INFLATION

Shortcomings of the Standard Big Bang Cosmological Model:

(α) Horizon problem

The Cosmic Background Radiation (CBR) received now was emitted at "decoupling" of matter and radiation (recombination of atoms) when $T_d \approx 3{,}000^\circ K$.

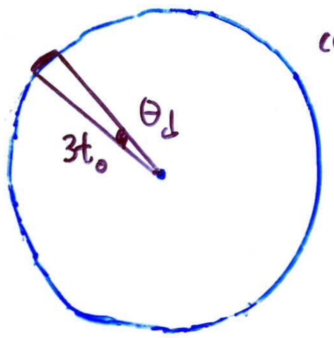
Calculate decoupling time t_d: $\frac{T_0}{T_d} = \frac{2.73}{3{,}000} = \frac{a(t_d)}{a(t_0)} = \left(\frac{t_d}{t_0}\right)^{2/3}$

$\Longrightarrow \quad t_d \approx 200{,}000\, h^{-1}$ years

Distance over which these photons travelled $\longrightarrow$

$$a(t_0)\int_{t_d}^{t_0} \frac{dt'}{a(t')} = 3t_0\left(1 - \left(\frac{t_d}{t_0}\right)^{1/3}\right) \approx 3t_0 \approx 6{,}000 h^{-1}\, Mpc$$

$\approx$ present particle horizon



"last scattering surface"

Horizon size at $t_d = 2H^{-1}(t_d) = 3t_d = 0.168\, h^{-1}$ Mpc

$\longrightarrow$ expands till now and becomes $= 0.168 h^{-1} \frac{a(t_0)}{a(t_d)}$ Mpc $\approx 184 h^{-1}$ Mpc

θ_d = angle subtended by decoupling horizon at present

$$= \frac{184}{6{,}000} = 0.03 \text{ rads} = 2^\circ$$

$\longrightarrow$ sky splits into $\frac{4\pi}{(0.03)^2} \approx 14{,}000$ patches that never communicated before sending light to us.

How come $\frac{\delta T}{T} \approx 6.6 \times 10^{-6}$ (COBE) ?

(β) Flatness problem

Today : $0.1\,\rho_c \lesssim \rho \lesssim 2\,\rho_c$

virial theorem on galactic clusters

galactic ~~\#~~ density at large distances ⟶ Volume exp. rate

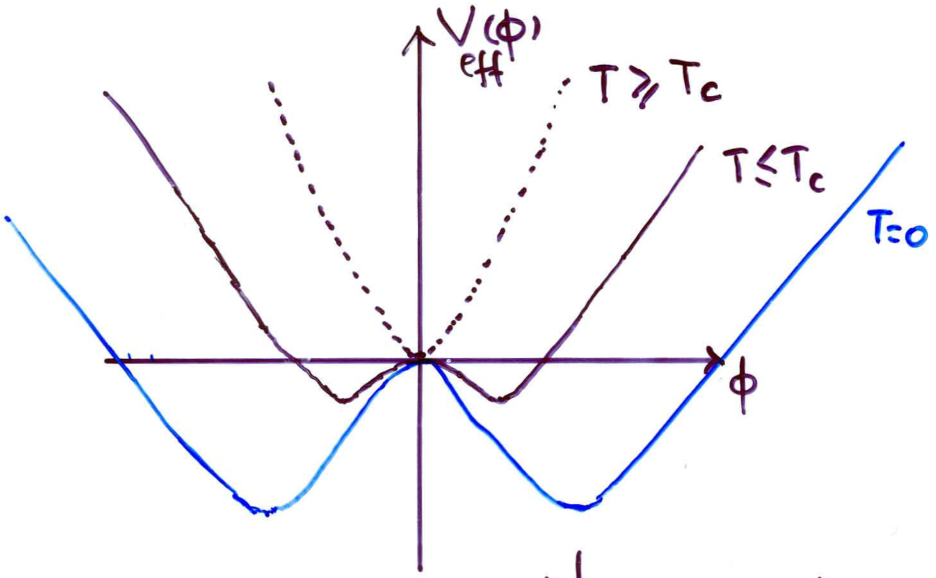
Friedmann equ: $H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} = \frac{8\pi G}{3}\rho_c$

$$\Longrightarrow \frac{\rho - \rho_c}{\rho_c} = \frac{3}{8\pi G \rho_c}\left(\frac{k}{a^2}\right) \propto a \longrightarrow$$

In the early Universe $\frac{\rho - \rho_c}{\rho_c} \ll 1$

(γ) Magnetic Monopole Problem (SBBCM + GUTs)

