CGAs and Invariant PDEs

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Based on:

N. Aizawa, Z. Kuznetsova and F. T., **JMP** 56, 031701 (2015), arXiv:1501.001201 & arXiv:1506.08488.

F.T., **JoP: CS** 597, (2015) 012071.

Symmetries of PDEs:

$$\Omega = \partial_t + a\partial_x^2 - aV(x),$$
$$\Omega\Psi(x,t) = 0.$$

The invariant condition

$$\Omega\delta\Psi(x,t) = 0,$$
$$\delta\Psi(x,t) = f(x,t)\Psi_t(x,t) + g(x,t)\Psi_x(x,t) + h(x,t)\Psi(x,t)$$

implies

$$\begin{aligned} f_x &= 0, \\ f_t - 2g_x + af_{xx} &= 0, \\ g_t + a(2aVf_x + 2h_x + g_{xx}) &= 0, \\ agV_x + h_t + 2a^2V_xf_x + aV(f_t + af_{xx}) + ah_{xx} &= 0. \end{aligned}$$

Maximal number of symmetry generators (non semi-simple) Schrödinger algebra:

i) $V(x) = x^0$ (**free** particle case),
ii) $V(x) = x^1$ (solved by **Airy** functions),
iii) $V(x) = x^2$ (**harmonic oscillator** case).

6-generator algebras $z_{\pm 1}, z_0, w_\pm, c$, with dimensions

$$[z_{\pm 1}] = \pm 1, \quad [w_\pm] = \pm\frac{1}{2}, \quad [z_0] = [c] = 0.$$
$$([t] = -1, \quad [x] = -\frac{1}{2})$$

$$\{z_0, z_{\pm 1}\} \in \mathfrak{sl}(2), \qquad \{w_\pm, c\} \in \mathfrak{h}(1),$$

$$\mathfrak{sch}(1) \;=\; \mathfrak{sl}(2) \ni\!\!\!+ \mathfrak{h}(1).$$

Linear Schrödinger Equation:

$$
\begin{aligned}
z_+ &= t^2\partial_t + (tx - \frac{1}{2}t^3)\partial_x + (\frac{1}{2}t - \frac{i}{8}t^4 + \frac{3i}{2}t^2x - \frac{i}{2}x^2),\\
z_0 &= t\partial_t + (\frac{1}{2}x - \frac{3}{2}t^2)\partial_x + \frac{3i}{2}tx - \frac{i}{4}t^3 + \frac{1}{4},\\
z_- &= \partial_t - t\partial_x + ix - \frac{i}{2}t^2,\\
w_+ &= t\partial_x - ix + \frac{i}{2}t^2,\\
w_- &= \partial_x + it,\\
c &= 1.
\end{aligned}
$$

$$
\begin{aligned}
[z_0, z_\pm] &= \pm z_\pm,\\
[z_+, z_-] &= -2z_0,\\
[z_0, w_\pm] &= \pm\tfrac{1}{2}w_\pm,\\
[z_\pm, w_\mp] &= \mp w_\pm,\\
[w_+, w_-] &= ic.
\end{aligned}
$$

Non-vanishing on-shell relations

$$
\begin{aligned}
[z_+, \Omega] &= -2t\Omega, \\
[z_0, \Omega] &= -\Omega.
\end{aligned}
$$

$\mathfrak{sl}(2)$ algebra produced by the $\Omega_0, \Omega_{\pm 1}$ invariant PDEs:

$$
\Omega_{-1} = \Omega, \quad [z_+, \Omega_{-1}] = \Omega_0, \quad [z_+, \Omega_0] = \Omega_{+1}, \quad ([z_+, \Omega_{+1}] = 0)
$$

$$
\Omega_{-1} = \Omega = iz_- + \frac{1}{2}w_-^2, \quad \Omega_0 = -2t\Omega_{-1}, \quad \Omega_{+1} = -2t^2\Omega_{-1}.
$$

$$
\begin{aligned}
[\Omega_0, \Omega_{\pm 1}] &= \mp 2i\Omega_{\pm 1}, \\
[\Omega_+, \Omega_-] &= 2i\Omega_0.
\end{aligned}
$$

