The phase diagram of the scalar field theory on the fuzzy sphere and the multi-trace matrix models

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Conclusions

- We will show that the triple point of the ϕ^4 theory on the fuzzy sphere is in the region of parameter space where the kinetic term can not be treated perturbatively.
- We will locate the triple point in a non-perturbative, but not complete, approximation of the kinetic term. This location will agree qualitatively, but not quantitatively with the available numerical results.

The scalar field theory on a fuzzy space is a multi-trace matrix model.

The scalar field theory on **a fuzzy space** is a multi-trace matrix model.

Fuzzy sphere Madore '92

- The fuzzy sphere S^2_F is a space, which has the algebra of functions generated by
$$x_i x_i = \rho^2 \quad , \quad x_i x_j - x_j x_i = i\theta\varepsilon_{ijk}x_k$$
- This can be realized as a $N = 2j+1$ dimensional representation of the $SU(2)$

$$x_i = \frac{2r}{\sqrt{N^2-1}}L_i \quad , \quad \theta = \frac{2r}{\sqrt{N^2-1}} \quad , \quad \rho^2 = \frac{4r^2}{N^2-1}j(j+1) = r^2$$

- The coordinates x_i still carry an action of $SU(2)$ and thus the space still has the symmetry of the sphere.
- Limit of large N reproduces the original sphere S^2.
- x_i are $N \times N$ matrices and functions on S^2_F are combinations of products $\rightarrow$ a herminitian matrix M

The scalar field theory on a fuzzy space is a multi-trace matrix model.

Scalar field on S^2_F

- Scalar field is a function on S^2_F, i.e. an $N \times N$ hermitian matrix.
- Derivatives become commutators with generators L_i, integrals become traces and we can write an Euclidean field theory action

$$S(M) = -\frac{4\pi R^2}{N}\mathrm{Tr}\left(\frac{1}{2R^2}[L_i, M][L_i, M] + \frac{1}{2}rM^2 + V(M)\right) =$$
$$= \mathrm{Tr}\left(\frac{1}{2}M[L_i,[L_i,M]] + \frac{1}{2}rM^2 + V(M)\right)$$

- The theory is given by functional correlation functions

$$\langle F\rangle = \frac{\int dM\, F(M)e^{-S(M)}}{\int dM\, e^{-S(M)}}$$

Balachandran, Kürkçüoğlu, Vaidya '05; Szabo '03

UV/IR mixing

- The trademark property of the noncommutative field theories is the UV/IR mixing, which arises as a consequence of the non-locality of the theory. Minwalla, Van Raamsdonk, Seiberg '00; Chu, Madore, Steinacker '01
- The commutative limit of such noncommutative theory is (very) different than the commutative theory we started with.
- In our setting the UV/IR mixing will result into an extra phase in the phase diagram, not present in the commutative case.

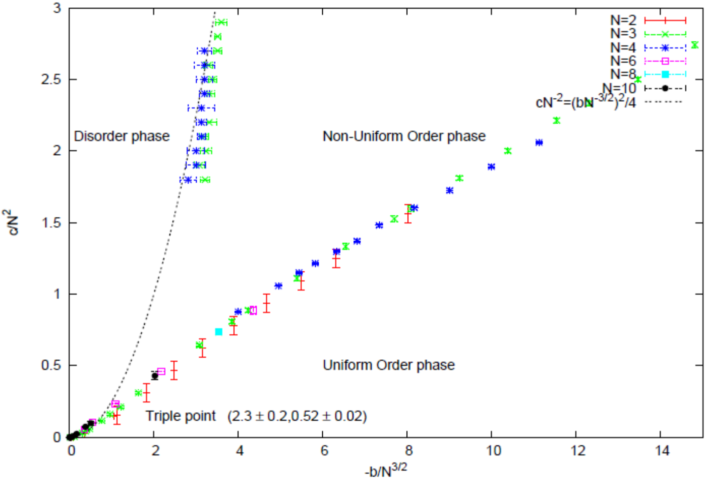
UV/IR mixing

- The commutative field theory has two phases in the phase diagram, disorder and uniform order phases.
  Glimm, Jaffe '74; Glimm, Jaffe, Spencer '75
- Noncommutative theories have a third phase, a striped or non-uniform order phase.
- This has been established computationally Gubser, Sondhi '01 and numerically for fuzzy sphere Martin '04; García Flores, Martin, O'Connor '06, '09; Panero '06, '07; Ydri '14.

