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Exploring fuzzy space through Monte Carlo Methods

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September 26, 2015

Outline

What I will explain

Monte Carlo simulations on fuzzy space:

- How do we implement them?
- What are our results?

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What I will explain

Monte Carlo simulations on fuzzy space:

- How do we implement them?
- What are our results?

Why are we doing this?

- Path integral of fuzzy space
- Quantum Non commutative geometry (see Johns talk)
- Quantum gravity through fuzzy space

Fuzzy geometry (p, q)

$$(s, \mathcal{H}, \mathcal{A}, \Gamma, J, \mathcal{D})$$

- $\mathcal{H} = V \otimes M(n, \mathbb{C})$
  where V is a (p, q)-Clifford module
  p-times $(\gamma^i)^2 = 1$ and q-times $(\gamma^i)^2 = -1$
- $\mathcal{A}$ is a $*-$ algebra $M(n, \mathbb{C})$
- $s = (q - p) \bmod 8$
- $\Gamma(v \otimes m) = \gamma v \otimes m$ with γ the chirality operator on V
- $J(v \otimes m) = Cv \otimes m^*$ where C is charge conjugation on V

Dirac operators

In general

$$\mathcal{D}(v \otimes m) = \sum_i \omega^i v \otimes \left(K_i m + \epsilon' m K_i^*\right)$$

ω^i is a product of γ^i

- if $\omega^i = \omega^i$ then $K_i = K_i^*$
- if $\omega^i = -\omega^i$ then $K_i = -K_i^*$

so that $\mathcal{D} = \mathcal{D}^*$

$\epsilon' = \pm 1$ depending on s

Dirac operators

In general

$$\mathcal{D}(v \otimes m) = \sum_i \omega^i v \otimes \left(K_i m + \epsilon' m K_i^*\right)$$

For the example of a $(1,1)$ geometry

$$D = \gamma^1 \otimes \{H, \cdot\} + \gamma^2 \otimes [L, \cdot]$$

With H hermitian and L anti-hermitian and traceless

Remark:

The trace of L decouples, since these matrices always contribute in commutators, so only the traceless part is physical & we can ignore the traceless condition for convenience.

The path integral over Fuzzy space

$$\langle f \rangle = \frac{\int f(\mathcal{D}) e^{\ i\mathcal{S}(\mathcal{D})} \mathrm{d}D}{\int e^{\ i\mathcal{S}(\mathcal{D})} \mathrm{d}D}$$

The path integral over Fuzzy space

$$\langle f \rangle = \frac{\int f(\mathcal{D}) e^{-\mathcal{S}(\mathcal{D})} \mathrm{d}D}{\int e^{-\mathcal{S}(\mathcal{D})} \mathrm{d}D}$$

The path integral over Fuzzy space

$$\langle f \rangle = \frac{\int f(\mathcal{D}) e^{-\mathcal{S}(\mathcal{D})} \mathrm{d}D}{\int e^{-\mathcal{S}(\mathcal{D})} \mathrm{d}D}$$

- What action $\mathcal{S}$ should we use?
- Which functions f should we measure, and what do they tell us?
- How do we choose the measure $\mathrm{d}D$ on the space of geometries?

One slide on Monte Carlo

$$\langle f \rangle = \frac{\sum_{\mathcal{D}} f(\mathcal{D}) e^{-\mathcal{S}(\mathcal{D})}}{\sum_{\mathcal{D}} e^{-\mathcal{S}(\mathcal{D})}}$$

Generate statistically independent Dirac operators $\mathcal{D}_n$ w. probability distribution

$$P(\mathcal{D}_n) = \frac{e^{-\mathcal{S}(\mathcal{D}_n)}}{\sum_i e^{-\mathcal{S}(\mathcal{D}_i)}}$$

Then

$$\langle f(\mathcal{D}) \rangle_N = \frac{1}{N} \sum_{n=1}^{N} f(\mathcal{D}_n) \xrightarrow[N \to \infty]{} \langle f(\mathcal{D}) \rangle$$

The action

$$\mathcal{S} = f(\mathcal{D})$$

What do we want from an action?