The Hamiltonian is

$$
H = -\frac{1}{2}{\partial_x}^2 + x = -\frac{1}{2}{w_-}^2 + iw_+.
$$

Free case:

$$
\begin{aligned}
z_{+1} &= \partial_t, \\
z_0 &= t\partial_t + \frac{1}{2}x\partial_x + \frac{1}{4}, \\
z_{-1} &= t^2\partial_t + tx\partial_x - \frac{ix^2}{4} + \frac{1}{2}t, \\
w_+ &= \partial_x, \\
w_- &= t\partial_x - \frac{ix}{2}, \\
c &= 1.
\end{aligned}
$$

Harmonic oscillator:

$$
\begin{aligned}
z_{+1} &= e^{2it}\left(\partial_t + ix\partial_x + \frac{i}{2} - ix^2\right),\\
z_0 &= \partial_t,\\
z_{-1} &= e^{-2it}\left(\partial_t - ix\partial_x - \frac{i}{2} - ix^2\right),\\
w_+ &= e^{it}\left(\partial_x - x\right),\\
w_- &= e^{-it}\left(\partial_x + x\right),\\
c &= 1.
\end{aligned}
$$

1^{st} and 2^{nd} order differential operators:

$$\begin{array}{rcl} w_{+1} & = & \{w_+, w_+\}, \\ w_0 & = & \{w_+, w_-\}, \\ w_{-1} & = & \{w_-, w_-\}. \end{array}$$

Algebra-superalgebra duality:

• the non-simple Lie algebra $\mathfrak{esch}$ with 9 generators $\{z_{\pm 1}, z_0, w_\pm, w_{\pm 1}, w_0, c\}$ and

• the Lie superalgebra $\mathfrak{ssch} = S_0 \oplus S_1$, with 7 even $(z_{\pm 1}, z_0, w_{\pm 1}, w_0, c \in S_0)$ and 2 odd generators $(w_\pm \in S_1)$.

In $\mathfrak{ssch}$ we have the anti-commutators

$$\{w_+, w_+\} = w_{+1}, \quad \{w_+, w_-\} = w_0, \quad \{w_-, w_-\} = w_{-1}.$$

$$\begin{aligned} x^0: \quad \Omega &= z_{+1} + \frac{1}{2} a w_{+1}, \\ x^1: \quad \Omega &= z_{-1} + \frac{a}{2} w_{-1}, \\ x^2: \quad \Omega &= z_0 + \frac{a}{2} w_0. \end{aligned}$$

$V(x) = x^0$ example: all commutators with Ω are vanishing, apart

$$\begin{aligned} [z_0, \Omega] &= -z_{+1} - \frac{a}{2} w_{+1}, \\ [z_{-1}, \Omega] &= -2z_0 - a w_0. \end{aligned}$$

Representation-dependent formulas for the commutators above:

$$\begin{aligned} [z_0, \Omega] &= -\Omega, \\ [z_{-1}, \Omega] &= -2t\Omega. \end{aligned}$$

Representation-dependent commutators (on-shell condition):

$$[g, \Omega] = f_g \cdot \Omega.$$

$d = 1$ $\ell = \frac{1}{2} + \mathbb{N}_0$ Centrally Extended CGAs

$$\mathfrak{cga}_\ell \quad = \quad \mathfrak{sl}(2) \ni \mathfrak{h}(\ell + \tfrac{1}{2}).$$

Features:

• Spin ℓ representation of $\mathfrak{sl}(2)$.
• $\ell + \frac{1}{2}$ copies of Heisenberg algebras.
• $\mathbb{Z}_2$-grading.
• Schrödinger algebra recovered at $\ell = \frac{1}{2}$.

Generators:

$\mathfrak{sl}(2)$: $z_\pm, z_0$,
central charge: c ,
creation/annihilation operators: w_j with $j = -\ell, \ell + 1, \ldots, \ell$.

Inverse problem: diff. realizations are given. Are there invariant PDEs?