Numerical phase diagram

For the fuzzy sphere, the following is numerically obtained phase diagram

García Flores, Martin, O'Connor '09



Numerical phase diagram

Numerical results agree on the critical value of the coupling

$$g_c \approx (0.125, 0.15)$$

The scalar field theory on a fuzzy space is a multi-trace **matrix model.**

(Usual) Hermitian matrix model

- The random variable is $N \times N$ hermitian matrix M and the mean value of matrix function $f(M)$ is

$$\langle f \rangle = \frac{1}{Z} \int dM \; e^{-N^2 S(M)} f(M)$$

where

$$dM = \prod_{i \leq j} d\mathrm{Re} M_{ij} \prod_{i < j} d\mathrm{Im} M_{ij}$$

Hermitian matrix model

- Fuzzy field theory = matrix model with

$$S(M) = \frac{1}{2}\mathrm{Tr}\left(M[L_i,[L_i,M]]\right) + \frac{1}{2} r \mathrm{Tr}\left(M^2\right) + g\mathrm{Tr}\left(M^4\right)$$

- What kind of matrix model is this? We diagonalize $M = U\Lambda U^\dagger$ for some $U \in SU(N)$ and $\Lambda = diag(\lambda_1, \ldots, \lambda_N)$, the integration measure becomes

$$dM = dU \left(\prod_{i=1}^{N} d\lambda_i\right) \times \prod_{i<j} (\lambda_i - \lambda_j)^2$$

and we are to compute integrals like

$$\langle f \rangle \sim \int \left(\prod_{i=1}^{N} d\lambda_i\right) f(\lambda_i)\, e^{-N^2\left[\frac{1}{2} r \sum \lambda_i^2 + g \sum \lambda_i^4 - \frac{2}{N^2} \sum_{i<j} \log|\lambda_i - \lambda_j|\right]}$$

$$\times \int dU\, e^{-N^2 \frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger [L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

Hermitian matrix model

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- What kind of matrix model is this? We diagonalize $M = U\Lambda U^\dagger$ for some $U \in SU(N)$ and $\Lambda = diag(\lambda_1, \ldots, \lambda_N)$, the integration measure becomes

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$$\langle f\rangle \sim \int \left(\prod_{i=1}^{N} d\lambda_i\right) f(\lambda_i)\, e^{-N^2\left[S_{eff}(\lambda_i) + \frac{1}{2}r\frac{1}{N}\sum\lambda_i^2 + g\frac{1}{N}\sum\lambda_i^4 - \frac{2}{N^2}\sum_{i<j}\log|\lambda_i - \lambda_j|\right]}$$

$$e^{-N^2 S_{eff}(\lambda_i)} = \int dU\, e^{-N^2\frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

Steinacker '05

The scalar field theory on a fuzzy space is a **multi-trace** matrix model.

Multitraces

- Perturbative calculation of the integral show that the S_{eff} contains products of traces of M. O'Connor, Sämann '07; Sämann '10
- The most recent result is Sämann '15

$$S_{eff}(M) = \frac{1}{2}\left[\varepsilon\frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2\frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4\frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] - \\ - \varepsilon^4\frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2\right]^2 - \\ - \varepsilon^3\frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2$$

where

$$c_n = \frac{1}{N}\mathrm{Tr}\left(M^n\right)$$

and

$$e^{-N^2 S_{eff}(\lambda_i)} = \int dU\, e^{-N^2\varepsilon\frac{1}{2}\mathrm{Tr}\left(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]]\right)}$$

