Fedosov quantization and noncommutative gravity

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Motivation - why Fedosov construction?

- Noncommutative versions of GR should behave well and follow some rules of classical general relativity. At least they should be coordinate covariant.
- Fedosov defromation quantization provides quite straightforward framework for such approach.

Fedosov construction

- On symplectic manifold $(\mathcal{M}, \omega)$ with symplectic connection ∂^S there exists canonical coordinate covariant Fedosov $*$-product of functions.
- But we have much more. Let $\mathcal{E}$ be a vector bundle over $\mathcal{M}$, with a connection $\partial^{\mathcal{E}}$. Let End$(\mathcal{E})$ be corresponding bundle of endomorphisms. Fedosov $*$-product can be covariantly generalized to sections of End$(\mathcal{E})$.

Fedosov product of endomorphisms

$$A * B = AB - \frac{\mathrm{i}h}{2}\omega^{ab}\partial_a A \partial_b B - \frac{h^2}{8}\omega^{ab}\omega^{cd}\Big(\{\partial_b A, R^{\mathcal{E}}_{ac}\}\partial_d B + \partial_b A\{R^{\mathcal{E}}_{ac}, \partial_d B\} + \partial_{(a}\partial_{c)} A \partial_{(b}\partial_{d)} B\Big) + O(h^3).$$

Trace functional

- Importantly, Fedosov algebras come with the trace functional.
- There is only one (up to normalizing constant) family of functionals satisfying
  - $\mathrm{tr}_*(A * B) = \mathrm{tr}_*(B * A)$
  - $\mathrm{tr}_{*_1}(F) = \mathrm{tr}_{*_2}(M(F))$ where M is arbitrary $*$-isomorphism between $*_1$ and $*_2$.

Trace functional

$$\mathrm{tr}_*(A) = \int_{\mathcal{M}} \mathrm{Tr}\left(A + \frac{\mathrm{i}h}{2}\omega^{ab} R^{\mathcal{E}}_{ab} A \right.$$

$$\left. + h^2\left(-\frac{3}{8}\omega^{[ab}\omega^{cd]} R^{\mathcal{E}}_{ab} R^{\mathcal{E}}_{cd} + s_2\right) A + O(h^3)\right)\frac{\omega^n}{n!},$$

where

$$s_2 = \frac{1}{64}\omega^{[ab}\omega^{cd]} \overset{S}{R}{}^k{}_{lab} \overset{S}{R}{}^l{}_{kcd} + \frac{1}{48}\omega^{ab}\omega^{cd} \overset{S}{\partial}_e \overset{S}{\partial}_a \overset{S}{R}{}^e{}_{bcd}\,,$$

Fedosov construction and NCFT

These structures allow for the following geometric (global, coordinate and gauge covariant) deformation scheme of classical field theories:

- take an action functional,
- represent Lagrangian as a trace of some endomorphism (or product of endomorphisms),
- replace product of endomorphisms by $*$-product of endomorphisms,
- replace integral by the trace functional.

Problem – volume form incompatibility.

Fedosov construction and SW map

- Important point - we are not inventing something radically different from known theories.
- Seiberg-Witten map can be understood as a $*$-isomorphism [*Jurčo, Schupp 2000*]. In fact, it can be understood as a local consequence of global Fedosov quantization of $\mathrm{End}(\mathcal{E})$ [*MD 2011*].
- Our approach guarantees that the deformed theory can be locally interpreted in terms of Seiberg-Witten map.
- This fact comes from $*$-isomorphism invariance of the trace functional and the $*$-isomorphism interpretation of SW map.

Example – Einstein-Hilbert action for GR

- Consider Einstein-Hilbert action $\mathcal{S}_{EH} = \int_{\mathcal{M}} R \,\mathrm{vol}_M$.
- Rewrite it as
$$\mathcal{S}_{EH} = \int_{\mathcal{M}} \mathrm{Tr}\, \breve{\underline{R}} \, \frac{\omega^n}{n!},$$
where $\underline{R}$ denotes endomorphism of $T\mathcal{M}$ given by Ricci tensor, i.e. $(\underline{R}X)^i = R^i{}_j X^j$.
- Here $\breve{A} := vA$, with $v : \mathcal{M} \to \mathbb{R}$ defined by $\mathrm{vol}_M = v \,\mathrm{vol}_S$.

