfeld; ...

In this talk

- Focus on 2D bosonic σ-models (LO, no dilaton)

Determine the conditions for existence of *some* gauge extension

- In other words, couple gauge fields A to the theory, valued in *some* "gauge" bundle
- For appropriate gauge transformations, determine the rhs of the Lie derivatives

$$\mathcal{L}_\rho g = \dots \quad \text{and} \quad \mathcal{L}_\rho B = \dots \quad \text{or} \quad \mathcal{L}_\rho H = \dots$$

such that the theory is gauge invariant.

Prospectus of results

- The rhs of the invariance conditions need not be zero
- The gauging is controlled by two (curved, in general) connections $\nabla^{\pm}$
- There exists a universal gauge theory with target $TM \oplus T^*M$
- In the "maximal" case it generalizes G/G WZW and WZ Poisson σ-models
- $\nabla^{\pm}$ are determined in closed form for the "maximal" case

Good old standard gauging of group actions - Global symmetry

Usual procedure: rigid symmetry $\longrightarrow$ (minimal coupling) $\longrightarrow$ local symmetry

Strings propagating in a target spacetime M $\rightsquigarrow$ σ-model of maps $X = (X^i) : \Sigma \to M$

$$S_0[X] = \int_\Sigma \tfrac{1}{2} g_{ij}(X) \mathrm{d}X^i \wedge * \mathrm{d}X^j + \int_\Sigma \tfrac{1}{2} B_{ij}(X) \mathrm{d}X^i \wedge \mathrm{d}X^j \ .$$

Consider Lie algebra $\mathfrak{g}$ with elements ξ_a mapped to vector fields $\rho_a = \rho_a^i(X)\partial_i$ of M:

$$M \times \mathfrak{g} \xrightarrow{\rho} TM \ , \quad \text{such that} \quad \rho(\xi_a) = \rho_a \ ,$$

In general, non-Abelian vector fields satisfying the algebra: $[\rho_a, \rho_b]_{\text{Lie}} = C_{ab}^c \rho_c$.

Then the action S_0 is invariant under the rigid symmetry $\delta_\epsilon X^i = \rho_a^i(X)\epsilon^a$ provided that:

$$\mathcal{L}_{\rho_a} g = 0 \ , \quad \mathcal{L}_{\rho_a} B = \mathrm{d}\beta_a \ .$$

Good old standard gauging of group actions - Gauging/Minimal Coupling

Gauging the symmetry requires coupling of $\mathfrak{g}$-valued 1-forms (gauge fields) $A = A^a \xi_a$

$$\mathrm{d}X^i \to DX^i = \mathrm{d}X^i - \rho_a^i(X) A^a \ .$$

The candidate gauged action is simply

$$S_{\text{m.c.}}[X, A] = \int_\Sigma \tfrac{1}{2} g_{ij}(X) DX^i \wedge *DX^j + \int_\Sigma \tfrac{1}{2} B_{ij}(X) DX^i \wedge DX^j \ .$$

The action is invariant under the (standard) infinitesimal gauge transformations:

$$\begin{aligned} \delta_\epsilon X^i &= \rho_a^i(X)\epsilon^a \ , \\ \delta_\epsilon A^a &= \mathrm{d}\epsilon^a + C^a_{bc} A^b \epsilon^c \ , \end{aligned}$$

with a Σ-dependent gauge parameter ϵ^a (and $\beta_a = 0$).

Note: For $\beta_a \neq 0$ minimal coupling is not sufficient. cf. Hull, Spence '89, ...

Beyond the Standard Gauging

Default: no requirement for a rigid symmetry/no initial assumptions for $g(X)$ and $B(X)$.

In other words, considering again the candidate (minimally-coupled) *gauged* action:

$$S_{\text{m.c.}}[X,A] = \int_\Sigma \tfrac{1}{2} g_{ij}(X) DX^i \wedge *DX^j + \int_\Sigma \tfrac{1}{2} B_{ij}(X) DX^i \wedge DX^j \ ,$$

Question

Under which conditions does $S_{\text{m.c.}}$ have a gauge symmetry $\delta_\epsilon X^i = \rho^i_a(X)\epsilon^a$?