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where

$$c_n = \frac{1}{N}\mathrm{Tr}\left(M^n\right)$$

- The model we wish to study is

$$S(M) = S_{eff}(M) + \frac{1}{2}r\mathrm{Tr}\left(M^2\right) + g\mathrm{Tr}\left(M^4\right)$$

Some basics of matrix models

Some basics of matrix models

- **In the large-N (commutative) limit** the eigenvalue measure

$$e^{-N^2[\dots]}$$

localizes on the extremal configuration $\tilde{\lambda}_i$

$$\left.\frac{\partial S}{\partial \lambda_i}\right|_{\tilde{\lambda}} = 0$$

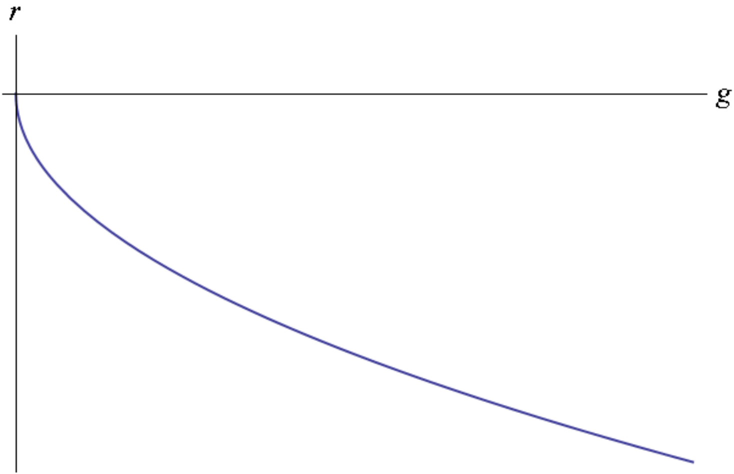
- The problem is equivalent to finding an equilibrium of N particles in an external potential

$$\frac{1}{2} r x^2 + g x^4$$

which logarithmically repel each other.
Brezin, Itzykson, Parisi, Zuber '78

Some basic matrix model techniques

$$r = -4\sqrt{g}$$



Some basic matrix model techniques

- Multi-traces introduce further selfinteraction.
- For the simplest multi-trace action term

$$c_2^2 + \frac{1}{2} r c_2 + g c_4$$

the equation determining $\tilde{\lambda}$ changes from

$$r\tilde{\lambda}_i + 4g\tilde{\lambda}_i^3 = \frac{2}{N} \sum_{j \neq i} \frac{1}{|\tilde{\lambda}_i - \tilde{\lambda}_j|}$$

to

$$2c_2\tilde{\lambda}_i + r\tilde{\lambda}_i + 4g\tilde{\lambda}_i = \frac{2}{N} \sum_{j \neq i} \frac{1}{|\tilde{\lambda}_i - \tilde{\lambda}_j|}$$

$$\left[r + 2c_2\right]\tilde{\lambda}_i + 4g\tilde{\lambda}_i = \frac{2}{N} \sum_{j \neq i} \frac{1}{|\tilde{\lambda}_i - \tilde{\lambda}_j|}$$

Some basic matrix model techniques

- This looks like a single trace model with a changed $r \to r_{reff}$

$$S(M) = \frac{1}{2} r_{reff} \mathrm{Tr}\left(M^2\right) + g\mathrm{Tr}\left(M^4\right)$$

and is solved by imposing self consistency conditions

$$c_2 = \frac{1}{N} \sum_i \tilde{\lambda}_i^2(c_2)$$

Das, Dhar, Sengupta, Wadia '90

Perturbative considerations

Perturbative effective action

- The model

$$S_{eff}(M) = \frac{1}{2}\left[\varepsilon\frac{1}{2}\left(c_2 - c_1^2\right) - \varepsilon^2\frac{1}{24}\left(c_2 - c_1^2\right)^2 + \varepsilon^4\frac{1}{2880}\left(c_2 - c_1^2\right)^4\right] -$$
$$- \varepsilon^4\frac{1}{3456}\left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4\right) - 2\left(c_2 - c_1^2\right)^2\right]^2 -$$
$$- \varepsilon^3\frac{1}{432}\left[c_3 - 3c_1c_2 + 2c_1^3\right]^2$$

  is very difficult to solve. The symmetric regime simplifies at the phase transition, but the equations are still quite complicated.