GUT-phase transition : $G \xrightarrow[M_X]{\langle\phi\rangle} G_S$



2^{nd} order phase transition

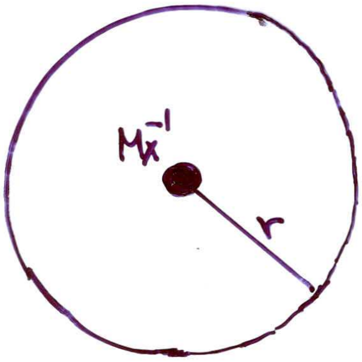
$$V_{eff}(\phi) = V_0(\phi) + \sigma^2 T^2 \phi^2 + \dots$$

$$\langle\phi\rangle(T) \simeq \langle\phi\rangle(T=0)\left(1 - \frac{T^2}{T_c^2}\right)^{1/2}$$

$T \lesssim T_c$

$$m_H(T) \sim h\,\langle\phi\rangle(T)$$

GUT-transition produces Magnetic monopoles = localized deviations from vacuum with radius $\sim M_X^{-1}$, energy $\sim \frac{M_X}{\alpha_G}$ and $\phi = 0$ at their centre



S^2 (sphere at ∞)

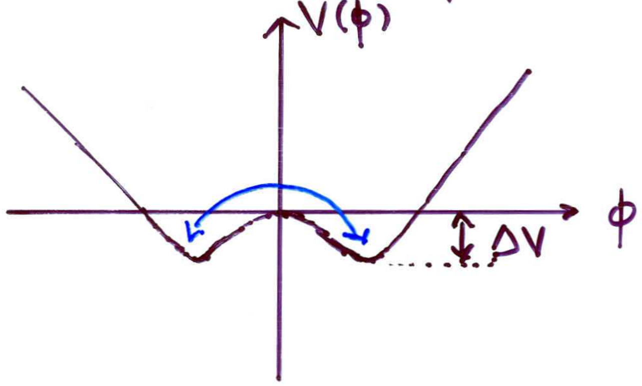
$\langle\phi\rangle \in G/G_S$

$r \gg M_X^{-1}$

$S^2 \longrightarrow G/G_S$

iff $\pi_2(G/G_S)$ is non-trivial $\longrightarrow$ topologically stable Magnetic Monopoles

Production of monopoles: At $T \leq T_c$



$\Delta V \sim h^2 \langle\phi\rangle^4$

= diffence in free energy density between $\phi = 0$ and $\phi = \langle\phi\rangle(T)$.

Higgs correlation length $\xi(T) = m_H^{-1}(T)$

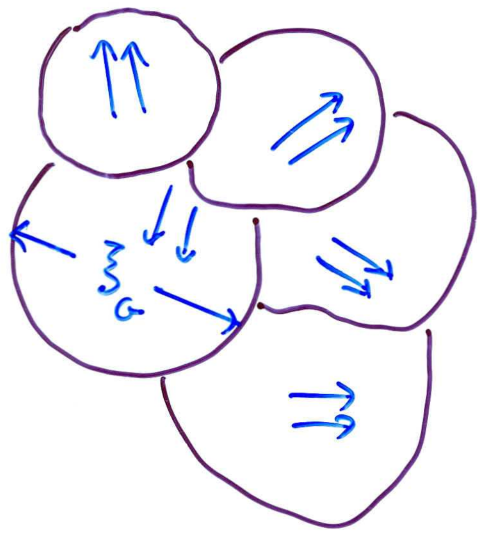
if $\frac{4\pi}{3} \xi^3 \Delta V \leq T$

then fluctuations back and forth over $\phi = 0$ are frequent.

Ginsburg temperature T_G corresponds to saturation of this inequality $\longrightarrow$ at $T \leq T_G$ fluctuations stop

$\Longrightarrow$ $\langle\phi\rangle$ settles on G/G_S $(G \longrightarrow G_S)$

At $T_G \longrightarrow \xi_G \sim \frac{1}{h^2 T_c}$



$$n_M \sim p\, \xi_G^{-3} \sim p\, h^6 T_c^3 \quad \left(p \sim \frac{1}{10}\right)$$

$$\Longrightarrow\ r_M = \frac{n_M}{T^3} \sim 10^{-6}$$

Causality Bound : $n_M \geqslant \dfrac{p}{\frac{4\pi}{3}(2t_G)^3} \Longrightarrow r_M \geqslant 10^{-10}$

Subsequent evolution of Monopoles after T_G:

$$\frac{d n_M}{dt} = -D\, n_M^2 - 3\frac{\dot{a}}{a}\, n_M$$

"monopole-antimonopole annihilation"

dilution due to expansion

Monopoles diffuse towards antimonopoles through the "plasma", capture each other in Bohr orbits and annihilate.