- physical motivation
- bounded from below
- need to be able to implement it

The action

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$$\mathcal{S} = g_4 \mathsf{Tr}\left(\mathcal{D}^4\right) + g_2 \mathsf{Tr}\left(\mathcal{D}^2\right)$$

What do we want from an action?

- physical motivation $\Rightarrow$ lowest order
- bounded from below $\Rightarrow$ for some g_2, g_4
- need to be able to implement it $\Rightarrow$ Yes, we did

Simplest cases for $\mathcal{S} = \text{Tr}\left(\mathcal{D}^2\right)$

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Type $(1,0)$ & $(0,1)$

$$D^{(1,0)} = \{H, \cdot\}$$

$$D^{(0,1)} = [H, \cdot]$$

Simplest cases for $\mathcal{S} = \mathrm{Tr}\left(\mathcal{D}^2\right)$

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Type $(1,0)$ & $(0,1)$

$$D^{(1,0)} = \{H, \cdot\} = \mathrm{I}_n \otimes H + H^T \otimes \mathrm{I}_n$$

$$D^{(0,1)} = [H, \cdot] = \mathrm{I}_n \otimes H - H^T \otimes \mathrm{I}_n$$

E.v. of $\mathcal{D}^{(1,0)}/\mathcal{D}^{(0,1)}$ are sum/ diff of e.v. of H

Simplest cases for $\mathcal{S} = \mathrm{Tr}\left(\mathcal{D}^2\right)$

Type $(1,0)$ & $(0,1)$

$$\mathcal{S}(D^{(1,0)}) = g_2\left(2n\mathrm{Tr}\left(H^2\right) + 2\mathrm{Tr}\left(H\right)^2\right)$$

$$\mathcal{S}(D^{(0,1)}) = g_2\left(2n\mathrm{Tr}\left(H^2\right) - 2\mathrm{Tr}\left(H\right)^2\right)$$

The action for H is close to a random matrix model.

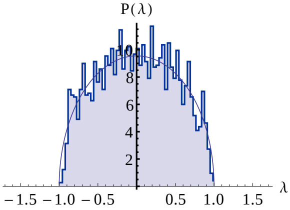
E.V. of $H, \mathcal{D}$ for $(1,0)$ and $(0,1)$

$\mathcal{S} = \mathrm{Tr}\left(\mathcal{D}^2\right)$

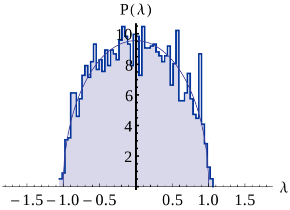
$n = 15$

200 runs

$Hist(Ev(H))$



$Hist(Ev(H))$



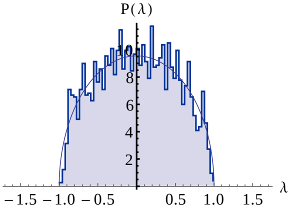
The eigenvalues of H follow the Wigner Semi circle law

We find this is also true for all further H, L with quadratic actions.

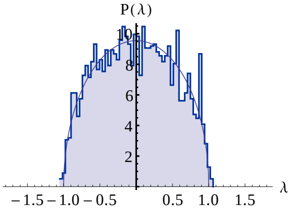
E.V. of $H, \mathcal{D}$ for $(1,0)$ and $(0,1)$