- One can try to solve the model pertubratively.

Perturbative effective action

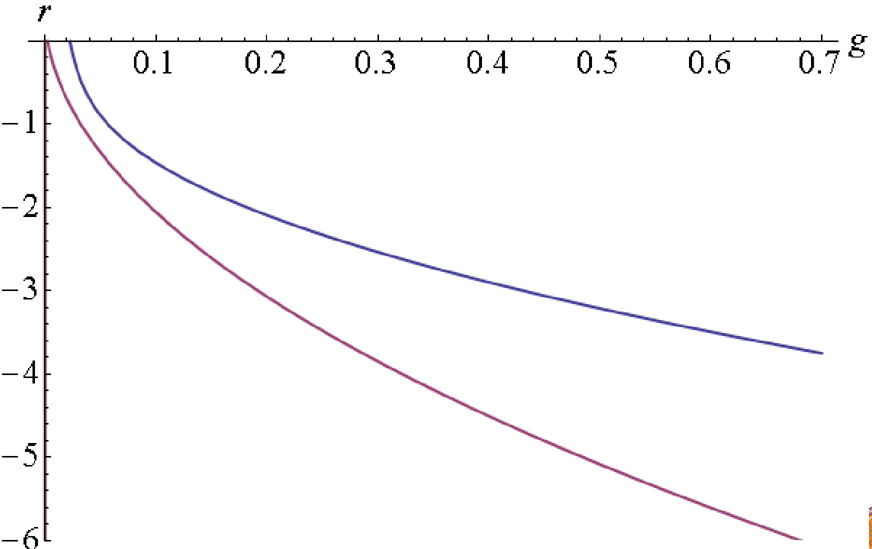
- The boundary of existence of a symmetric one cut solution turns out to be JT '14,'15

$$r = -4\sqrt{g} - \varepsilon \frac{1}{2} + \varepsilon^2 \frac{1}{12\sqrt{g}} + \varepsilon^4 \frac{7}{5760 g^{3/2}} + \varepsilon^6 \frac{29}{1935360 g^{5/2}}$$

- The boundary of existence of an asymmetric single cut solution is

$$r = -2\sqrt{15}\sqrt{g} + \varepsilon \frac{2}{5} - \varepsilon^2 \frac{19}{18000\sqrt{15}\sqrt{g}} + \varepsilon^3 \frac{29}{1125000 g} - \varepsilon^4 \frac{7886183}{4374000000000\sqrt{15} g^{3/2}}$$

Perturbative phase diagram



(Some) Non-perturbative considerations

Effective action

- For the free theory $g = 0$ the kinetic term just rescales the eigenvalues $\tilde{\lambda}$.
  Steinacker '05
- There is a unique parameter independent effective action that reconstructs this rescaling. Polychronakos '13

$$S_{eff} = \frac{1}{2} F(c_2) + \mathcal{R} = \frac{1}{2} \log \left(\frac{c_2}{1 - e^{-c_2}} \right) + \mathcal{R}$$

- Recall the perturbative action

$$\begin{aligned} S_{eff}(M) =& \frac{1}{2} \left[\varepsilon \frac{1}{2} \left(c_2 - c_1^2 \right) - \varepsilon^2 \frac{1}{24} \left(c_2 - c_1^2 \right)^2 + \varepsilon^4 \frac{1}{2880} \left(c_2 - c_1^2 \right)^4 \right] - \\ & - \varepsilon^4 \frac{1}{3456} \left[\left(c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4 \right) - 2 \left(c_2 - c_1^2 \right)^2 \right]^2 - \\ & - \varepsilon^3 \frac{1}{432} \left[c_3 - 3c_1c_2 + 2c_1^3 \right]^2 \end{aligned}$$

these are exactly the first terms of the small c_2 expansion with $c_2 \to c_2 - c_1^2$.

