Nonassociative Weyl star product

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Deformation quantization: BFFLS, '77

Let A be an algebra of functions on $\mathbb{R}^N$, e.g., $C^\infty(\mathbb{R}^N)$, $\mathrm{Poly}(\mathbb{R}^N)$. **Star product** is a formal deformation of the pointwise product on A in the direction of a given Poisson bivector field $P^{ij}(x)$.

1. A formal deformation,

$$f \cdot g \to f \star g = f \cdot g + \sum_{r=1}^{\infty} (i\alpha)^r C_r(f,g)\,.$$

2. The "Initial condition",

$$C_1(f,g) - C_1(g,f) = 2\{f,g\} = 2P^{ij}(x)\partial_i f\ \partial_j g\,.$$

3. The associativity condition, $(f \star g) \star h = f \star (g \star h)$.

The last condition
– allows to proceed to higher orders, $C_r(f,g)$, $r > 1$,
– requires Jacobi identity on P^{ij} for consistency.

Existence: Formality theorem by M. Kontsevich, '97

Nonassociative version

String theory: star products from D-branes [Schomerus, '99].
But: for a non-constant B-field this product is non-associative, since P^{jk} is not Poisson [Cornalba and Schiappa; Kreuzer et al, '01].

More recent: non-geometric fluxes in closed string theory [Blumenhagen and Plauschinn; Lust, '11].

Main problem: for a non-Poisson P^{jk} what can be used instead of the associativity condition to restrict the higher order terms in star products?

Proposals:
– use properties of particular configurations (like the R-fluxes).
– the Konstevich formula [Kreuzer et al, '04; Mylonas et al, '12].
– quasi-Hopf twist deformations [Mylonas et al, '14].

Weyl star products

Any star product defines a correspondence between functions and formal differential operators $f \to \hat{f}$,

$$(f \star g)(x) = \hat{f} \triangleright g(x) \;, \quad x^i \star g = \hat{x}^i \triangleright g(x) \;.$$

If $\star$ is nonassociative, $f \to \hat{f}$, is not an algebra representation,

$$\hat{f}\hat{g} \neq \widehat{f \star g}.$$

For Weyl $\star$ we suppose the Weyl symmetric ordering of operators $\hat{x}$,

$$(f \star g)(x) = \hat{f} \triangleright g(x) = W(f) \triangleright g(x) = \int \frac{d^N p}{(2\pi)^N} \tilde{f}(p)\, e^{-ip_m \hat{x}^m} \triangleright g(x).$$

def. Weyl star products satisfy

$$(x^{i_1} \dots x^{i_n}) \star f = \sum_{P_n} \frac{1}{n!} P_n(x^{i_1} \star (\dots \star (x^{i_n} \star f) \dots)),$$

e.g.,

$$(x^i x^j) \star f = \frac{1}{2} \left(x^i \star (x^j \star f) + x^j \star (x^i \star f) \right).$$

The formal differential operator $\hat{x}$ can be written as

$$\hat{x}^j = x^j + \sum_{n=1}^{\infty} \Gamma^{j(n)}(\alpha, x)\,(i\alpha\partial)^n \ ,$$

where (n) is a multiindex (that is automatically symmetrized).
Stability of the unity: $f \star 1 = 1 \star f = f$.
Hermiticity: $(g \star f)^* = f^* \star g^*$.
Weak hermiticity: if for all x^j, $(x^j \star f)^* = f^* \star x^j$.
We call $\star$ **triangular** if each $C_r(f, g)$ contains no more than r derivatives on f and no more than r on g, which implies

$$\Gamma^{j(n)}(\alpha, x) = \sum_{k=0}^{\infty} (i\alpha)^k \Gamma_k^{j(n)}(x) \ .$$

$\star$ is **strictly triangular** if $\Gamma^{i(n)}$ with $n = 1$ has no α-corrections, i.e.,

$$\Gamma^{jk}(\alpha, x) = \Gamma_0^{jk}(x).$$

Proposition

For any bivector field $P^{ij}(x)$ there is unique weakly Hermitian strictly triangular Weyl star product satisfying the stability of unity.