Annihilation is effective $\Longleftrightarrow$ mean free path $\leqslant$ capture distance

$\Longleftrightarrow\ T \geqslant 10^{12}\,\mathrm{GeV}$

After this essentially no annihilation occurs

Final result: if $r_{in} \geqslant 10^{-9} \longrightarrow r_{fin} \sim 10^{-9}$

if $r_{in} \leqslant 10^{-9} \longrightarrow r_{fin} \sim r_{in}$

Causality Bound $r_{fin} \geqslant 10^{-10}$

Bound from Nucleosynthesis (they should not dominate)

$$r_M(T \approx 1\,\text{MeV}) \lesssim 10^{-19}$$

(5) Density flutations

For structure formation we need $\delta\rho/\rho$ at all scales with a nearly flat spectrum

Also we need some explanation of $\delta T/T$ of CBR at $\theta \geq \theta_d$ which violates causality.

Expand $\delta\rho/\rho$ in plane waves

$$\frac{\delta\rho}{\rho}(\bar{x},t) = \int d^3k\, \delta_{\bar{k}}(t)\, e^{i\bar{k}\bar{x}}$$

k = comoving wave # $\longrightarrow$ $\lambda = \dfrac{2\pi}{k}$ = comoving wave length

$$\lambda_{phys.} = a(t)\lambda$$

For $\lambda_{phys} \lesssim H^{-1} \longrightarrow$ Newtonian equ:

$$\ddot{\delta}_{\bar{k}} + 2H\dot{\delta}_{\bar{k}} + \frac{v_s^2 k^2}{a^2}\delta_{\bar{k}} = 4\pi G\rho\,\delta_{\bar{k}} \qquad \left(v_s^2 = \frac{dp}{d\rho}\right)$$

cosmological expansion — pressure term — gravitational attraction — velocity of sound

For the moment put $H=0$ (static universe)

k_J = Jeans wavenumber : $k_J^2 = \dfrac{4\pi G a^2 \rho}{v_s^2}$

for $k \geq k_J$, pressure dominates $\rightarrow$ pert. just oscillates

for $k \leq k_J$, gravity dominates $\rightarrow$ pert. grows exponentially

In particular, for $p=0$ (CDM), $v_s=0$

$\longrightarrow$ all scales are Jeans unstable

$$\delta_{\bar{k}} \propto \exp(t/\tau) \quad , \quad \tau = (4\pi G\rho)^{-1/2}$$

Now take $H \neq 0$. Since expansion "pulls particles apart" we get smaller growth:

$$\delta_{\bar{k}} \propto a \propto t^{2/3} \quad \text{(matter domination)}$$

No essential growth (radiation domination, $p \neq 0$)

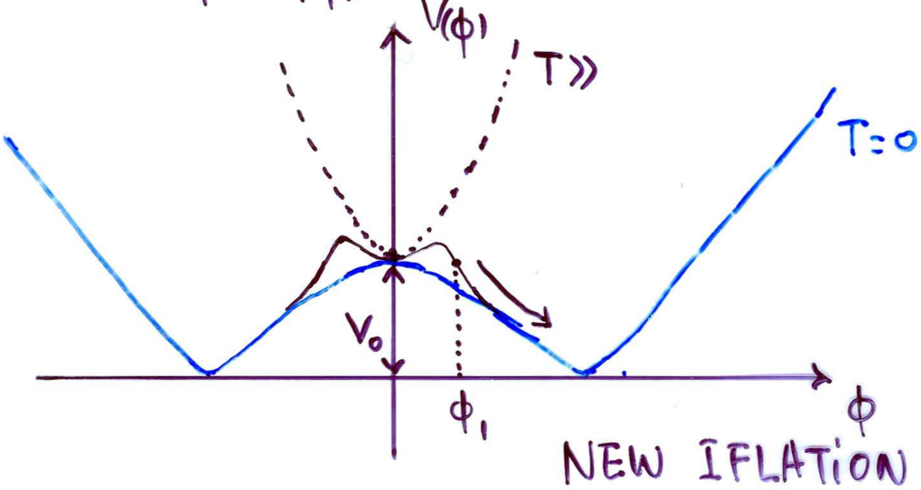
SO WE NEED $\left(\frac{\delta\rho}{\rho}\right)_{equ.} \sim 4\times10^{-5}(\Omega_0 h)^{-2}$

since the available growth is $\frac{a_0}{a_{equ.}} \sim 2.5\times10^4(\Omega_0 h)^2$

INFLATION

Inflation is an idea which solves simultaneously all four cosmological puzzles in one go!!