Also, replace Lie algebra $\mathfrak{g}$ by *some* vector bundle $L \xrightarrow{\pi} M$ with an *almost Lie* bracket

cf. Strobl '04

$$L \xrightarrow{\rho} TM \ , \quad [\cdot,\cdot]_L$$

In a local basis of sections e_a of L:

$$[e_a, e_b]_L = C^c_{ab}(X) e_c \quad \xrightarrow{\rho} \quad [\rho_a, \rho_b]_{\text{Lie}} = C^c_{ab}(X)\rho_c$$

$\rho_a = \rho(e_a)$: involutive vector fields generating a (possibly singular) foliation $\mathcal{F}$ on M.

Invariance Conditions

Let us now make a general Ansatz for the gauge transformation of $A = A^a e_a \in \Gamma(L)$:

$$\delta_\epsilon A^a = \mathrm{d}\epsilon^a + C^a_{bc}(X) A^b \epsilon^c + \Delta A^a \,.$$

The worldsheet covariant derivative transforms as

$$\delta_\epsilon DX^i = \epsilon^a \rho^i_{a,j} DX^j - \rho^i_a(X) \Delta A^a \,.$$

Transformation of the action:

$$\delta_\epsilon S_{\text{m.c.}} = \int_\Sigma \epsilon^a \Big(\tfrac{1}{2} (\mathcal{L}_{\rho_a} g)_{ij} DX^i \wedge * DX^j + \tfrac{1}{2} (\mathcal{L}_{\rho_a} B)_{ij} DX^i \wedge DX^j \Big) - \int_\Sigma g_{ij} \rho^i_a \Delta A^a \wedge * DX^j + B_{ij} \rho^i_a \Delta A^a \wedge DX^j \,.$$

Considering $\Delta A^a = \omega^a_{bi}(X) \epsilon^b DX^i + \phi^a_{bi}(X) \epsilon^b * DX^i$, invariance of $S_{\text{m.c.}}$ requires:
$(*^2 = \mp 1)$

$$\mathcal{L}_{\rho_a} g = \omega^b_a \vee \iota_{\rho_b} g - \phi^b_a \vee \iota_{\rho_b} B \,,$$
$$\mathcal{L}_{\rho_a} B = \omega^b_a \wedge \iota_{\rho_b} B \pm \phi^b_a \wedge \iota_{\rho_b} g \,.$$

Geometric Interpretation

What happens under a change of basis $e_a \to \Lambda(X)_a^b e_b$ in L?

$$\begin{aligned} \omega_{bi}^a &\to (\Lambda^{-1})_c^a \omega_{di}^c \Lambda_b^d - \Lambda_b^c \partial_i (\Lambda^{-1})_c^a \ , \\ \phi_{bi}^a &\to (\Lambda^{-1})_c^a \phi_{di}^c \Lambda_b^d \ . \end{aligned}$$

$\rightsquigarrow$ ω_{bi}^a are the coefficients of a connection 1-form on the vector bundle L:

$$\nabla^\omega e_a = \omega_a^b \otimes e_b \ ,$$

and ϕ_{bi}^a are the coefficients of an endomorphism-valued 1-form: $\phi \in \Gamma(T^*M \otimes L^* \otimes L)$.

Since the difference of two vector bundle connections is an endomorphism 1-form,

$\rightsquigarrow$ the gauging is controlled by two connections $\nabla^\pm = \nabla^\omega \pm \phi$ on L

Mixing of g and B - Generalized Geometry

Consider the following two maps, defined via the interior product:

$$E^{\pm} := B \pm g : TM \to T^*M$$

Additionally, define the following linear combinations of ω^a_b and ϕ^a_b:

$$(\Omega^{\pm})^a_b := (\omega \pm \phi)^a_b$$

Then the invariance conditions for Lorentzian world sheets are re-expressed as

$$\mathcal{L}_{\rho_a} E^{\pm} = (\Omega^{\mp})^b_a \otimes \iota_{\rho_b} E^{\pm} - \iota_{\rho_b} E^{\mp} \otimes (\Omega^{\pm})^b_a \,.$$

The graphs of $E^{\pm}$ are identified with n-dimensional sub-bundles $C_{\pm}$ of $TM \oplus T^*M$.