Differential realizations from

$$e^{\mathcal{G}_-} e^{\mathcal{G}_0} e^{\mathcal{G}_+} | l.w.r >$$

Invariant PDEs:

- Verma modules (difficult),

- on-shell condition (easy).

$d = 1$ $\ell = \frac{3}{2}$ - deformation of the free system:

$$
\begin{aligned}
\overline{z}_{+} &= \partial_\tau, \\
\overline{z}_{0} &= -2i\tau\partial_\tau - ix\partial_x - 3iy\partial_y - 2i, \\
\overline{z}_{-} &= -4\tau^2 - 4(\tau x - \frac{3}{\gamma}y)\partial_x - 12\tau y\partial_y - 8(\tau - ix^2), \\
\overline{w}_{+3} &= \partial_y, \\
\overline{w}_{+1} &= -2i\tau\partial_y + \frac{2i}{\gamma}\partial_x, \\
\overline{w}_{-1} &= -4\tau^2\partial_y + \frac{8}{\gamma}\tau\partial_x - \frac{8i}{\gamma}x, \\
\overline{w}_{-3} &= 8i\tau^3\partial_y - \frac{24i}{\gamma}\tau^2\partial_x - 48(\frac{1}{\gamma}\tau x + \frac{1}{\gamma^2}y), \\
\overline{c} &= 1.
\end{aligned}
$$

Algebra:

$$
\begin{aligned}
[\overline{z}_0, \overline{z}_\pm] &= \pm 2i\overline{z}_\pm,\\
[\overline{z}_+, \overline{z}_-] &= -4i\overline{z}_0,\\
[\overline{z}_\pm, \overline{w}_k] &= (k \mp 3)i\overline{w}_{k\pm 2},\\
[\overline{w}_{|k|}, \overline{w}_{-|k|}] &= (3-2k)\frac{16}{\gamma^2}c.
\end{aligned}
$$

$$
\begin{aligned}
\overline{\Omega}_{+1} &= i\partial_\tau - i\gamma\partial_y + \frac{1}{2}\partial_x^2 = i\overline{z}_+ - \overline{H}_+ = i\overline{z}_+ + \frac{\gamma^2}{16}(\{\overline{w}_{+3},\overline{w}_{-1}\} - \{\overline{w}_{+1},\overline{w}_{+1}\}),\\
\overline{\Omega}_0 &= -2i\tau\overline{\Omega}_{+1} = i\overline{z}_0 - \overline{H}_0 = i\overline{z}_0 + \frac{\gamma^2}{32}(\{\overline{w}_{+3},\overline{w}_{-3}\} - \{\overline{w}_{+1},\overline{w}_{-1}\}),\\
\overline{\Omega}_{-1} &= -4\tau^2\overline{\Omega}_{+1} = i\overline{z}_- - \overline{H}_- = i\overline{z}_- + \frac{\gamma^2}{16}(\{\overline{w}_{+1},\overline{w}_{-3}\} - \{\overline{w}_{-1},\overline{w}_{-1}\}).
\end{aligned}
$$

Non-vanishing (on-shell invariant) commutators involving the $\overline{\Omega}$'s:

$$
\begin{aligned}
[\overline{z}_0,\overline{\Omega}_{+1}] &= 2i\overline{\Omega}_{+1},\\
[\overline{z}_-,\overline{\Omega}_{+1}] &= 4i\overline{\Omega}_0 = 8\tau\overline{\Omega}_{+1},\\
[\overline{z}_+,\overline{\Omega}_0] &= -2i\overline{\Omega}_{+1} = \tau^{-1}\overline{\Omega}_0,\\
[\overline{z}_-,\overline{\Omega}_0] &= 2i\overline{\Omega}_{-1} = 4\tau\overline{\Omega}_0,\\
[\overline{z}_+,\overline{\Omega}_{-1}] &= -4i\overline{\Omega}_0 = 2\tau^{-1}\overline{\Omega}_{-1},\\
[\overline{z}_0,\overline{\Omega}_{-1}] &= -2i\overline{\Omega}_{-1}.
\end{aligned}
$$

$$
\begin{aligned}
[\overline{\Omega}_0,\overline{\Omega}_{\pm1}] &= \mp2\overline{\Omega}_{\pm1},\\
[\overline{\Omega}_{+1},\overline{\Omega}_{-1}] &= 4\overline{\Omega}_0.
\end{aligned}
$$