Example – Einstein-Hilbert action for GR

After deformation [*MD 2011*], the action reads

$$\widehat{\mathcal{S}}_{EH} = \mathrm{tr}_*(\underline{\check{R}}) =$$
$$= \int_{\mathcal{M}} \left(R - \frac{3}{8} h^2 X^{k\ l}_{\ l\ m} R^m{}_k + h^2 s_2 R + O(h^3) \right) \mathrm{vol}_M ,$$

where

$$X^{ijkl} := \omega^{[ab}\omega^{cd]} R^{ij}{}_{ab} R^{kl}{}_{cd}$$

and

$$s_2 = \frac{1}{64} \omega^{[ab}\omega^{cd]} \overset{S}{R}{}^k{}_{lab} \overset{S}{R}{}^l{}_{kcd} + \frac{1}{48} \omega^{ab}\omega^{cd} \partial^S_e \partial^S_a \overset{S}{R}{}^e{}_{bcd}$$

Example – deformed field equations

Variation of the metric yields field equations

$$R^{ab} - \frac{1}{2}g^{ab}R + h^2\Bigg[\frac{3}{8}\Big(- R^{(a}{}_k X_l{}^{b)kl} + \frac{1}{2}R^k{}_l X^l{}_m{}^m{}_k g^{ab} + \nabla_k \nabla^{(a} X^{b)}{}_l{}^{lk}$$

$$- \frac{1}{2}\nabla_l \nabla^l X^a{}_k{}^{kb} - \frac{1}{2}g^{ab}\nabla_k \nabla_l X^k{}_m{}^{ml} - 2\nabla_k \nabla^l \left(R^{(a}{}_m Y_l{}^{b)mk}\right)$$

$$+ 2\nabla_k \nabla_l \left(R^{km} Y^{l(a}{}_m{}^{b)}\right)\Big) - \frac{1}{2}g^{ab}Rs_2 + R^{ab}s_2 + g^{ab}\nabla_l \nabla^l s_2$$

$$- \nabla^a \nabla^b s_2\Bigg] + O(h^3) = 0,$$

for

$$Y^{ijkl} := \omega^{[ij}\omega^{ab]}R^{kl}{}_{ab}.$$

Example – deformed solutions

Write a metric as a formal series $g_{ab} = \overset{(0)}{g}_{ab} + h\overset{(1)}{g}_{ab} + h^2\overset{(2)}{g}_{ab} + \dots$ and put it into field equations.

- $\overset{(0)}{g}_{ab}$ is just classical Ricci-flat metric.
- $\overset{(1)}{g}_{ab}$ is just classical first order perturbation of $\overset{(0)}{g}_{ab}$
- for $\overset{(1)}{g}_{ab} = 0$ (no classical first order perturbation)

$$\overset{(2)}{g}_{ab} = -\frac{3}{8}\overset{(0)}{X}_{ak}{}^{k}{}_{b} - \frac{1}{n-1}\left(s_2 - \frac{3}{16}\overset{(0)}{X}_{mk}{}^{km}\right)\overset{(0)}{g}_{ab}$$

with $\overset{(0)}{X}{}^{ijkl} = \omega^{[ab}\omega^{cd]}\overset{(0)}{R}{}^{ij}{}_{ab}\overset{(0)}{R}{}^{kl}{}_{cd}$ and

$$s_2 = \frac{1}{64}\omega^{[ab}\omega^{cd]}\overset{S}{R}{}^{k}{}_{lab}\overset{S}{R}{}^{l}{}_{kcd} + \frac{1}{48}\omega^{ab}\omega^{cd}\partial^{S}_{e}\partial^{S}_{a}\overset{S}{R}{}^{e}{}_{bcd}$$

Problems

One can identify number of (related) problems with such approach:

- incompatible volume forms,
- unrelated structures – metric and symplectic,
- fixed background of symplectic geometry: ω,∂^S.