Suppose there is a scalar field ϕ = inflaton with potential $V(\phi)$ which is quite "flat" near $\phi=0$ and has a minimum at $\phi = \langle\phi\rangle$.

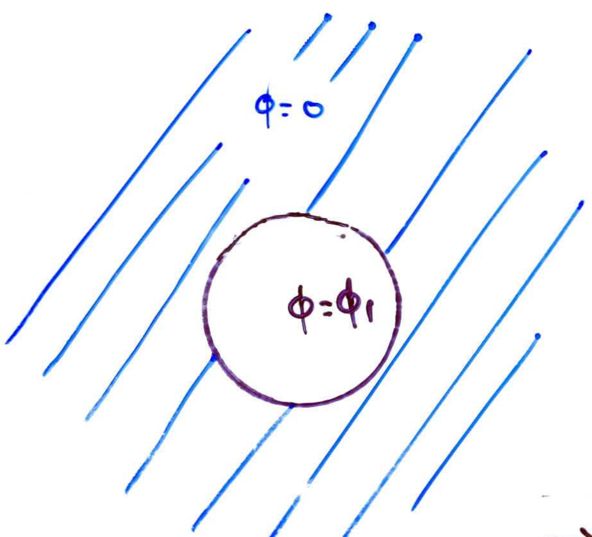


NEW IFLATION

At $T\gg$ Universe at $\phi=0$

As T drops, $V(\phi)$ approaches the $T=0$ potential but a little barrier still remains

At some point ϕ tunnels out to $\phi_1 \ll \langle\phi\rangle$ and a bubble with $\phi = \phi_1$ is created. The field then rolls over to the minimum very slowly (flat potential). During this slow roll-over $\rho \approx V_0 \approx$ const. for quite some time.



The Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)$$

$$\rightarrow T_\mu{}^\nu = -\partial_\mu\phi\,\partial^\nu\phi + \delta_\mu{}^\nu\left(\tfrac{1}{2}\partial_\lambda\phi\partial^\lambda\phi - V(\phi)\right)$$

During the slow roll-over : $T_\mu{}^\nu \approx -V_0\,\delta_\mu{}^\nu$

$$\Longrightarrow \quad \rho = -p = V_0$$ (negative pressure equal in magnitude with ρ)

Consistent with continuity equ. $\dot\rho = -3H(\rho + p)$

Friedmann eq. $\longrightarrow \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}V_0 - \frac{k}{a^2}$

$a(t)$, as we will see, grows fast and the 2^{nd} term becomes subdominant

$$\Longrightarrow \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}V_0 \rightarrow a(t) \propto e^{Ht},\ H^2 = \frac{8\pi G}{3}V_0 = \text{const.}$$

SO THE BUBBLE EXPANDS EXPONENTIALLY AND

$$\frac{a(t_f)}{a(t_i)} = \exp H(t_f - t_i) = \exp H\tau\ ,\quad H\tau = \#\text{ of e-foldings}$$

WITH ENOUGH E-FOLDINGS THE FIRST THREE PUZZLES ARE EASILY RESOLVED !!

(α) Horizon problem

The particle horizon

$$d(t) = e^{Ht} \int_{t_i}^{t} \frac{dt'}{e^{Ht'}} = \frac{1}{H} e^{Ht}\left(e^{-Ht_i} - e^{-Ht}\right) \approx H^{-1} \exp H(t-t_i)$$

for $t - t_i \gg H^{-1}$

So the particle horizon grows as fast as $a(t)$
At the end of inflation ($t = t_f$), $d(t_f) = H^{-1} \exp H\tau$ and ϕ starts oscillating about the min. at $\phi = \langle\phi\rangle$. It then decays and "reheats" the Universe at a temperature T_r ($\sim 10^9$ GeV) and go back to normal cosmology.

The horizon $d(t_f)$ is streched during this oscillation by some factor ($\sim 10^9$) depending on details and between T_r and present by T_r / T_0. So it becomes =

$$H^{-1} e^{H\tau} 10^9 \frac{T_r}{T_0} \geqslant 2 H_0^{-1}$$

to solve the horizon problem

So with $V_0 \approx M_X^4$, $M_X \sim 10^{16}$ GeV $\rightarrow$ $H\tau \geqslant 55$

(β) Monopole problem

With ~~#~~ of e-foldings $\geqslant 55$, the primordial monopole density is diluted by $\geqslant 70$ orders of magnitude and they become totally irrelevant. Also, since $T_r \ll m_M$, there is no production of monopoles after a reheat".