$d = 1$ $\ell = \frac{3}{2}$ - deformation of the "oscillator" system:

$$
\begin{aligned}
z_0 &= \partial_t,\\
z_+ &= e^{2it}(\partial_t + ix\partial_x + 3iy\partial_y + ix^2 + 2i),\\
z_- &= e^{-2it}(\partial_t - ix\partial_x - 3iy\partial_y + \frac{12}{\gamma}y\partial_x + 7ix^2 + \frac{12}{\gamma}xy - 2i),\\
w_{+3} &= e^{3it}\partial_y,\\
w_{+1} &= e^{it}(\partial_y + \frac{2i}{\gamma}\partial_x + \frac{2i}{\gamma}x),\\
w_{-1} &= e^{-it}(\partial_y + \frac{4i}{\gamma}\partial_x - \frac{4i}{\gamma}x),\\
w_{-3} &= e^{-3it}(\partial_y + \frac{6i}{\gamma}\partial_x - \frac{18i}{\gamma}x - \frac{48i}{\gamma^2}y),\\
c &= 1.
\end{aligned}
$$

$$
\begin{aligned}
\Omega_{+1} &= e^{2it}\Omega_0 = iz_+ - H_+ = iz_+ + \frac{\gamma^2}{16}\left(\{w_{+3}, w_{-1}\} - \{w_{+1}, w_{+1}\}\right), \\
\Omega_0 &= i\partial_t + \frac{1}{2}{\partial_x}^2 - \frac{1}{2}x^2 - 3y\partial_y - i\gamma x\partial_y - \frac{3}{2} = \\
&= iz_0 - H_0 = iz_0 + \frac{\gamma^2}{32}\left(\{w_{+3}, w_{-3}\} - \{w_{+1}, w_{-1}\}\right), \\
\Omega_{-1} &= e^{-2it}\Omega_0 = iz_- - H_- = iz_- + \frac{\gamma^2}{16}\left(\{w_{+1}, w_{-3}\} - \{w_{-1}, w_{-1}\}\right).
\end{aligned}
$$

$$
\begin{aligned}
[z_0, \Omega_{+1}] &= 2i\Omega_{+1}, \\
[z_-, \Omega_{+1}] &= 4i\Omega_0 = 4ie^{-2it}\Omega_{+1}, \\
[z_+, \Omega_0] &= -2i\Omega_{+1} = -2ie^{2it}\Omega_0, \\
[z_-, \Omega_0] &= 2i\Omega_{-1} = 2ie^{-2it}\Omega_0, \\
[z_+, \Omega_{-1}] &= -4i\Omega_0 = -4ie^{2it}\overline{\Omega}_{-1}, \\
[z_0, \Omega_{-1}] &= -2i\Omega_{-1}; \\
[\Omega_0, \Omega_{\pm 1}] &= \mp 2\Omega_{\pm 1}, \\
[\Omega_{+1}, \Omega_{-1}] &= 4\Omega_0.
\end{aligned}
$$

Connection of the two systems:

$g \mapsto \overline{g}$ via similarity transform. and $t \mapsto \tau = \frac{i}{2}e^{-2it}$ redefinition of time.

To the "oscillatorial" D-module rep $z_\pm = e^{\pm 2it}(\partial_t + X_\pm), \quad z_0 = \partial_t$
we apply the similarity transformation

$$
\begin{aligned}
g \mapsto \overline{g} &= e^S g e^{-S}, \qquad (e^S = e^{S_2} e^{S_1}),\\
S_1 &= tX_+,\\
S_2 &= \frac{1}{2}x^2
\end{aligned}
$$