(Slightly) generalized Fedosov theory

- Idea: to put fields *into* deformation quantization.
- Generic Fedosov $*$-product is prototyped by Moyal product in the fibers of $T\mathcal{M}$. But one can consider different prototypes [*MD 2015*].
- Among other, one can consider symmetric part of noncommutativity tensor

$$a \overset{\sim}{\circ} b = \sum_{m=0}^{\infty} \left(-\frac{\mathrm{i}h}{2}\right)^m \frac{1}{m!} \frac{\partial^m a}{\partial y^{i_1} \dots \partial y^{i_m}} s^{i_1 j_1} \dots s^{i_m j_m} \frac{\partial^m b}{\partial y^{j_1} \dots \partial y^{j_m}}$$

where $s^{ij} = \omega^{ij} + g^{ij}$

$$
\begin{aligned}
A * B = {} & AB - \frac{\mathrm{i}}{2} h \big(\omega^{ab} + g^{ab}\big) \partial_a A \partial_b B + h^2 \Bigg(- \frac{1}{8} g^{rs} \overset{S}{R}_{rsab} \big(\omega^{a(p} g^{q)b} + \omega^{pa}\omega^{qb}\big) \partial_p A \partial_q B \\
& + \frac{1}{8} (\omega^{ps} + g^{ps}) (\omega^{qr} + g^{qr}) \partial_p A \partial_q B R^{\mathcal{E}}_{rs} + \frac{1}{4} (\omega^{ps} + g^{ps}) (\omega^{qr} - g^{qr}) \partial_p A R^{\mathcal{E}}_{rs} \partial_q B \\
& + \frac{1}{8} (\omega^{ps} - g^{ps}) (\omega^{qr} - g^{qr}) R^{\mathcal{E}}_{rs} \partial_p A \partial_q B - \frac{1}{8} \partial_s g^{qr} \Big((\omega^{ps} - g^{ps}) \partial_r \partial_q A \partial_p B - (\omega^{ps} + g^{ps}) \partial_p A \partial_r \partial_q B \Big) \\
& - \frac{1}{8} (\omega^{ps} + g^{ps}) (\omega^{qr} + g^{qr}) \partial_{(p} \partial_{q)} A \partial_{(r} \partial_{s)} B \Bigg) + O(h^3)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr}_*(A) = {} & \int_{\mathcal{M}} \mathrm{Tr} \Bigg(A + \frac{\mathrm{i}h}{2} \Big(R^{\mathcal{E}}_{ab} \omega^{ab} + \frac{1}{2} \partial_b \partial_a g^{ab} \Big) A + h^2 \Bigg(- \frac{3}{8} R^{\mathcal{E}}_{ab} R^{\mathcal{E}}_{cd} \omega^{[ab} \omega^{cd]} \\
& - \frac{1}{8} \omega^{cd} \partial_b \partial_a \Big(g^{ab} R^{\mathcal{E}}_{cd} \Big) - \frac{1}{16} \omega^{cd} \partial_b \partial_d \Big(g^{ab} R^{\mathcal{E}}_{ac} \Big) + \frac{1}{4} \omega^{cd} \partial_d \Big(g^{ab} \partial_b R^{\mathcal{E}}_{ac} \Big) \\
& - \frac{3}{16} \omega^{cd} \partial_d \partial_b \Big(g^{ab} R^{\mathcal{E}}_{ac} \Big) + \frac{1}{48} \omega^{ab} \omega^{cd} \omega^{ep} \partial_d \partial_b \overset{S}{R}_{acep} + \frac{1}{64} \overset{S}{R}{}^{d}{}_{aeq} \overset{S}{R}{}^{a}{}_{dpr} \omega^{[eq} \omega^{pr]} \\
& - \frac{1}{32} \partial_b \partial_p \Big(g^{(cd} g^{ep)} \overset{S}{R}_{cdae} \omega^{ab} + g^{ab} g^{cd} \Big(3 \overset{S}{R}_{acde} + \overset{S}{R}_{cead} \Big) \omega^{ep} \Big) \\
& + \frac{1}{32} \omega^{ep} \partial_p \Big(g^{(cb} g^{ad)} \partial_d \overset{S}{R}_{cbea} + 2 g^{ab} g^{cd} \partial_d \overset{S}{R}_{abce} \Big) \\
& - \frac{1}{16} \omega^{cd} \partial_d \Big(g^{ab} \Big(\frac{2}{3} \overset{S}{R}_{cepb} \partial_a g^{ep} + \frac{1}{2} \overset{S}{R}_{aebp} \partial_c g^{ep} \Big) \Big) + \frac{1}{48} \partial_b \partial_d \partial_c \Big(2 g^{ab} \partial_a g^{cd} + 4 g^{ac} \partial_a g^{bd} \Big) \\
& - \frac{1}{16} \partial_d \partial_c \Big(g^{ab} \partial_b \partial_a g^{cd} \Big) - \frac{1}{32} \partial_a \partial_b \partial_c \partial_d \Big(g^{(ab} g^{cd)} \Big) \Bigg) A + O(h^3) \Bigg) \frac{\omega^n}{n!}
\end{aligned}
$$