Gualtieri '04

$\rightsquigarrow$ reduction of structure group $O(n,n) \to O(n) \times O(n)$/generalized Riemannian metric

$\rightsquigarrow$ a metric $\mathcal{H} : TM \oplus T^*M \to TM \oplus T^*M$ on the generalized tangent bundle of M

cf. Gualtieri '14 for this perspective

$$\mathcal{H} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}$$

Beyond Minimal Coupling - WZ terms

In the presence of a Wess-Zumino term in the action, minimal coupling is not enough.

$$S_{0,WZ}[X] = \int_\Sigma \tfrac{1}{2} g_{ij}(X)\mathrm{d}X^i \wedge * \,\mathrm{d}X^j + \int_{\hat{\Sigma}} \tfrac{1}{3!} H_{ijk}\mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \,.$$

The candidate gauged action functional, at least for minimally coupled kinetic sector, is

$$S_{WZ}[X,A] = \int_\Sigma \tfrac{1}{2} g_{ij}(X) DX^i \wedge * DX^j + \int_{\hat{\Sigma}} H + \int_\Sigma A^a \wedge \theta_a + \tfrac{1}{2}\gamma_{ab}(X) A^a \wedge A^b \,,$$

where $\theta_a = \theta_{ai}(X)\mathrm{d}X^i$ are 1-forms and $\gamma_{ab}(X)$ are functions, pulled back from M via X.

Such an action was first studied in Hull-Spence '89 and more recently also in Plauschinn '14; Kotov, Salnikov, Strobl '14; Bakas, Lüst, Plauschinn '15

Conditions for gauge invariance

As in the minimally coupled case, we examine the gauge invariance of S_{WZ} under

$$\begin{aligned} \delta_\epsilon X^i &= \rho^i_a(X)\epsilon^a \ , \\ \delta_\epsilon A^a &= \mathrm{d}\epsilon^a + C^a_{bc}(X)A^b\epsilon^c + \omega^a_{bi}(X)\epsilon^b\, DX^i + \phi^a_{bi}(X)\epsilon^b \ast DX^i \ . \end{aligned}$$

Invariance Conditions

$$\begin{aligned} \mathcal{L}_{\rho_a} g &= \omega^b_a \vee \iota_{\rho_b} g + \phi^b_a \vee \theta_a \ , \\ \iota_{\rho_a} H &= \mathrm{d}\theta_a - \omega^b_a \wedge \theta_b \pm \phi^b_a \wedge \iota_{\rho_a} g \ . \end{aligned}$$

Obstructing Constaints

$$\gamma_{(ab)} = \iota_{\rho_a}\theta_b + \iota_{\rho_b}\theta_a = 0 \ , \quad \iota_{\rho_b}\iota_{\rho_a} H = C^d_{ab}\theta_d + \mathrm{d}\iota_{\rho_{[a}}\theta_{b]} - 2\mathcal{L}_{\rho_{[a}}\theta_{b]} \ .$$

Addendum to the Geometric Interpretation

There is a map θ from the vector bundle L to the cotangent bundle of M:

$$\begin{array}{ccc} L & \stackrel{\theta}{\to} & T^*M \\ e_a & \mapsto & \theta(e_a) := \theta_a = \theta_{ai}\mathrm{d}x^i \ . \end{array}$$

Combining this with the map ρ, we obtain a map to the generalized tangent bundle:

$$\begin{array}{ccc} L & \stackrel{\rho\oplus\theta}{\to} & TM \oplus T^*M \\ e_a & \mapsto & (\rho\oplus\theta)(e_a) := \rho_a + \theta_a = \rho_a^i\partial_i + \theta_{ai}\mathrm{d}x^i \ . \end{array}$$

$\rightsquigarrow$ H-twisted Courant algebroid structure on the $TM \oplus T^*M$, with bracket and bilinear:

$$\begin{array}{rcl} [\xi_a, \xi_b] & = & [\rho_a, \rho_b] + \mathcal{L}_{\rho_a}\theta_b - \mathcal{L}_{\rho_b}\theta_a - \frac{1}{2}\mathrm{d}\left(\iota_{\rho_a}\theta_b - \iota_{\rho_b}\theta_a\right) - \iota_{\rho_a}\iota_{\rho_b}H \ , \\ \langle \xi_a, \xi_b \rangle & = & \iota_{\rho_a}\theta_b + \iota_{\rho_b}\theta_a \ . \end{array}$$

Meaning of the two Constraints

- Vanishing of the bilinear form
- Closure of the bracket

$\rightsquigarrow$ (Small) Dirac Structures

So, when can we really find $\nabla^{\pm}/\omega^a_b$ and ϕ^a_b?