Proof: constructive, by an iterative procedure that allows to compute the product to any order. For the Weyl star product,

$$(f \star g)(x) = \int \frac{d^N p}{(2\pi)^N} \tilde{f}(p)\, e^{-ip_m \hat{x}^m} \triangleright g(x).$$

using the conditions of the stability of unity, $f \star 1 = 1 \star f = f$, and the weak hermiticity, $(x^j \star f)^* = f^* \star x^j$, one defines order by order in α the terms Γ of the expansions

$$\hat{x}^j = x^j + \sum_{n=1}^{\infty} (i\alpha)^n \sum_{k=0}^{n-1} \Gamma_k^{j(n-k)}(x)\, (\partial)^{n-k} \quad ,$$

and then construct the star product.

Details of the proof

A Weyl star product satisfies the stability of the unity iff the totally symmetric parts of all Γ's vanish [KVG and Vassilevich, '08]:

$$p_j p_{j_1} \dots p_{j_k} \Gamma^{j j_1 \dots j_k} = 0 \ .$$

The weak hermiticity, $(x^j \star f)^* = f^* \star x^j$, implies

$$e^{-i p_j \hat{x}^j} \triangleright x^i = \left(\hat{x}^i \triangleright e^{i p_j x^j} \right)^* .$$

Expanded:

$$e^{-i p_j x^j} \left[x^i + \alpha p_{j_1} \Gamma_0^{j_1 i} + \alpha^2 p_{j_1} p_{j_2} \Gamma_0^{j_1 j_2 i} + \frac{\alpha^2}{2} p_{j_1} p_{j_2} \Gamma_0^{j_1 l_1} \partial_{l_1} \Gamma_0^{j_2 i} \right]$$
$$= e^{-i p_j x^j} \left[x^i - \alpha p_{j_1} \Gamma_0^{i j_1} + \alpha^2 p_{j_1} p_{j_2} \Gamma_0^{i j_1 j_2} \right] + o(\alpha^2).$$

First order: $\Gamma_0^{j_1 i} + \Gamma_0^{i j_1} = 0$. From $C_1(f,g) - C_1(g,f) = 2\{f,g\}$, one obtaines $\Gamma_0^{ij}(x) = P^{ij}(x)$.

Details of the proof

Second order:

$$(\Gamma_0^{ij_1j_2} - \Gamma_0^{j_1j_2i})p_{j_1}p_{j_2} = \frac{1}{2}\Gamma_0^{j_1l_1}\partial_{l_1}\Gamma_0^{j_2i}p_{j_1}p_{j_2} \,.$$

Since, $\Gamma^{jj_1\ldots j_k}$ is symmetric in the last k indices by the construction and $\Gamma^{(jj_1\ldots j_k)} = 0$, due to the stability of unity, one has:

$$\Gamma_k^{(m)i}(p)^m = -\frac{1}{m}\Gamma_k^{i(m)}(p)^m \,.$$

In particular, $\Gamma_0^{j_1j_2i}p_{j_1}p_{j_2} = -1/2\Gamma_0^{ij_1j_2}p_{j_1}p_{j_2}$. Which gives,

$$\Gamma_0^{ijk} = \frac{1}{6}\left(P^{kl}\partial_l P^{ji} + P^{jl}\partial_l P^{ki}\right) .$$

In higher orders the equations

$$\left(\Gamma_k^{i(m)} - (-1)^{k+m}\Gamma_k^{(m)i}\right)(p)^m = (-1)^{k+m}F_k^{i(m)}(p)^m, \;\; k+m=n,$$

yield the iteration relation,

$$\Gamma_k^{i(m)}(p)^m = \frac{m}{1+(-1)^{m+k}m}\,F_k^{i(m)}(p)^m \,.$$

The star product

$$
\begin{aligned}
(f \star g)(x) &= f \cdot g + i\alpha P^{ij} \partial_i f \partial_j g \\
&- \frac{\alpha^2}{2} P^{ij} P^{kl} \partial_i \partial_k f \partial_j \partial_l g - \frac{\alpha^2}{3} P^{ij} \partial_j P^{kl} \left(\partial_i \partial_k f \partial_l g - \partial_k f \partial_i \partial_l g \right) \\
&- i\alpha^3 \left[\frac{1}{3} P^{nl} \partial_l P^{mk} \partial_n \partial_m P^{ij} \left(\partial_i f \partial_j \partial_k g - \partial_i g \partial_j \partial_k f \right) \right. \\
&+ \frac{1}{6} P^{nk} \partial_n P^{jm} \partial_m P^{il} \left(\partial_i \partial_j f \partial_k \partial_l g - \partial_i \partial_j g \partial_k \partial_l f \right) \\
&+ \frac{1}{3} P^{ln} \partial_l P^{jm} P^{ik} \left(\partial_i \partial_j f \partial_k \partial_n \partial_m g - \partial_i \partial_j g \partial_k \partial_n \partial_m f \right) \\
&+ \frac{1}{6} P^{jl} P^{im} P^{kn} \partial_i \partial_j \partial_k f \partial_l \partial_n \partial_m g \\
&\left. + \frac{1}{6} P^{nk} P^{ml} \partial_n \partial_m P^{ij} \left(\partial_i f \partial_j \partial_k \partial_l g - \partial_i g \partial_j \partial_k \partial_l f \right) \right] + \mathcal{O}\left(\alpha^4\right) .
\end{aligned}
$$