$$\mathcal{S} = \text{Tr}\left(\mathcal{D}^2\right)$$
$$n = 15$$
200 runs

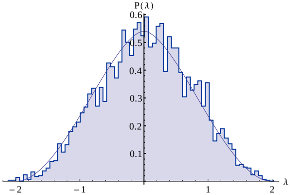
$Hist(Ev(H))$



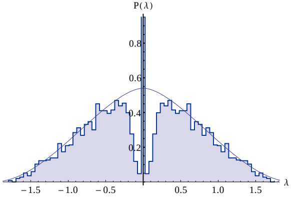
$Hist(Ev(H))$



$Hist(Ev(\mathcal{D}^{(1,0)}))$

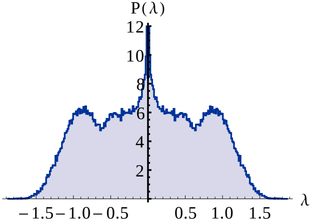


$Hist(Ev(\mathcal{D}^{(0,1)}))$

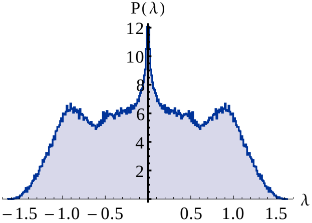


2 d cases for $\mathcal{S} = \mathrm{Tr}\left(\mathcal{D}^2\right)$

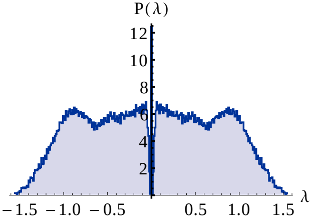
Type (2,0)



Type (1,1)



Type (0,2)



- shoulders + central peak
- $2n$ e.v. 0 for (0,2)
- slight gap around 0 for (2,0),(1,1)

2 d cases for $\mathcal{S} = \mathrm{Tr}\left(\mathcal{D}^2\right)$

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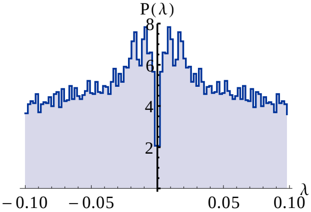
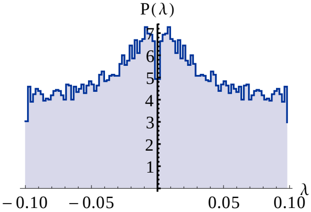
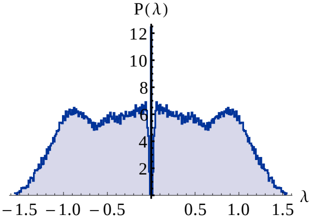
Type (2,0)



Type (1,1)

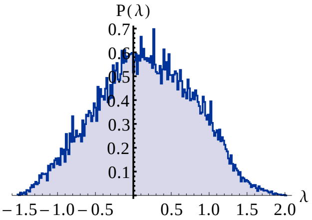


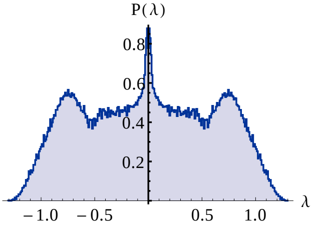
Type (0,2)

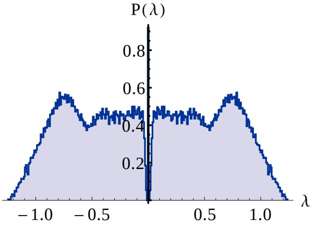


- shoulders + central peak
- $2n$ e.v. 0 for (0,2)
- slight gap around 0 for (2,0),(1,1)

$\mathcal{S} = \mathsf{Tr}\left(\mathcal{D}^4\right)$



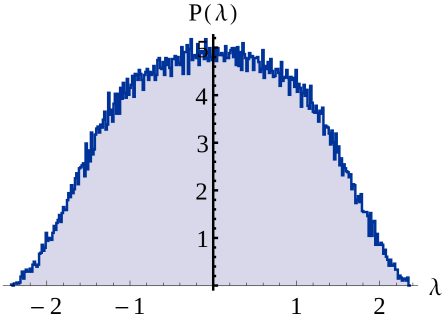




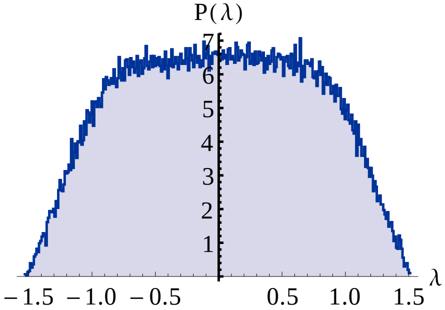
Not much changed.

