A solvable quantum field theory in 4 dimensions

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(based on joint work with Harald Grosse,
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Clay Mathematics Institute Millennium Prize Problem (2000)

5. Yang-Mills Existence and Mass Gap

Prove that for any compact simple gauge group $G$, a non-trivial quantum Yang-Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$. Existence includes establishing axiomatic properties at least as strong as those of [Wightman, Osterwalder-Schrader].

- This problem is out of reach for our century.
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- This problem is out of reach for our century.
- There is no solution for a far more modest problem:

*Prove that a non-trivial toy model for a quantum field theory on $\mathbb{R}^4$ exists and satisfies [Wightman, Osterwalder-Schrader]*.
Vanishing $\beta$-function

It is probably a good idea to try first QFT models with vanishing $\beta$-function. They are nice both in UV and IR.

Candidates

1. $\mathcal{N} = 4$ super Yang-Mills theory
   - very active subject with many strong results
   - Wightman axioms are not made for gauge theory, but $\mathcal{N} = 4$ SYM might suggest reasonable axioms
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2. self-dual noncommutative $\phi^4_4$-theory
   - perturbatively renormalisable (Grosse-W. 2004)
   - β-function vanishes to all orders in perturbation theory
     (Disertori-Gurau-Magnen-Rivasseau, 2006)
   - model is relatively simple

→ Can we construct it?
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→ Can we construct it? … YES (as statistical physics model)
To our big surprise, Wightman axioms seem to be satisfied!
Regularisation of $\lambda \phi^4_4$ on noncommutative space

$$S[\phi] = \int_{\mathbb{R}^4} \frac{dx}{64\pi^2} \left( \frac{1}{2} \phi ( -\Delta + \mu^2 ) \phi + \frac{\lambda}{4} \phi^4 \right)(x)$$
Regularisation of $\lambda \phi^4$ on noncommutative space

$$S[\phi] = \int_{\mathbb{R}^4} \frac{dx}{64\pi^2} \left( \frac{1}{2} \phi \left( -\Delta + \Omega^2 (x)^2 + \mu^2 \right) \phi + \frac{\lambda}{4} \phi \phi \phi \phi \right)(x)$$
Regularisation of $\lambda \phi^4$ on noncommutative space

$$S[\phi] = \int \frac{dx}{64\pi^2} \left( \frac{1}{2} \phi( -\Delta + \Omega^2 (2\Theta^{-1}x)^2 + \mu^2 ) \phi + \frac{\lambda}{4} \phi \star \phi \star \phi \star \phi \right)(x)$$

with Moyal product $(f \star g)(x) = \int \frac{dy}{(2\pi)^4} \frac{dk}{(2\pi)^4} f(x + \frac{1}{2} \Theta k) g(x+y) e^{i\langle k,y \rangle}$
Regularisation of $\lambda \phi^4$ on noncommutative space

\[
S[\phi] = \int_{\mathbb{R}^4} \frac{dx}{64\pi^2} \left( \frac{Z}{2} \phi \star \left( -\Delta + \Omega^2 (2\Theta^{-1}x)^2 + \mu_{\text{bare}}^2 \right) \phi + \frac{\lambda Z^2}{4} \phi \star \phi \star \phi \star \phi \right)(x)
\]

with Moyal product $(f \star g)(x) = \int_{\mathbb{R}^4 \times \mathbb{R}^4} \frac{dy \, dk}{(2\pi)^4} f(x + \frac{1}{2} \Theta k) \, g(x+y) \, e^{i \langle k, y \rangle}$
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$$S[\phi] = \int_{\mathbb{R}^4} \frac{dx}{64\pi^2} \left( \frac{Z}{2} \phi^* (-\Delta + \Omega^2 (2\Theta^{-1} x)^2 + \mu_{\text{bare}}^2) \phi + \frac{\lambda Z^2}{4} \phi^* \phi^* \phi^* \phi \right)(x)$$

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matrix basis $f_{mn}(x) = f_{m_1n_1}(x^0, x^1) f_{m_2n_2}(x^3, x^4)$

$$f_{mn}(y^0, y^1) = 2(-1)^m \sqrt{\frac{m!}{n!}} \left( \sqrt{\frac{2}{\theta}} y \right)^{n-m} L_{m}^{n-m} \left( \frac{2|y|^2}{\theta} \right) e^{-\frac{|y|^2}{\theta}}$$

due to $f_{mn} \star f_{kl} = \delta_{nk} f_{ml}$ and $\int dx \, f_{mn}(x) = 64\pi^2 V \delta_{mn}$
Regularisation of $\lambda \phi_4^4$ on noncommutative space

$$S[\phi] = \int_{\mathbb{R}^4} \frac{dx}{64\pi^2} \left( \frac{Z}{2} \phi^*( -\Delta + \Omega^2 (2\Theta^{-1} x)^2 + \mu_{\text{bare}}^2 \right) \phi + \frac{\lambda Z^2}{4} \phi^* \phi \phi^* \phi (x)$$