Explanation: $z_+ \mapsto \hat{z}_+ = e^{S_1} z_+ e^{-S_1} = e^{2it}\partial_t = \partial_\tau$,

$$
\begin{aligned}
\Omega_{+1} \mapsto \widehat{\Omega}_{+1} &= e^{S_1}\Omega_{+1}e^{-S_1} = ie^{2it}\partial_t - \widehat{H}_{+1},\\
\widehat{H}_{+1} &= e^{2it}\left(iX_+ + e^{tX_+}H_0 e^{-tX_+}\right).
\end{aligned}
$$

Magic identity: $[X_+, H_0] = 2iK_+$, $[X_+, K_+] = -2iK_+ \Rightarrow$
$\Rightarrow iX_+ + H_0 + K_+ = 0 \Rightarrow \widehat{H}_{+1}$ **does not depend on time.**

The commutative diagram

$$\begin{array}{lccc} \textit{coupled } (\gamma \neq 0): & \mathbf{Free}_\gamma^{0,\pm1}(\tau) & \overset{\mathbf{S}}{\longleftrightarrow} & \mathbf{Osc}_\gamma^{0,\pm1}(t) \\ & {}^{\mathbf{r}}\downarrow & & \downarrow^{\mathbf{r}} \\ \textit{decoupled } (\gamma = 0): & \mathbf{Free}^{0,\pm1}(\tau) & \overset{\mathbf{S}}{\longleftrightarrow} & \mathbf{Osc}^{0,\pm1}(t) \end{array}$$

- left: equations from the "free" D-module rep.
- right: equations obtained from the "oscillator" D-module rep.

Each three invariant PDEs (deg $0, \pm1$) are mapped into each other.

Left: Schrödinger-type invariant PDE corresponds to *deg* 1 and possesses a continuous spectrum.

Right: Schrödinger-type invariant PDE corresponds to *deg* 0 and possesses a discrete spectrum.

Vertical arrows: $g \mapsto RgR^{-1}$, $R = e^{\alpha y \partial_y}$.

Therefore $\gamma \to e^{-\alpha}\gamma$. The $\alpha \to \infty$ limit is singular.

Four Schrödinger-type equations:

- deformed oscillator:

$$\Omega_0(\gamma)\Psi(t,x,y)=0 \quad \equiv \quad (i\partial_t+\frac{1}{2}\partial_x^2-\frac{1}{2}x^2-3y\partial_y-i\gamma x\partial_y-\frac{3}{2})\Psi(t,x,y);$$

- decoupled oscillator:

$$\Omega_0\Psi(t,x,y)=0 \quad \equiv \quad (i\partial_t+\frac{1}{2}\partial_x^2-\frac{1}{2}x^2-3y\partial_y-\frac{3}{2})\Psi(t,x,y);$$

- free equation:

$$\overline{\Omega}_1\Psi(\tau,x,y)=0 \quad \equiv \quad (i\partial_\tau+\frac{1}{2}\partial_x^2)\Psi(\tau,x,y)=0;$$

- deformed free equation:

$$\overline{\Omega}_1(\gamma)\Psi(\tau,x,y)=0 \quad \equiv \quad (i\partial_\tau+\frac{1}{2}\partial_x^2+i\gamma x\partial_y)\Psi(\tau,x,y)=0.$$

Fundamental domain: $\gamma \in]0, +\infty[$

- Left PDEs: hermitian.
- Right PDEs: non-hermitian.

All operators $K = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 + \omega y\partial_y - i\gamma x\partial_y + C$, $\forall C$ and $\forall \gamma \neq 0$ correspond to $\ell = \frac{3}{2}$ CGA invariant PDE if

$$\omega = \pm\frac{1}{3}, \pm 3$$

The $\omega \leftrightarrow -\omega$ change of sign explained by the hermitian conjugation.
The $\omega \leftrightarrow \frac{1}{\omega}$ transformation explained by $x \leftrightarrow y$ exchange.
Explicit check: to get $z_+ = e^{i\lambda t}\partial_t + \ldots$, the following necessary (and sufficient) condition has to be satisfied:

$$\begin{aligned} \lambda(\omega^2 + 1 - \frac{5}{2}\lambda^2) &= 0, \\ -3\lambda^2 + 3\lambda^4 + 2\lambda\omega + 4\lambda^3\omega - \lambda^2\omega^2 - 2\lambda\omega^3 &= 0. \end{aligned}$$

All three equations in $1+1$ invariant under the Schrödinger algebra are recovered as contractions of the $\ell = \frac{3}{2}$ invariant PDEs.