Suppose $L = D$ and $(\rho \oplus \theta)(D) = \widetilde{D}$, with full Dirac structures $D, \widetilde{D} \subset (TM \oplus T^*M)_H$:

$$\mathrm{rk}\, D = \tfrac{1}{2}\mathrm{rk}\, TM \oplus T^*M\,, \quad [\Gamma(D), \Gamma(D)] \subset \Gamma(D)\,, \quad \langle \Gamma(D), \Gamma(D) \rangle = 0\,.$$

- First, we proved the invertibility of the operators: $\theta^* \pm \rho : D \to TM$ $\quad (\theta^* = g^{-1} \circ \theta)$
- Note: this holds regardless of the invertibility or not of ρ and θ.

Then we showed that the following coefficients solve the invariance conditions:

$$\begin{aligned} \omega^a_{bi} &= \Gamma^a_{bi} - \phi^a_{bi} + T^a_{bi}\,, \\ \phi^a_{bi} &= [(\theta^* - \rho)^{-1}]^a_k \left(\mathring{\nabla}_i \rho^k_b - \rho^k_c T^c_{bi} \right)\,, \end{aligned}$$

where $T^a_{bi} = [(\theta^* + \rho)^{-1}]^a_k \left(\mathring{\nabla}_i (\theta^* + \rho)^k_b - \tfrac{1}{2} \rho^l_b H^k_{li} \right)$.

Here $\mathring{\nabla}$ is the LC connection on TM and Γ^a_{bi} are the coefficients of ∇^{LC} on D.

Dirac σ-models as Universal Gauge Theory

In intrinsic geometric terms, defining $T(\rho) = T_a^b \otimes e^a \otimes \rho_b \in \Gamma(T^*M \otimes D^* \otimes TM)$:

$$\begin{aligned} \nabla^+ &= \nabla^{LC} + T \, , \\ \nabla^- &= \nabla^{LC} + T - 2\iota_{(\dot{\nabla}-T)(\rho)}(\theta^* - \rho)^{-1} \, . \end{aligned}$$

Returning to the gauged action functional, defining the field $\upsilon \oplus \eta \in \Omega^1(\Sigma, X^*D)$ as:

$$\upsilon = \rho(A) \Rightarrow \upsilon^i = \rho^i_a(X)A^a \quad \text{and} \quad \eta = \theta(A) \Rightarrow \eta_i = \theta_{ai}(X)A^a \, ,$$

the $S_{\rm WZ}[X, A]$ becomes identical to the action for the topological Dirac Sigma Model

Kotov, Schaller, Strobl '04

$$S_{\rm DSM}[X, \upsilon \oplus \eta] = \int_\Sigma \tfrac{1}{2} g_{ij}(X) DX^i \wedge *DX^j + \int_\Sigma \left(\eta_i \wedge \mathrm{d}X^i - \tfrac{1}{2}\eta_i \wedge \upsilon^i\right) + \int_{\hat{\Sigma}} H$$

A non-topological analog may be obtained for small Dirac structures.

Application: the H-twisted Poisson Sigma Model - Motivation

Suppose (M, π) is a Poisson manifold with Poisson structure π $\quad (\pi^{l[i}\partial_l \pi^{jk]} = 0)$

Ikeda '93; Schaller, Strobl '94

$$S_{\text{PSM}}[X, A] = \int_\Sigma \left(A_i \wedge \mathrm{d}X^i + \tfrac{1}{2}\pi^{ij}(X) A_i \wedge A_j \right) \ .$$

- Equivalent (for a bivector linear in x) to 2D YM in the 1st order formalism
- Its path integral quantization yields Kontsevich $\star$ product Cattaneo-Felder '01
- From a different viewpoint, it can be related to Q flux string backgrounds

Application: the H-twisted Poisson Sigma Model - Initial data

Suppose (M, π) is a Poisson manifold with Poisson structure π, and we choose:

$$L = T^*M \ , \quad \rho = \pi^\sharp \quad \text{and} \quad \theta = \mathrm{id} \ \Rightarrow \ \theta^* = g^{-1} \ .$$