Effective action

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- There is a unique parameter independent effective action that reconstructs this rescaling. Polychronakos '13

$$S_{eff} = \frac{1}{2}F(c_2) + \mathcal{R} = \frac{1}{2}\log\left(\frac{c_2}{1-e^{-c_2}}\right) + \mathcal{R}$$

- Introducing the asymmetry $c_2 \to c_2 - c_1^2$ and interaction we obtain a matrix model

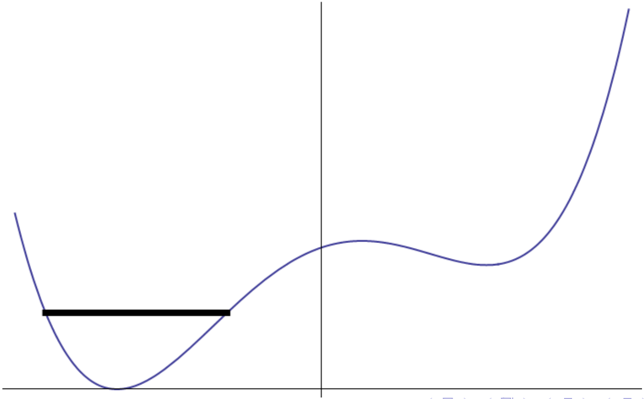
$$S(M) = \frac{1}{2}F(c_2 - c_1^2) + \frac{1}{2}\mathrm{Tr}\left(M^2\right) + g\mathrm{Tr}\left(M^4\right)$$

Effective action

- Such F corresponds to an attractive interaction.
- Effectively particles in an asymmetric potential

$$S(M) = a_{eff}\mathrm{Tr}\,(M) + \frac{1}{2}r_{eff}\mathrm{Tr}\,\left(M^2\right) + g\mathrm{Tr}\,\left(M^4\right)$$

where asymmetric one cut is possible



Symmetric regime

- Transition line for the symmetric regime $c_1 = 0$ can be computed analytically Polychronakos '13

$$r = -5\sqrt{g} - \frac{1}{1 - e^{1/\sqrt{g}}}$$

Note, that at the phase transition $c_2 = 1/\sqrt{g}$, which explains the problems with the perturbative results.

Asymmetric regime

- Asymmetric regime is more complicated and can not be solved analytically, but can be treated numerically. JT '15

Interplay of symmetric and asymmetric regime

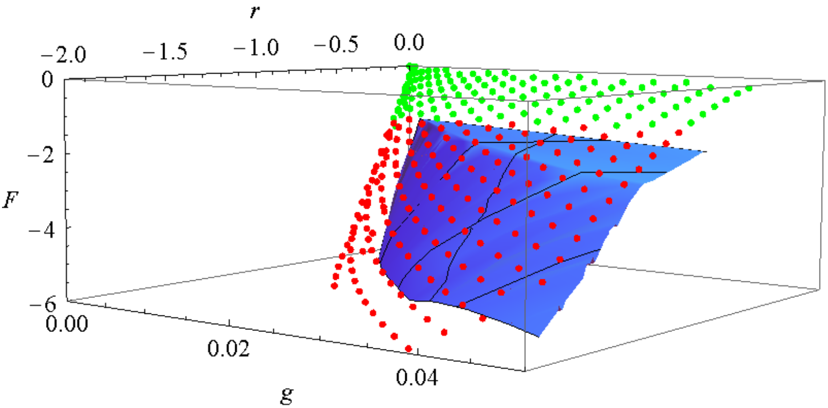
- More than one solution in a region of the parameter space. The solution with a lower free energy

$$\mathcal{F} = S(\tilde{\lambda}) - \frac{2}{N^2} \sum_{i<j} \log |\tilde{\lambda}_i - \tilde{\lambda}_j|$$

is realized.