Symmetries of two decoupled oscillators

Without loss of generality, $\omega \geq 1$, for

$$\Omega \;=\; i\partial_t + \frac{1}{2}\partial_x^2 - \frac{1}{2}x^2 + \omega y\partial_y .$$

At generic ω, nine invariant operators can be encountered at degree $0, \pm\frac{1}{2}, \pm\frac{\omega}{2}, \pm 1$ ($\overline{d}$ is the degree operator):

$$\begin{aligned}
\overline{z}_\pm &= e^{\pm 2it}(\partial_t \pm ix\partial_x + i\omega y\partial_y + ix^2 \pm \tfrac{i}{2}),\\
\overline{z}_0 &= \partial_t + i\omega y\partial_y,\\
\overline{d} &= -\frac{i}{2}\partial_t,\\
\overline{c} &= 1,\\
\overline{w}_\omega &= e^{i\omega t}\partial_y,\\
\overline{w}_1 &= e^{it}(\partial_x + x),\\
\overline{w}_{-1} &= e^{-it}(\partial_x - x),\\
\overline{w}_{-\omega} &= e^{-i\omega t}y.
\end{aligned}$$

The 9-generator algebra closes the $\mathfrak{u}(1)\ni(\mathfrak{sch}(1)\oplus\mathfrak{h}(1))$ algebra.

Enhanced symmetry at the critical values $\omega = 1$ and $\omega = 3$.

- $\omega = 3$: three extra generators $\bar{r}_{-j}$, $j = 1, 2, 3$, of degree $-j$,

$$\begin{aligned} \bar{r}_{-1} &= e^{-2it} y(\partial_x + x), \\ \bar{r}_{-2} &= e^{-4it} y(\partial_x - x), \\ \bar{r}_{-3} &= e^{-6it} y^2. \end{aligned}$$

- $\omega = 1$: three extra generators at degree 0 and -1:

$$\begin{aligned} q_1 &= y(\partial_x + x), \\ q_2 &= e^{-2it} y^2, \\ q_3 &= e^{-2it} y(\partial_x - x). \end{aligned}$$

In both cases, we obtain a (different) 12-generator closed symmetry algebra.

The contraction algebra

In the $\gamma \to 0$ limit, a contraction algebra is recovered by rescaling the generators ($g \mapsto \widetilde{g} = \gamma^s g$), where

$$\begin{array}{rcl} s = 0 & : & z_0, z_+, w_3, c, \\ s = 1 & : & z_-, w_1, w_{-1}, \\ s = 2 & : & w_{-3}. \end{array}$$

$$\begin{array}{rcl}
\widetilde{z}_+ & = & e^S \overline{z}_+ e^{-S} = e^{2it}(\partial_t + ix\partial_x + 3iy\partial_y + ix^2 + 2i), \\
\widetilde{z}_0 & = & e^S(az_0 + bd)e^{-S} = \partial_t, \\
\widetilde{z}_- & = & e^S(12i\overline{r}_{-1})e^{-S} = 12ie^{-2it}y(\partial_x + x), \\
\widetilde{w}_{+3} & = & e^S \overline{w}_{+3} e^{-S} = e^{3it}\partial_y, \\
\widetilde{w}_{+1} & = & e^S(-2\overline{w}_{+1})e^{-S} = -2e^{it}y(\partial_x + x), \\
\widetilde{w}_{-1} & = & e^S(-4\overline{w}_{-1})e^{-S} = -4e^{-it}y(\partial_x - x), \\
\widetilde{w}_{-3} & = & e^S(48\overline{w}_{-3})e^{-S} = 48e^{-3it}y, \\
\widetilde{c} & = & e^S \overline{c} e^{-S} = 1.
\end{array}$$

The contraction algebra is $\mathfrak{e}(2) \Supset \mathfrak{h}(2)$.