The 3d case $(0, 3)$

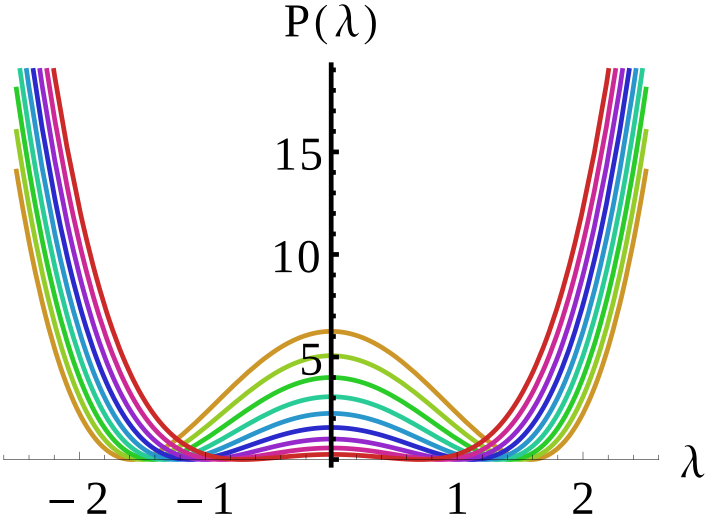
$\mathcal{S} = \mathsf{Tr}\left(\mathcal{D}^2\right)$



$\mathcal{S} = \mathsf{Tr}\left(\mathcal{D}^4\right)$



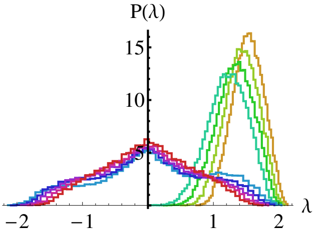
$\mathcal{S} = g_2 \text{Tr}\left(\mathcal{D}^2\right) + \text{Tr}\left(\mathcal{D}^4\right)$



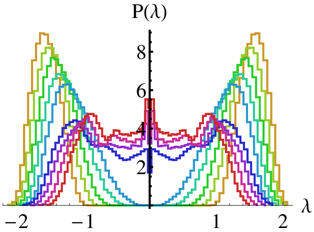
$g_2 = \{-1, -1.5, \cdots, -4.5, -5\}$

$$\mathcal{S} = g_2 \mathrm{Tr}\left(\mathcal{D}^2\right) + \mathrm{Tr}\left(\mathcal{D}^4\right)$$

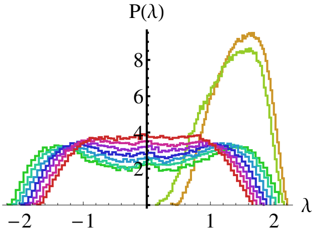
Type (1,0)



Type (2,0)



Type (0,3)



Interesting change as g_2 becomes more negative

$$\mathcal{S} = g_2 \text{Tr}\left(\mathcal{D}^2\right) + \text{Tr}\left(\mathcal{D}^4\right)$$

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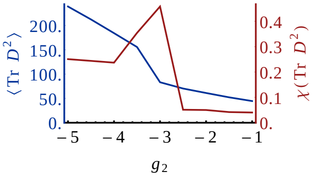
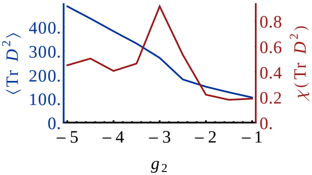
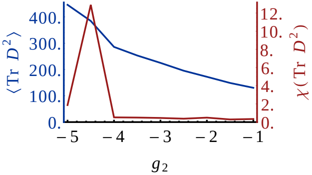
Type (1,0)



Type (2,0)



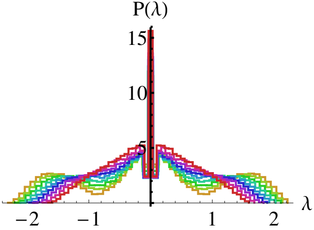
Type (0,3)



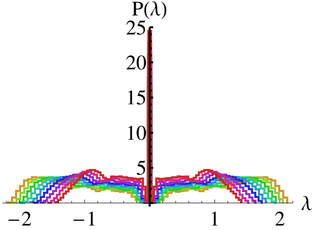
Interesting change as g_2 becomes more negative
Also in the order parameter $\text{Tr}\left(\mathcal{D}^2\right)$