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takes at $\Omega = 1$ in matrix basis $f_{mn}(x) = f_{m_1 n_1}(x^0, x^1) f_{m_2 n_2}(x^3, x^4)$

$$f_{mn}(y^0, y^1) = 2(-1)^m \sqrt{\frac{m!}{m^n}} \left( \sqrt{\frac{2}{\Theta}} y \right)^{n-m} L_{n-m}^{m} \left( \frac{2|y|^2}{\Theta} \right) e^{-\frac{|y|^2}{\Theta}}$$

due to $f_{mn} \star f_{kl} = \delta_{nk} f_{ml}$ and $\int dx \ f_{mn}(x) = 64\pi^2 V \delta_{mn}$ the form

$$S[\Phi] = V \left( \sum_{m,n \in \mathbb{N}_2} E_m \Phi_{mn} \Phi_{nm} + \frac{Z^2 \lambda}{4} \sum_{m,n,k,l \in \mathbb{N}_2} \Phi_{mn} \Phi_{nk} \Phi_{kl} \Phi_{lm} \right)$$

$$E_m = Z \left( \frac{|m|}{\sqrt{V}} + \frac{\mu_{\text{bare}}^2}{2} \right) , \quad |m| := m_1 + m_2 \leq \mathcal{N}$$

$V = (\frac{\Theta}{4})^2$ is for $\Omega = 1$ the volume of the nc manifold.
More generally: field-theoretical matrix models

**Euclidean quantum field theory**

- **action** \( S[\Phi] = V \text{tr}(E\Phi^2 + P[\Phi]) \)
  for unbounded positive selfadjoint operator \( E \) with compact resolvent, and \( P[\Phi] \) a polynomial

- **partition function** \( Z[J] = \int \mathcal{D}[\Phi] \exp(-S[\Phi] + V \text{tr}(\Phi J)) \)
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For \( P[\Phi] = \frac{i}{6} \Phi^3 \) this is the Kontsevich model which computes the intersection theory on the moduli space of complex curves. We choose \( P[\Phi] = \frac{\lambda}{4} \Phi^4 \).
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Perturbative expansion \( e^{-V \text{tr}(P[\Phi])} = \sum_{n=0}^{\infty} (-1)^n \frac{(V \text{tr}(P[\Phi]))^n}{n!} \)
leads to ribbon graphs. They encode genus-\( g \) Riemann surface with \( B \) boundary components.

We avoid the expansion, but keep the topological structure:
Topological expansion

- Choosing $E = \text{diag}(E_a)$, matrix index conserved along every strand.

- The $k$th boundary component carries a cycle $\mathcal{J}^{N_k}_{p_1...p_{N_k}} := \prod_{j=1}^{N_k} \mathcal{J}_{p_j p_{j+1}}$ of $N_k$ external sources, $N_k + 1 \equiv 1$. 

![](image.png)
Topological expansion

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The $k^{\text{th}}$ boundary component carries a cycle $J_{p_1\ldots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of $N_k$ external sources, $N_k + 1 \equiv 1$.

Expand $\log Z[J] = \sum \frac{1}{S} V^{2-B} G_{|p_1\ldots p_{N_1}|\ldots|p_1\ldots p_{N_B}|} \prod_{\beta=1}^{B} J_{p_1^\beta \ldots p_{N_\beta}^\beta}$ according to the cycle structure.

QFT of matrix models determines the weights of Riemann surfaces with decorated boundary components compatible with 1 gluing (of fringes, not boundaries!)