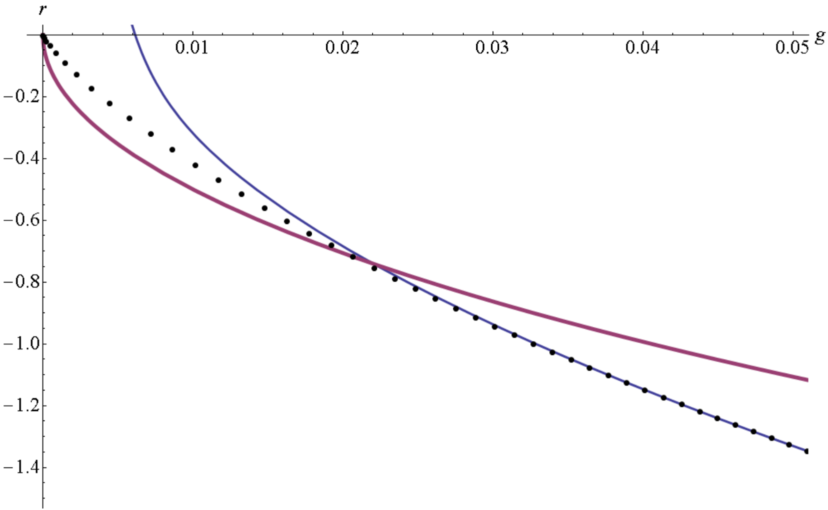
Interplay of symmetric and asymmetric regime

Comparison of the symmetric regime and the asymmetric regime free energy. The free energy of the asymmetric single cut is lower everywhere it exists.



Interplay of symmetric and asymmetric regime

The phase diagram



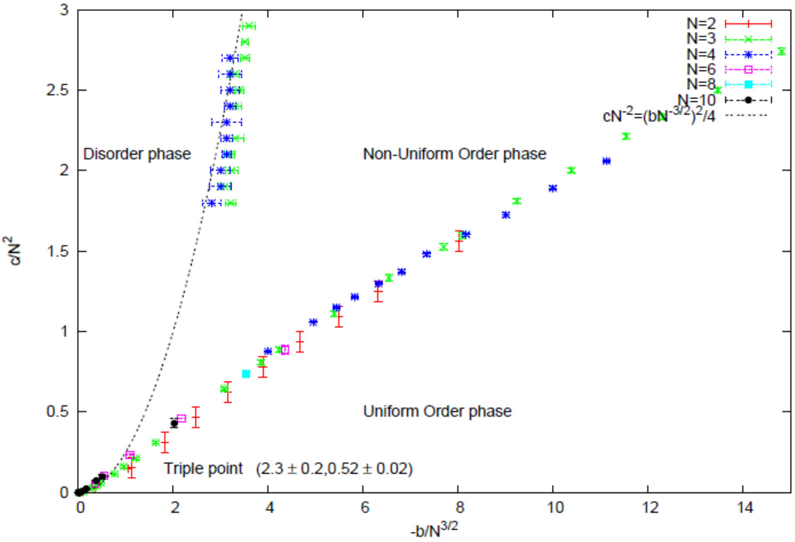
Interplay of symmetric and asymmetric regime

- The triple point is located at approximately

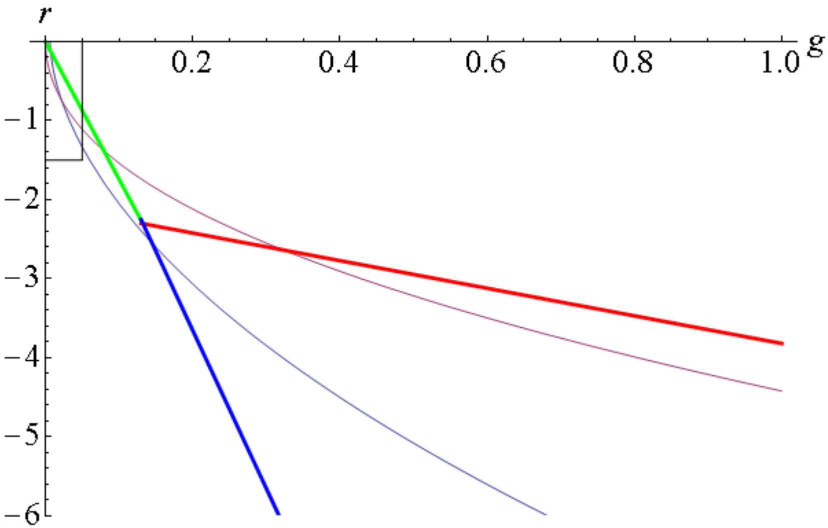
$$g_c \approx 0.02$$

which differs from the numerical value by a factor of 7.
- Terms depending on higher multitraces will deform the phase transition lines.
- Numerical data linearly extrapolate from little too far out.