But some show no phase transition

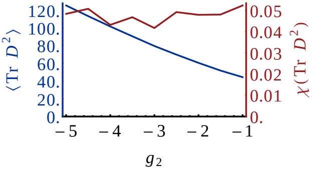
Type (0,1)



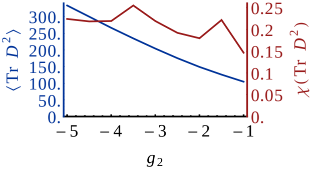
Type (0,2)



Type (0,1)

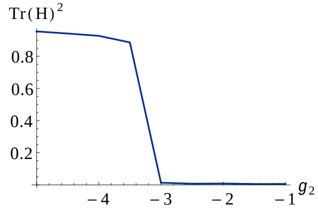


Type (0,2)

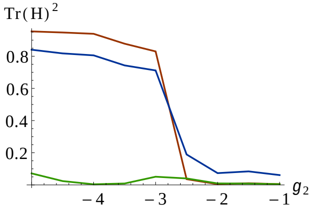


What is the difference?

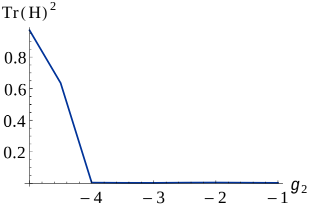
Type (1,0)



Type (2,0)



Type (0,3)

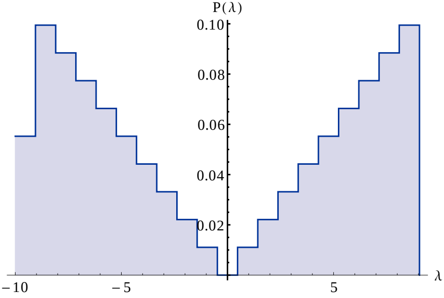


H is the reason

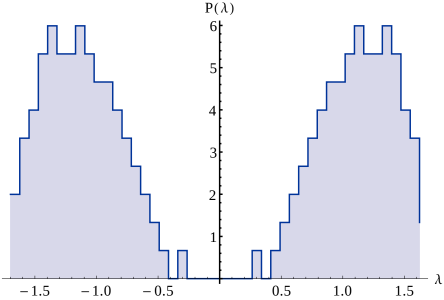
$Tr(H)$ developes a non-zero expectation value.

Signs of a continuum?

Fuzzy sphere



Type (1,1) at $g_2 = -3$



Summary & Conclusion

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What did we do?

- Monte Carlo simulation on fuzzy geometry
- Examined type $(1,0),(0,1),(2,0),(1,1),(0,2),(0,3)$

What did we find

- We can explain the eigenvalue density
- Phase transition depending on g_4
- Possible continuum behavior?

Summary & Conclusion

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What did we do?

- Monte Carlo simulation on fuzzy geometry
- Examined type $(1,0), (0,1), (2,0), (1,1), (0,2), (0,3)$

What are we looking for

- $\mathcal{S}$ that peaks around the fuzzy S^2
- Other observables to look at
- A better computer to run with 96×96 matrices ☺

Can we hear the geometry?

Can we hear the geometry?

YES!

Can we hear the geometry?

YES!

Thank you for your attention!

Diracs we studied

For this project we have examined the simplest cases

$$D^{(1,0)} = \{H, \cdot\} \qquad D^{(0,1)} = [H, \cdot]$$

$$D^{(2,0)} = \gamma^1 \otimes \{H_1, \cdot\} + \gamma^2 \otimes \{H_2, \cdot\}$$

$$D^{(1,1)} = \gamma^1 \otimes \{H, \cdot\} + \gamma^2 \otimes [L, \cdot]$$

$$D^{(0,2)} = \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot]$$

$$D^{(0,3)} = \sum_{j<k=1}^{3} \gamma^0 \gamma^j \gamma^k \otimes [L_{jk}, \cdot] + \gamma^1 \gamma^2 \gamma^3 \otimes \{H_{123}, \cdot\}$$

$$+ \gamma^0 \otimes \{L_0, \cdot\} + \sum_{i=1}^{3} \gamma^i \otimes [L_i, \cdot]$$