The almost Lie bracket on T^*M is the Koszul-Schouten bracket of 1-forms $\alpha, \widetilde{\alpha}$:

$$[\alpha, \widetilde{\alpha}]_{\mathrm{KS}} := \mathcal{L}_{\pi^\sharp(\alpha)}\widetilde{\alpha} - \iota_{\pi^\sharp(\widetilde{\alpha})}\mathrm{d}\alpha - H(\pi^\sharp(\widetilde{\alpha}), \pi^\sharp(\alpha), \cdot) \ ,$$

which in a basis e^i of local sections of T^*M satisfies

$$[e^i, e^j]_{\mathrm{KS}} = C^{ij}_k(X) e^k \ ,$$

with structure functions

$$C^{ij}_k = \partial_k \pi^{ij} + H_{kmn}\pi^{mi}\pi^{nj} \ .$$

Application: the H-twisted Poisson Sigma Model - Action and Symmetry

The gauge field $A = (A_i) \in \Gamma(T^*M)$ is now encoded in the identifications

$$\eta_i = A_i \quad \text{and} \quad \upsilon^i = \pi^{ji} A_j \ .$$

The corresponding gauged action functional, with $DX^i = \mathrm{d}X^i + \pi^{ij} A_j$, is

$$S_{g\mathrm{WZPSM}}[X, A] = \int_\Sigma \Big(A_i \wedge \mathrm{d}X^i + \tfrac{1}{2} \pi^{ij} A_i \wedge A_j \Big) + \int_\Sigma \tfrac{1}{2} g_{ij}(X) DX^i \wedge * DX^j + \int_{\hat{\Sigma}} H(X) \ .$$

The general gauge symmetries of the model are controlled by the coefficients

$$\begin{aligned} \omega^j_{ik} &= \Gamma^j_{ik} + g_{il} \pi^{lm} \phi^j_{mk} + \tfrac{1}{2} \pi^{jl} H_{lik} \ , \\ \phi^j_{ik} &= -[(1 - g\pi g\pi)^{-1}]^l_i g_{lm} (\mathring{\nabla}_k \pi^{mj} + \tfrac{1}{2} H_{knp} \pi^{nm} \pi^{pj}) \ . \end{aligned}$$

$\rightsquigarrow$ this gauging of the (g, H) model led to the H-PSM with extended local symmetries

Integration of gauge fields and reduction

In the (g, B) case, recalling that $E := E^+ = g + B$ and in light-cone coordinates,

$$S[X,A] = \int_\Sigma E_{ij}(X) D_+ X^i D_- X^j \mathrm{d}\sigma^+ \wedge \mathrm{d}\sigma^- \ ,$$

where $D_\pm X^i = \partial_\pm X^i - \rho^i_a(X) A^a_\pm$.

In adapted coordinates $(X^i) = (X^I, X^\alpha)$, integration of the gauge fields leads to

$$S^{\text{red}} = \int_\Sigma E^{\text{red}}_{IJ} \partial_+ X^I \partial_- X^J \mathrm{d}\sigma^+ \wedge \mathrm{d}\sigma^- \ ,$$

where $E^{\text{red}}_{IJ} = E_{IJ} - E_{I\alpha} E^{\alpha\beta} E_{\beta J}$. Moreover, $E^{\text{red}}_{IJ} = E^{\text{red}}_{IJ}(X^I) \rightsquigarrow X^\alpha$-independent.

This is a reduced action with target space the quotient $Q = M/\mathcal{F}$.

Strict vs. non-strict gauging

When such a reduced action exists for a (locally) smooth Q, the gauging is called *strict*.

Thus, the gauging of (g, B) with minimal coupling is always strict.

However, e.g. the G/G WZW models correspond to a *non-strict* gauging.

The usefulness of this distinction lies in capturing cases where $\ker\rho \neq \emptyset$

$$F \xrightarrow{t} L \xrightarrow{\rho} TM\,, \qquad \text{with} \qquad \rho \circ t = 0$$

Then the gauging is strict whenever the action functional has a λ-symmetry:

$$\delta_\lambda A^a = t^a_M(X)\lambda^M\,, \qquad \lambda \in \Gamma(T^*\Sigma \otimes X^*F)\,.$$

This is automatic for minimal coupling, but not for the general (g, H) case.

Concluding remarks

Take-home messages from this talk

- ✓ Universal 2D Gauge Theory for general background fields $g(X)$ and $B(X)/H(X)$
- ✓ The gauging is controlled by 2 (curved) connections on an almost Lie algebroid
- ✓ The connections are fully determined for the case of H-twisted Dirac structures

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Other results

- ✿ A framework for non-Abelian T-duality without isometry and beyond group actions
- ✿ A framework for string theories in quotient spaces $M/\mathcal{F}$ by a general foliation $\mathcal{F}$

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Current and future work

- (BV) Quantization - CFT viewpoint? Does the procedure survive quantization?
- Applications to true string solutions, e.g. S^3 with H flux?
- Our procedure as solution generating technique?