2 covariance (under $\Phi \mapsto U^* \Phi U$, which is not a symmetry!)
Schwinger-Dyson equations (for $S_{int}[\Phi] = \frac{\lambda}{4} \text{tr}(\Phi^4)$)

In a scaling limit $V \to \infty$ and $\frac{1}{V} \sum_{p \in I}$ finite, we have:

1. A closed non-linear equation for $G_{|ab|}$

$$G_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{(E_a + E_b)} \frac{1}{V} \sum_{p \in I} \left( G_{|ab|} G_{|ap|} - \frac{G_{|pb|} - G_{|ab|}}{E_p - E_a} \right)$$
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\]

2. For $N \geq 4$ a universal algebraic recursion formula

\[
G_{|b_0b_1\ldots b_{N-1}|} = (-\lambda) \sum_{l=1}^{N-2} \frac{2}{2} \left( G_{|b_0b_1\ldots b_{2l-1}|} G_{|b_{2l}b_{2l+1}\ldots b_{N-1}|} - G_{|b_{2l}b_1\ldots b_{2l-1}|} G_{|b_0b_{2l+1}\ldots b_{N-1}|} \right) \frac{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})}{E_{b_0} - E_{b_{2l}}}
\]

- scaling limit corresponds to restriction to genus $g = 0$
- similar formulae for $B \geq 2$
- no index summation in $G_{|abcd|} \Rightarrow \beta$-function zero!
Back to $\lambda \Phi^4_4$ on Moyal space

- Infinite volume limit (i.e. $\theta \to \infty$) turns discrete matrix indices into continuous variables $a, b, \cdots \in \mathbb{R}_+$ and sums into integrals.
- Need energy cutoff $a, b, \cdots \in [0, \Lambda^2]$ and normalisation of lowest Taylor terms of two-point function $G|_{nm} \mapsto G_{ab}$.
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- Carleman-type singular integral equation for $G_{ab} - G_{a0}$.

Theorem (2012/13) (for $\lambda < 0$, using $G_{b0} = G_{0b}$)

Let $\mathcal{H}_a^\Lambda(f) = \frac{1}{\pi} \mathcal{P} \int_0^\Lambda \frac{f(p) \, dp}{p-a}$ be the finite Hilbert transform.
Introduction

\[ \Phi_4^4 \text{ on Moyal space} \]

Integral representation

\[ \text{Schwinger functions} \]

Summary

Back to \( \lambda \Phi_4^4 \) on Moyal space

- Infinite volume limit (i.e. \( \theta \to \infty \)) turns discrete matrix indices into continuous variables \( a, b, \cdots \in \mathbb{R}_+ \) and sums into integrals
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Let \( H^\Lambda_a(f) = \frac{1}{\pi} \mathcal{P} \int_0^{\Lambda^2} \frac{f(p) \, dp}{p-a} \) be the finite Hilbert transform. Then

\[
G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda| \pi a} e^{\text{sign}(\lambda)(H^\Lambda_0[\tau_0(\bullet)] - H^\Lambda_a[\tau_b(\bullet)])}
\]

where \( \tau_b(a) := \arctan \left[ \frac{|\lambda| \pi a}{b + \frac{1+\lambda \pi a H^\Lambda_a[G_{0b}]}{G_{a0}}} \right] \) and \( G_{a0} \) solution of

\[
G_{b0} = G_{0b} = \frac{1}{1+b} \exp \left( -\lambda \int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda \pi p)^2 + (t + \frac{1+\lambda \pi p H^\Lambda_p[G_{0b}]}{G_{p0}})^2} \right)
\]
Discussion

Together with explicit (but complicated for $G_{ab|cd}$, $G_{ab|cd|ef}$, ...) formulae for higher correlation functions, we have exact solution of $\lambda \phi^4_4$ on extreme Moyal space in terms of

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Possible treatments

1. perturbative solution: reproduces all Feynman graphs, generates polylogarithms and $\zeta$-functions
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Possible treatments

1. **perturbative solution**: reproduces all Feynman graphs, generates polylogarithms and $\zeta$-functions

2. **iterative solution on computer**: nicely convergent, find interesting phase structure

3. **rigorous existence proof** of a solution

4. **work in progress**: try to guess the solution; should give uniqueness as by-product
Computer simulation: evidence for phase transitions

piecewise linear approximation of $G_{0b}$, $G_{ab}$ for $\Lambda^2 = 10^7$ and 2000 sample points. Consider $1 + \mathcal{Y} := -\frac{dG_{0b}}{db} \bigg|_{b=0}$

- $(1 + \mathcal{Y})' (\lambda)$
  - discontinuous at $\lambda_c = -0.39$
- order parameter $b_\lambda = \sup\{b : G_{0b} = 1\}$
  - non-zero for $\lambda < \lambda_c$
- A key property for Schwinger functions is realised in $[\lambda_c, 0]$, not outside!
  - The critical couplings coincide!
Fixed point theorem

Theorem (2015)

Let \(-\frac{1}{6} \leq \lambda \leq 0\). Then the equation has a $C^1_0$-solution

\[
\frac{1}{(1+b)^{1-|\lambda|}} \leq G_0b \leq \frac{1}{(1+b)^{1-\frac{|\lambda|}{1-2|\lambda|}}}
\]

Proof via Schauder fixed point theorem.