Cryptohermiticity

Two types of operators, same CCR, but different conjugation properties

$$K(\overline{\gamma}) = a^\dagger a + 3b^\dagger b + \frac{1}{2} + \overline{\gamma}(a+a^\dagger)b.$$

$$[a, a^\dagger] = [b, b^\dagger] = 1 \quad .$$

The operator $K(\overline{\gamma})$ acts on the Hilbert space $\mathcal{L}^2(\mathbb{R}^2)$, spanned by the (unnormalized) states $|n, m >= (a^\dagger)^n (b^\dagger)^m |vac>$, where $|vac> \equiv |0, 0>$ is the Fock vacuum ($a|vac>= b|vac>= 0$).

Excitation mode creation $[K(\overline{\gamma}), A_\lambda] = \lambda A_\lambda$.
For any $\overline{\gamma} \neq 0$, $\lambda = \pm 3, \pm\frac{1}{3}$:

$$\begin{aligned}
A_{-3} &= b, \\
A_{-1} &= a + \frac{1}{2}\overline{\gamma} b, \\
A_{+1} &= a^\dagger - \frac{1}{4}\overline{\gamma} b, \\
A_{+3} &= b^\dagger - \frac{1}{2}\overline{\gamma} a^\dagger - \frac{1}{4}\overline{\gamma} a + \frac{1}{24}\overline{\gamma}^2 b.
\end{aligned}$$

Non-vanishing commutators

$$[A_{-i}, A_j] = \delta_{ij}, \qquad (i,j = 1,3).$$

The non-hermitian operator $K(\overline{\gamma})$ commutes with the “non-hermitian analog of the Number operator”, $N(\overline{\gamma})$.

$$\begin{aligned} K(\overline{\gamma}) &= 3A_3A_{-3} + A_1A_{-1} + \frac{1}{2}, \\ N(\overline{\gamma}) &= A_3A_{-3} + A_1A_{-1}, \\ [K(\overline{\gamma}), N(\overline{\gamma})] &= 0. \end{aligned}$$

The Fock vacuum $|vac>$ satisfies

$$a|vac>= b|vac>= 0 \quad , \quad A_{-1}|vac>= A_{-3}|vac>= 0.$$

The Hilbert space $\mathcal{L}^2(\mathbb{R}^2)$ can be spanned by both sets of (unnormalized) states,

$$\begin{aligned} |n,m> &= (a^\dagger)^n(b^\dagger)^m|vac>, \\ |\overline{n},\overline{m}> &= A_1^n A_3^m|vac>, \end{aligned}$$

so that $|vac>= |0,0>= |\overline{0},\overline{0}>$.

The spectrum of $K(\overline{\gamma})$, $N(\overline{\gamma})$ is real, discrete and bounded. It coincides with the spectrum of the Hamiltonian and Number operator of the decoupled harmonic oscillators.

$|\overline{n},\overline{m}>$ is an eigenvector for $K(\overline{\gamma})$, $N(\overline{\gamma})$ The respective eigenvalues are $n+3m+\frac{1}{2}$ and $n+m$.

$$
\begin{aligned}
(\frac{1}{2},0): &\qquad |\overline{0},\overline{0}>=|0,0>=|vac>,\\
(\frac{3}{2},1): &\qquad |\overline{1},\overline{0}>=|1,0>,\\
(\frac{5}{2},2): &\qquad |\overline{2},\overline{0}>=|2,0>,\\
(\frac{7}{2},1): &\qquad |\overline{0},\overline{1}>=|0,1>-\frac{1}{2}\overline{\gamma}|1,0>,\\
(\frac{7}{2},3): &\qquad |\overline{3},\overline{0}>=|3,0>,\\
(\frac{9}{2},2): &\qquad |\overline{1},\overline{1}>=|1,1>-\frac{1}{2}\overline{\gamma}|2,0>-\frac{1}{4}\overline{\gamma}|0,0>,\\
(\frac{9}{2},4): &\qquad |\overline{4},\overline{0}>=|4,0>\ .
\end{aligned}
$$

Since the operators are non-hermitian, their eigenvectors are non-orthogonal.