(γ) Flatness problem

The curvature term at present

$$\frac{k}{a_0^2} \approx \left(\frac{k}{a^2}\right)_{\text{before infl.}} \cdot \underset{\uparrow\ \text{infl.}}{e^{-2H\tau}} \cdot \underset{\uparrow\ \text{oscillation}}{10^{-18}} \left(\frac{10^{-13}\,\text{GeV}}{10^{9}\,\text{GeV}}\right)^2 \underset{\uparrow\ \text{after "reheat"}}{}$$

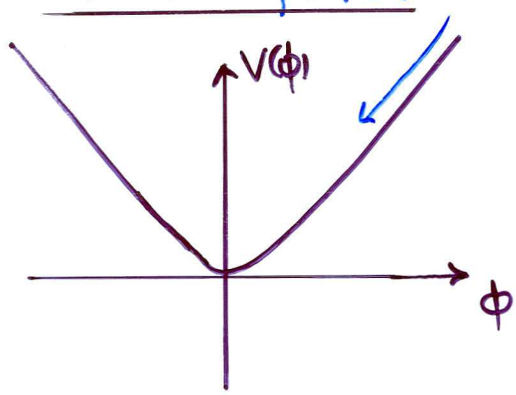
Assuming $\left(\frac{k}{a^2}\right)_{b.i} \sim \frac{8\pi G}{3}\rho \sim H^2 \quad (\rho = V_0)$

$$\longrightarrow \quad \frac{k}{a_0^2 H_0^2} \sim 10^{48}\, e^{-2H\tau}$$

$$\longrightarrow \quad \Omega_0 - 1 = \frac{k}{a_0^2 H_0^2} \ll 1 \qquad \text{for} \qquad H\tau \gg 55$$

STRONG INFLATION $\longrightarrow$ PRESENT UNIVERSE is FLAT !! $(\rho = \rho_c)$

Chaotic inflation



A region in the Universe starts with a big and almost uniform value of ϕ and inflation takes place as ϕ rolls down towards $\phi = 0$

DETAILED ANALYSIS OF INFLATION

Hubble parameter is not exactly const. during inflation as we naively assumed so far.

$$H^2(\phi) = \frac{8\pi G}{3} V(\phi)$$

To find the evolution of φ during inflation we vary the action

$$\int \sqrt{-\det g}\, d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) + M(\phi) \right)$$

coupling of ϕ to "light" matter causing its decay.

$$\Longrightarrow \quad \ddot{\phi} + 3H\dot{\phi} + \Gamma_\phi \dot{\phi} + V'(\phi) = 0$$

Γ_ϕ = decay width of the inflaton $\longrightarrow$ Assume, for the moment, that its decay time $t_d = \Gamma_\phi^{-1} \gg H^{-1}$ = expansion time for inflation

$\longrightarrow$ $\Gamma_\phi \dot{\phi}$ - term can be ignored.

$$\longrightarrow \quad \ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

Inflation is by definition when $\ddot{\phi}$ is subdominant to the "friction" term $3H\dot{\phi}$

$$\Longrightarrow \quad \text{Inflationary equ:} \quad 3H\dot{\phi} = -V'(\phi)$$

$$\longrightarrow \quad \dot{\phi} = -\frac{V'(\phi)}{3H(\phi)}$$

$$\Longrightarrow \quad \ddot{\phi} = -\frac{V''(\phi)\,\dot{\phi}}{3H(\phi)} + \frac{V'(\phi)}{3H^2(\phi)} H'(\phi)\,\dot{\phi}$$

Then for inflation to hold

$$\left| \frac{V''(\phi)\,\dot{\phi}}{3H(\phi)} \right| \lesssim H(\phi)|\dot{\phi}| \quad , \quad \left| \frac{V'(\phi)}{3H^2(\phi)} H'(\phi)\dot{\phi} \right| \lesssim H(\phi)|\dot{\phi}|$$

$$\Rightarrow \eta \equiv \frac{M_P^2}{8\pi}\left|\frac{V''(\phi)}{V(\phi)}\right| \lesssim 1 \quad , \quad \varepsilon \equiv \frac{M_P^2}{16\pi}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \lesssim 1$$

The end of the slow "roll-over" occurs when either of these inequalities is saturated.