This involves continuity and compactness of a certain operator (in norm topology)
Relativistic and Euclidean quantum field theory

- We view QFT as defined by Wightman’s axioms for distributions \( \mathcal{W}_N(x_1, \ldots, x_N) = W_N(x_1 - x_2, \ldots, x_{N-1} - x_N) \)
- Theorem: The \( W_N \) are boundary values of holomorphic functions (on permuted extended forward tube \( \subset \mathbb{C}^{4(N-1)} \))
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- Restriction of $W_N$ to real subspace of Euclidean points (minus diagonals) defines Schwinger functions

- Schwinger functions inherit properties such as real analyticity, Euclidean invariance and complete symmetry

- Hence, moments of probability distributions provide candidate Schwinger functions (link to statistical physics)
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Theorem (Osterwalder-Schrader, 1974)

One additional requirement, reflection positivity, leads back to Wightman theory
From matrix model to Schwinger functions on $\mathbb{R}^4$

reverting harmonic oscillator basis, $1+\mathcal{Y} := -\frac{dG_{0b}}{db} |_{b=0} \ldots$

**Theorem (2013):** connected Schwinger functions

$$S_c(\mu x_1, \ldots, \mu x_N) = \frac{1}{64\pi^2} \sum_{N_1 + \ldots + N_B = N} \sum_{\sigma \in S_N} \left( \prod_{\beta=1}^{B} \frac{4^{N_\beta}}{N_\beta} \right) \int_{\mathbb{R}^4} \frac{d^4 p_\beta}{4\pi^2 \mu^4} e^{i \left( \frac{p_{\beta}}{\mu} \cdot \sum_{i=1}^{N_\beta} (-1)^{i-1} \mu x_{\sigma(N_1 + \ldots + N_\beta - 1 + i)} \right)} \left\langle \left( \sum_{i=1}^{N_\beta} \frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}, \ldots, \frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})} \right) \right\rangle$$
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$$\times G \left[ \frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}, \ldots, \frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})} \right] \ldots \left[ \frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}, \ldots, \frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})} \right]_{N_1} \ldots \left[ \frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}, \ldots, \frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})} \right]_{N_B}$$

**Confinement of noncommutativity:** have internal interaction of matrices; commutative subsector propagates to outside world
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Confinement of noncommutativity: have internal interaction of matrices; commutative subsector propagates to outside world

- Schwinger functions are symmetric and invariant under full Euclidean group (completely unexpected for NCQFT!)
- remains: reflection positivity
- finally: Is it non-trivial?
Connected (4+2+2)-point function

\[ \sum_{\sigma \in S_8} x_{\sigma(1)} x_{\sigma(4)} x_{\sigma(3)} x_{\sigma(2)} x_{\sigma(6)} x_{\sigma(5)} x_{\sigma(8)} x_{\sigma(7)} \]

\[ G_{aaaa|bb|cc} \]

1. individual Euclidean symmetry in every boundary component (no clustering)
2. particle scattering without momentum exchange in 4D a sign of triviality (mind assumptions!)
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2. particle scattering without momentum exchange
   - in 4D a sign of triviality (mind assumptions!)
   - familiar in 2D models with factorising S-matrix
   - a consequence of integrability
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familiar in 2D models with factorising S-matrix


Is there a precise link between exact solution of our 4D model and traditional integrability known from 2D?
Osterwalder-Schrader reflection positivity

Proposition (2013)

\[ S(x_1, x_2) \text{ is reflection positive iff } a \mapsto G_{aa} \text{ is a Stieltjes function,} \]

\[ G_{aa} = \int_0^\infty \frac{d(\rho(t))}{a + t}, \quad \rho - \text{positive measure.} \]
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Excluded for any \( \lambda > 0 \) (unless rescued by winding number)

- naïve anomalous dimension \( \eta \) positive for \( \lambda > 0 \), but \( \eta \) diverges to \( +\infty \)
- renormalisation oversubtracts: \( \eta_{\text{ren}}, \lambda \) of opposite sign
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- model-independent: \( \rho \)-space 2-point function

\[ \frac{1}{(p^2 + m^2)^{1 - \eta/2}} \]

Positivity and convergence contradict each other!