Interplay of symmetric and asymmetric regime



Interplay of symmetric and asymmetric regime



Outlook

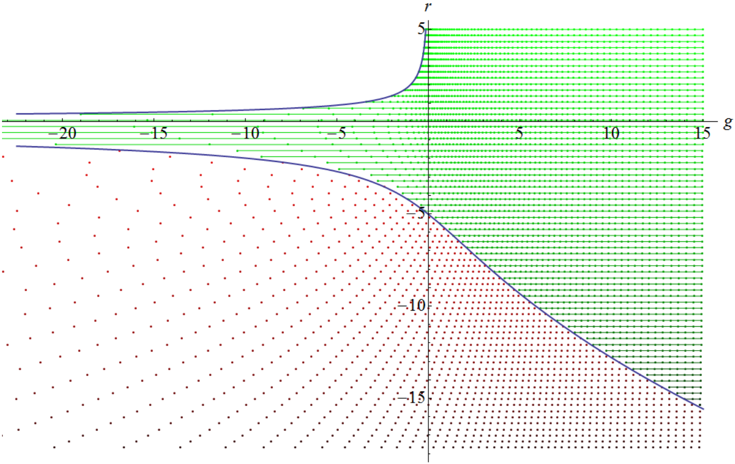
- Find a (more) complete treatment of the kinetic term effective action, as this could explain the difference from the numerical results.
- Generalize these ideas to non-commutative plane and other non-commutative spaces where numerical data is available.
- Study phase structure of theories free of the UV/IR mixing, as these should not have the third phase in the phase diagram.

Thank you for your attention!

If time permits

Multitrace matrix models with have very interesting phase structure on their own

$$S(M) = \frac{1}{2} r \mathrm{Tr}\left(M^2\right) + g \mathrm{Tr}\left(M^4\right) + \left(\mathrm{Tr}\left(M^4\right)\right)^2$$



If time permits

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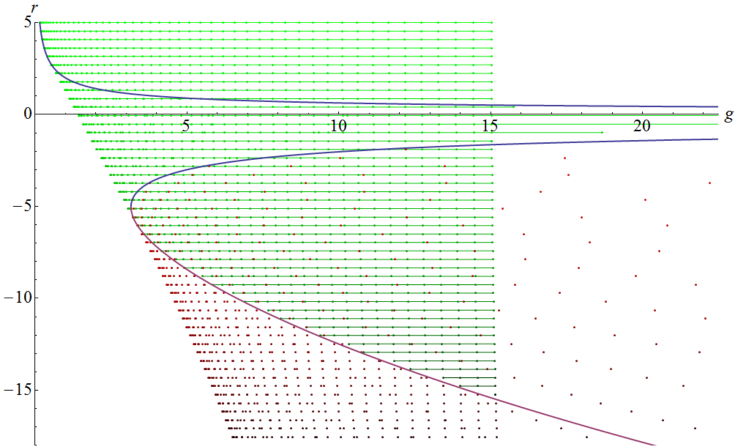
$$S(M) = \frac{1}{2} r \mathrm{Tr}\left(M^2\right) + g \mathrm{Tr}\left(M^4\right) - \left(\mathrm{Tr}\left(M^4\right)\right)^2$$



If time permits

The phase diagram of the asymmetric regime of the system

JT '15

$$S(M) = \frac{1}{2}F(c_2 - c_1^2) + \frac{1}{2}\mathrm{Tr}\left(M^2\right) + g\mathrm{Tr}\left(M^4\right)$$