If ϕ_f = value of ϕ at the end of inflation

$$\longrightarrow t_f \sim H^{-1}(\phi_f)$$

The ~~#~~ e-foldings during inflation is calculated as follows:

$$\frac{\dot a(t)}{a(t)} = H \Rightarrow \frac{da(t)}{a(t)} = H\,dt \Longrightarrow \ln\left(\frac{a(t_f)}{a(t_i)}\right) = \int_{t_i}^{t_f} H\,dt$$

$$= \int_{\phi_i}^{\phi_f} \frac{H(\phi)}{\dot\phi}\,d\phi = -\int_{\phi_i}^{\phi_f} \frac{3H^2(\phi)\,d\phi}{V'(\phi)} \equiv N(\phi_i \longrightarrow \phi_f)$$

Since $\dfrac{a(t_f)}{a(t_i)} = \exp N(\phi_i \rightarrow \phi_f)$

We shift the field ϕ so that its "global minimum" is at $\phi = 0$.

Then if $V(\phi) = \lambda\phi^\nu$ during inflation

$$N(\varphi_i \rightarrow \varphi_f) = -\int_{\phi_i}^{\phi_f} \frac{3H^2(\phi)\,d\phi}{V'(\phi)} = -8\pi G \int_{\phi_i}^{\phi_f} \frac{V(\phi)\,d\phi}{V'(\phi)}$$

$$= \frac{4\pi G}{\nu}\left(\phi_i^2 - \phi_f^2\right)$$

Assuming $\phi_i \gg \phi_f \longrightarrow N(\phi) = \dfrac{4\pi G}{\nu}\phi^2$

After the end of inflation the $\ddot\phi$-term takes over

$\longrightarrow \quad \ddot{\phi} + V'(\phi) = 0 \longrightarrow$ ϕ oscillates about its global minimum.

with $\frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \equiv \frac{d\rho}{dt} = 0$ (const. energy density)

In reality, due to the friction term, ϕ performs damped oscillations

$$\frac{d\rho}{dt} = \frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) = -3H\dot{\phi}^2 = -3H(\rho+p)$$

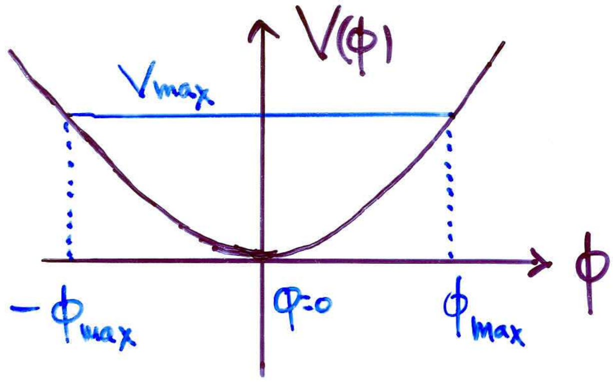
$$\longrightarrow \rho + p = \dot{\phi}^2 \qquad \left\{ \begin{array}{l} \rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ \\ p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{array} \right\}$$

Averaging over one oscillation : $\rho + p = \gamma\rho$

$$\rightarrow \frac{d\rho}{dt} = -3H\gamma\rho \rightarrow \frac{d\rho}{\rho} = -3\gamma\frac{da}{a} \rightarrow \rho \propto a^{-3\gamma}$$

Friedman equ: $\longrightarrow \quad \frac{\dot{a}}{a} \propto a^{-3\gamma/2} \Longrightarrow \quad a(t) \propto t^{2/3\gamma}$

The number γ for an oscillating field $\Longrightarrow$



$$\gamma = \frac{\int_0^T \dot{\phi}^2 dt}{\int_0^T \rho\, dt} = \frac{\int_0^{\phi_{max}} \dot{\phi}\, d\phi}{\int_0^{\phi_{max}} \frac{\rho}{\dot{\phi}}\, d\phi} \quad \left(\text{for sym. potential}\right)$$

Now $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) = V_{max} \Longrightarrow \quad \dot{\phi} = \sqrt{2(V_{max} - V(\phi))}$

and $\gamma = 2\int_0^{\phi_{max}} (1 - V/V_{max})^{1/2} d\phi \Big/ \int_0^{\phi_{max}} (1 - V/V_{max})^{-1/2} d\phi$

For a potential $V(\phi) = \lambda \phi^{\nu} \Longrightarrow \gamma = \frac{2\nu}{\nu+2}$

$$\Longrightarrow \rho \propto a^{-\frac{6\nu}{\nu+2}} \quad , \quad a(t) \propto t^{\frac{\nu+2}{3\nu}}$$