Physical consequence. Let us suppose we prepare a system in a given common eigenvector of $K(\overline{\gamma})$, $N(\overline{\gamma})$, let's say the state $|\overline{1}, \overline{1} >$. We can compute for instance compute the probability that in a measurement the state can be found in the vacuum state. A simple quantum mechanical computation gives for this probability $p = |_N < \overline{1}, \overline{1} | \overline{0}, \overline{0} >_N |^2$ ($|\overline{0}, \overline{0} >_N$, $|\overline{1}, \overline{1} >_N$ are the normalized states). We obtain

$$p \;=\; \frac{\overline{\gamma}^2}{16 + 9\overline{\gamma}^2}.$$

This probability is restricted in the range $0 \leq p < \frac{1}{9} < 1$. The parameter $\overline{\gamma}^2$ has measurable consequence.

Any $\ell = \frac{1}{2} + \mathbb{N}_0$:

$$i\partial_\tau \Psi(\tau, \mathbf{x}_j) \quad = \quad \mathbf{H}^{(\ell)}\Psi(\tau, \mathbf{x}_j),$$

for the ℓ-oscillator

$$\mathbf{H}^{(\ell)} \quad = \quad -\frac{1}{2m}\partial^2_{\mathbf{x}_1} + \frac{m}{2}{\mathbf{x}_1}^2 + \sum_{j=1}^{\ell-\frac{1}{2}}\left((2j+1)\mathbf{x}_{j+1}\partial_{\mathbf{x}_{j+1}} - i\gamma_j\mathbf{x}_j\partial_{\mathbf{x}_{j+1}}\right) + C$$

Spectrum:

$$E_{\vec{n}} \quad = \quad \sum_{j=1}^{\ell+\frac{1}{2}}\omega_j n_j + \omega_0, \quad \text{with} \quad \omega_j = (2j-1).$$

Energy modes: $1, 3, 5, 7, \ldots$.
For $\ell = \frac{5}{2}$: $\pm 1, \pm 3, \pm 5$; sign flipping for generic ℓ? Pais-Uhlenbeck Hamiltonian: $\omega_1, -\omega_2, \omega_3, -\omega_4, \ldots$ ($\omega_j \in \mathbb{R}^+, \quad \omega_{j+1} > \omega_j$).

Spectrum generating off-shell invariant algebra:

$B(0,n) = osp(1|2n)$ superalgebra in terms of n bosonic generators,

$$n = \ell + \frac{1}{2}.$$

- First-order differential operators w_j

(n copies of creation/annihilation operators).

- Second-order differential operators $\{w_i, w_j\} = w_{i,j}$

closing the $sp(2\ell+1)$ algebra, while $w_{i,j}, w_j$ close $osp(1|2\ell+1)$.

In the $\ell \to \infty$ limit we obtain $sp(\infty)$.

The spectrum is recovered from $osp(1|2n)$ l.w.r. .

Conclusions

Boring extensions: centrally extended CGAs $\ell \in \frac{1}{2} + \mathbb{N}_0$, $d > 1$: several copies of the $d = 1$ systems at given ℓ.

New Features: the exceptional centrally extended CGAs at $d = 2$ and $\ell \in \mathbb{N}_0$ ($\ell = 1, 2, 3, \ldots$).

Future investigations:

- connection with higher spin theory,
- non-linear Equations invariant under CGAs,
- Virasoro-Galilei algebra (BMS, asymptotic symmetries and soft theorems),
- ...

Thanks for the attention!

Announcement: GROUP31

The 31st International Colloquium on Group Theoretical Methods in Physics

will be held at **CBPF**, Urca, **Rio de Janeiro, Brazil**
Monday 20th June - Friday 24th June, 2016

The ICGTMP series webpage:
http://icgtmp.blogs.uva.es/conferences/

The Colloquium is traditionally dedicated to the application of symmetry and group theoretical methods in physics, chemistry and mathematics, and to the development of mathematical tools and theories for progress in group theory and symmetries.

You are all invited to Rio next year!