- Need (analytical?) continuation between
  – one regime where existence can be proved and
  – another regime where positivity holds.
Partition function is merely a guiding principle to identify the quantum equations of motion (and combinatorics).

These Schwinger-Dyson equations, and not the partition function, define the field theory!
Dismiss the partition function

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Non-triviality (proposed by J. Schlemmer)

- *If* our Schwinger functions do *not* arise from a Euclidean measure, there is no reason they satisfy Nelson-Symanzik positivity.

- Since the free field is Nelson-Symanzik positive, our Schwinger functions cannot be trivial.
Reflection positivity simplifies the problem

If $G_{x0}$ is Stieltjes, then Hilbert transform can be avoided:

$$
\frac{G_{xy}}{G_{x0}} = \exp \left( -\frac{1}{\pi} \int_{1}^{\infty} \frac{dt}{t+x} \arctan \left( \frac{y \text{ Im}(G_{-(t+i\epsilon),0})}{1 - \lambda t \int_{0}^{\infty} ds \frac{G_{s0}}{t+s} + y \text{ Re}(G_{-(t+i\epsilon),0})} \right) \right)
$$

Which class of functions has desired analyticity+holomorphicity and manageable integral transforms?
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hypergeometric functions \( G_{x0} = _nF_{n-1} \left( \begin{array}{c} a, b_1, \ldots b_{n-1} \\ c_1, \ldots, c_{n-1} \end{array} \right | -x \) if \( a \in [0, 1] \) and \( c_i > b_i > a \)

- holomorphicity at \( y > 0 \): determine \( a, b_i, c_i \) by \( G_{0y}^{(k)} = G_{y0}^{(k)} \)
- find: \( a = 1 + \frac{1}{\pi} \arcsin(\lambda \pi) \), \( \prod_{i=1}^{n} \frac{c_i - 1}{b_i - 1} = \frac{\arcsin(\lambda \pi)}{\lambda \pi} \)
- critical coupling constant is \( \lambda_c = -\frac{1}{\pi} = -0.3183 \ldots \)
- also take \( c_1 = 2 \) (nicer Stieltjes and Hilbert transforms)
for $G(x, 0) = {}_3F_2(\frac{a, b_1, b_2}{c_1, c_2} | -x)$

for $G(x, 0) = {}_4F_3(\frac{a, b_1, b_2, b_3}{c_1, c_2, c_3} | -x)$
Convergence of the method

$$3 F_2$$

at \( \lambda = -0.1 \)

\[ G_{4,0}^{(0..1)} = G_{0,4}^{(0..1)} \]


$$4 F_3$$

at \( \lambda = -0.1 \)

\[ G_{3/2,0}^{(0..3)} = G_{0,3/2}^{(0..3)} \]

improvement by factor \( \sim 250 \)

ready to compute the Schwinger 4-point function!
Källén-Lehmann spectrum

- these results make is completely clear (but don’t prove) that $G_{x0}$ is Stieltjes
- reflection positivity equivalent to $G_{xx}$ a Stieltjes function
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the shape makes this plausible:

$$G_{x0} = 4 F_3(\ldots | -x)$$

Stieltjes measure $\rho$
for $G_{x0} = \int_0^\infty dt \, \rho(t)/(t + x)$
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\[
G_{x0} = 4 {F}_3(\ldots | -x)
\]

\[
G_{x x} = \frac{x}{2 + x}
\]

Stieltjes measure $\rho$
for $G_{x0} = \int_0^\infty dt \frac{\rho(t)}{(t + x)}$
at $\lambda = -0.1$

- measure for $G_{x0}$ has mass gap $[0, 1]$, but no further gap (remnant of UV/IR-mixing)
- absence of the second gap (usually $]1, 4]$) circumvents triviality theorems
Summary

\(\lambda \phi^4\) on nc Moyal space is, at infinite noncommutativity, exactly solvable in terms of a fixed point problem

- theory defined by quantum equations of motion (= Schwinger-Dyson equations), not by a measure
- existence proved within small region
- phase transitions and critical phenomena
**Summary**

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2. Projection to Schwinger functions for scalar field on $\mathbb{R}^4$
   - confinement of noncommutativity
   - full Euclidean symmetry (completely unexpected)
   - no momentum exchange (close to triviality), possibly a consequence of integrability
   - numerical approach with tiny error: leaves no doubt that Schwinger 2-point function is reflection positive for $-\frac{1}{\pi} < \lambda \leq 0$
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3. ready to embark on higher Schwinger functions