(Slightly) generalized Fedosov theory

Message: *you can covariantly put a metric inside Fedosov's theory, if you really want*.

Incompatibility of volume forms

Let us have a look at the Einstein-Hilbert action

$$\frac{1}{16\pi G}\int_{\mathcal{M}} \underline{R}^{i}{}_{i}\sqrt{-g}\,\mathrm{d}^4 x + S_{\mathrm{matter}}$$

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Interpret (1) positively - it is the correct action integral but there are some constraints.

Yet another action for GR

Our variational problem is now given by the integral

$$\frac{1}{16\pi G}\int_{\mathcal{M}} \underline{R}^i{}_i \frac{\omega^n}{n!} + S_{\mathrm{matter}}$$

together with the set of constraints

1. $\partial_i^S \omega_{jk} = 0$ (there is symplectic connection)
2. $T^i{}_{jk} = \overset{S}{\Gamma}{}^i{}_{jk} - \overset{S}{\Gamma}{}^i{}_{kj} = 0$ (which is torsionfree)
3. $\Delta\Gamma^i{}_{ik} = \Gamma^i{}_{ik} - \overset{S}{\Gamma}{}^i{}_{ij} = 0$ (and generates correct volume form)

Everything is dynamical here: metric, symplectic form and symplectic connection.

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Condition 3 yields that $\frac{\omega^n}{n!} = \sqrt{\det\omega}\,\mathrm{d}^4x = \alpha\sqrt{-g}\,\mathrm{d}^4x$ because $\Gamma^i{}_{ik} = \frac{\partial \log\sqrt{-g}}{\partial x^k}$ and $\overset{S}{\Gamma}{}^i{}_{ik} = \frac{\partial \log\sqrt{\det\omega}}{\partial x^k}$.

Yet another action for GR

Using Lagrange multipliers method one can derive field equations

$$R^{ab} - \frac{1}{2}Rg^{ab} + \lambda(x)g^{ab} = 8\pi G T^{ab}$$

Using Bianchi identity and energetic condition $\nabla_a T^{ab} = 0$ we get $\lambda(x) = const$. Thus, we have obtained GR with cosmological constant.

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Now, one can use the same variational procedure for the noncommutative case with the action

$$\frac{1}{16\pi G} \operatorname{tr}_* \underline{R} + \widehat{S}_{\mathrm{matter}}$$

Again – everything, including noncommutativity, is dynamical.

Field equations – work in progres...

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Seiberg-Witten equations from Fedosov deformation quantization of endomorphism bundle

Int. J. Geom. Meth. Mod. Phys. **8** (2011), 411, [arXiv:0904.4409]

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Some models of geometric noncommutative general relativity

Phys. Rev. D **84** (2011), 065005 [arXiv:1011.0165]

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Remarks on generalized Fedosov algebras

Int. J. Geom. Meth. Mod. Phys. **12** (2015), 1550096, [arXiv:1411.4769]