and so on.

- The star product is very likely Hermitian.
- For the weak Hermitean Weyl star product the *associator* of three coordinates is exactly equals to the *jacobiator:*

$$x^i \star (x^j \star x^k) - (x^i \star x^j) \star x^k = [x^i, x^j, x^k]_\star ,$$
$$[x^i, x^j, x^k]_\star = \frac{1}{6}\left\{[x^i, [x^j, x^k]_\star]_\star + \mathrm{cycl.}(ijk)\right\} \ .$$

- Strictly triangular star product does not necessarily become an associative if P^{ij} is a Poisson bivector, i.e., if classical jacobiator vanishes,

$$\Pi^{ijk} = \{x^i, x^j, x^k\} = \frac{1}{3}\left(P^{il}\partial_l P^{jk} + P^{kl}\partial_l P^{ij} + P^{jl}\partial_l P^{ki}\right) = 0 \, .$$

Properties

To get an associative $\star$ one has to allow corrections in α to Γ^{jk}, corresponding to Kontsevich diagrams with wheels [Dito, '15],

$$\Gamma^{jk} = P^{jk} + \alpha^2 P_2^{jk} + \mathcal{O}\left(\alpha^4\right),$$
$$P_2^{jk} = \frac{1}{6}\partial_m P^{nl}\partial_n P^{mi}\partial_l\partial_i P^{jk} - \frac{1}{3}P^{il}\partial_n\partial_i P^{jm}\partial_m\partial_l P^{kn}.$$

Physically, one may say that the higher order corrections to Γ^{jk}, are absorbed in renormalization of the bivector $P^{jk} \to P_r^{jk}$.
For the star product $\star_r$, corresponding to the renormalized field P_r^{jk}, one calculates

$$[x^a, x^b, x^c]_{\star_r} = -\alpha^2\Pi^{abc} + \alpha^4\left(\frac{1}{3}P^{kl}\partial_k\partial_m P^{ad}\partial_l\partial_d\Pi^{mbc}\right.$$
$$+\frac{1}{18}\partial_m P^{nl}\partial_n P^{mk}\partial_l\partial_k\Pi^{abc} - \frac{1}{3}\partial_n\partial_k P^{bm}\partial_m\partial_l P^{cn}\Pi^{kla}$$
$$\left.+\frac{1}{3}\partial_k P^{ml}\partial_d\partial_l P^{bc}\partial_m\Pi^{kda}\right) + \mathrm{cycl.}(abc) + \mathcal{O}\left(\alpha^5\right).$$

Discussion

- For any quasi-Poisson bracket $\{f, g\} = P^{ij}(x)\partial_i f \; \partial_j g$, the deformation quantization exists and is essentially unique if one requires stability of unity, (weak) Hermiticity and the Weyl condition.
- We propose an iterative procedure that allows one to compute the star product up to any desired order.
- Imposing the Jacobi identity on P^{ij} does not make our star product associative, one sill needs additional corrections terms in Γ^{ij}, corresponding to the renormalization of the field P^{ij}.

Discussion

Conjecture: the associativity condition in deformation quantization of Poisson manifolds can be substituted by the condition:

$$f \star (g \star h) - (f \star g) \star h = [f, g, h]_\star,$$

to restrict the higher order terms of the star product in nonassociative case.

Because:
– in the associative case the both sides are zeros;
– it is invariant under the gauge transformation:

$$f \circ g = \mathcal{D}^{-1}\left(\mathcal{D}f \star \mathcal{D}g\right);$$

– our star product and the Kontsevich one [Mylonas et al, '12] satisfy it.