Examples: $\nu = 2$: $\gamma = 1 \longrightarrow \rho \propto a^{-3}, \, a(t) \propto t^{2/3}$ (pressureless "matter")

This is expected since a "coherent" oscillating massive free field is like a distribution of static massive particles

$\nu = 4$: $\gamma = 4/3 \longrightarrow \rho \propto a^{-4}, \, a(t) \propto t^{1/2}$ (radiation)

$\nu = 6$: $\gamma = 3/2 \longrightarrow \rho \propto a^{-4.5}, \, a(t) \propto t^{4/9}$ (slower than radiation)

($\gamma - 1 = 1/2$: more pressure than radiation)

N.B. $\nu = 4, 6, \ldots$ there is enough pressure <u>not</u> to let perturbations grow!

$\nu = 2$ perturbations grow like in "matter"

The period of "coherent oscillation" extends from the end of inflation till the decay of ϕ at $t_d = \Gamma_\phi^{-1}$.

The total expansion during this period is

$$\frac{a(t_d)}{a(t_f)} = \left(\frac{t_d}{t_f}\right)^{\frac{\nu+2}{3\nu}}$$

<u>Decay of the ϕ-field</u>

Reintroduce the Γ_ϕ-term $\rightarrow$

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma_\phi \dot{\phi} + V'(\phi) = 0 \quad \Longrightarrow$$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) = -\left(3H + \Gamma_\phi\right)\dot{\phi}^2$$

$$\Longrightarrow \quad \dot{\rho} = -(3H + \Gamma_\phi)\,\gamma\rho$$

$$\Longrightarrow \quad \rho(t) = \rho_f\left(\frac{a(t)}{a(t_f)}\right)^{-3\gamma}\exp\left[-\gamma\,\Gamma_\phi\,(t-t_f)\right]$$

Usual term due to expansion

exponential decay law of ϕ

Assuming the decay products are "photons" ("new radiation")

$$\dot{\rho}_r = -4H\rho_r + \gamma\,\Gamma_\phi\,\rho$$

dilution due to expansion

energy transfer from ϕ to γ

Taking $\rho_r(t_f) = 0$, this equ. gives

$$\rho_r(t) = \rho(t_f)\left(\frac{a(t)}{a(t_f)}\right)^{-4}\int_{\gamma\Gamma_\phi t_f = u_0}^{\gamma\Gamma_\phi t}\left(\frac{a(t')}{a(t_f)}\right)^{4-3\gamma} e^{u_0 - u}\,du$$

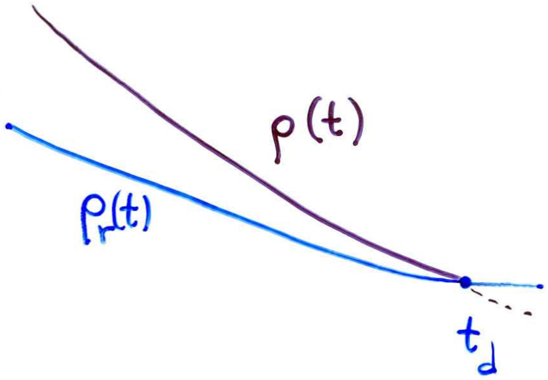
For $t_f \ll t_d$ and $\nu = 2 \Longrightarrow$ approximated

$$\rho_r(t) = \rho(t_f)\left(\frac{t}{t_f}\right)^{-8/3}\int_0^t\left(\frac{t'}{t_f}\right)^{2/3} e^{-\Gamma_\phi t'}\,dt'$$

Using : $\displaystyle\int_0^u x^{p-1}e^{-x}\,dx = e^{-u}\sum_{k=0}^{\infty}\frac{u^{p+k}}{p(p+1)\cdots(p+k)}$

$$\Longrightarrow \rho_r = \frac{3}{5}\,\rho\,(\Gamma_\phi t)\left[1 + \frac{3}{8}(\Gamma_\phi t) + \frac{9}{88}(\Gamma_\phi t)^2 + \cdots\right]$$

where $\displaystyle \rho = \rho(t_f)\left(\frac{t}{t_f}\right)^{-2} e^{-\Gamma_\phi t}$



$$\rho(t_d) = \rho_r(t_d)$$

$$\Leftrightarrow \quad \Gamma_\phi t_d = 1$$

$$1 + \frac{3}{8} + \frac{9}{88} + \cdots = \frac{3}{2}$$

$$\Rightarrow \quad t_d = \Gamma_\phi^{-1}$$

For $t \geq t_d$, "new radiation" dominates $\longrightarrow$ NORMAL DEVELOPMENT

The temperature at t_d:

$$T(t_d) = T_r = \text{"reheat temperature" (historically)}$$

Using:
$$T_r^2 = \frac{M_P}{2(8\pi c/3)^{1/2} t_d} = \frac{M_P \Gamma_\phi}{2(8\pi c/3)^{1/2}}, \quad c = \frac{\pi^2}{30}\left(N_b + \frac{7}{8} N_f\right)$$

$$\Rightarrow \quad T_r = \left(\frac{32\pi c}{3}\right)^{-1/4} (M_P \Gamma_\phi)^{1/2} \propto (M_P \Gamma_\phi)^{1/2}$$

DENSITY PERTURBATIONS

Inflation provides us also with the density fluctuations
$\longrightarrow$ STRUCTURE FORMATION

Let us first introduce the notion of "event horizon":

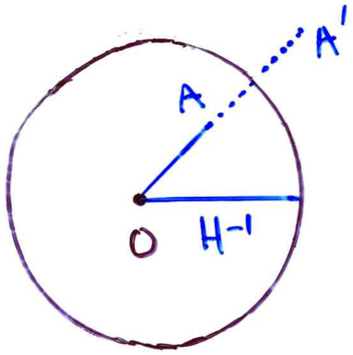
The "event horizon" at cosmic time t includes all points with which we will eventually communicate sending signals now

The instantaneous (at t) radius of "event horizon" is

$$d_e(t) = a(t) \int_t^\infty \frac{dt'}{a(t')}$$

The "event horizon" is ∞ for "matter" or "radiation"
For "inflation" though $\longrightarrow$

$$d_e(t) = \frac{1}{H} < \infty \quad \text{and slowly varying}$$



Points in the "event horizon" at t with which we can still communicate sending signals at t are eventually pulled out by the "exponential" expansion and we cannot communicate with them again starting at later times t'.

$$OA(t) \longrightarrow OA'(t') = e^{H(t'-t)}\, OA(t) \gg H^{-1}$$

We say that A (and the corresponding scale) crossed outside the "event horizon"

The situation is very similar to that of a "black hole" with the only difference that it is up-side-down: We are inside and the black hole surrounds us from all sides.

Then exactly as in a black hole there are "thermal" fluctuations governed by the "Hawking temperature"

$$T_H = \frac{H}{2\pi}$$

It turns out that the quantum fluctuations of all massless fields (inflaton is nearly massless $\longleftrightarrow$ "flat" potential) are

$$\delta\phi = \frac{H}{2\pi} = T_H$$

Fluctuations of ϕ lead to density fluctuations

$$\delta\rho = V'(\phi)\delta\phi$$

As the scale of this perturbations crosses outside the event horizon they become classical metric perturbations.

Their evolution outside the "inflationary horizon" is quite subbtle and involved due to gauge freedom in general relativity.

However, there is a simple gauge invariant quantity

$$\mathcal{J} \simeq \frac{\delta\rho}{\rho + p}$$

which remains const. outside the horizon:

$$\frac{\delta\rho}{\rho}\left(\text{when the scale crosses inside the postinflationary horizon; remember } p=0\right)$$

$$= \mathcal{J}\left(\text{when the scale crosses outside the inflationary horizon}\right)$$

$$\Longrightarrow \left(\frac{\delta\rho}{\rho}\right)_\ell = \left(\frac{\delta\rho}{\dot\phi^2}\right)_{\ell\sim H^{-1}} = \left(\frac{V'(\phi)\delta\phi}{\dot\phi^2}\right)_{\ell\sim H^{-1}}$$

$$= \left(\frac{V'(\phi)H(\phi)}{2\pi\dot\phi^2}\right)_{\ell\sim H^{-1}} = -\left(\frac{9H^3(\phi)}{2\pi V'(\phi)}\right)_{\ell\sim H^{-1}}$$

$$= 24\left(\frac{2\pi}{3}\right)^{1/2} \frac{V^{3/2}(\phi_\ell)}{M_p^3 V'(\phi_\ell)}\Bigg|_{\ell\sim H^{-1}}$$

Taking into account of an extra 2/5 factor

$$\Longrightarrow \left(\frac{\delta\rho}{\rho}\right)_\ell = \frac{16\sqrt{6\pi}}{5}\frac{V^{3/2}(\phi_\ell)}{M_p^3 V'(\phi_\ell)}\Bigg|_{\ell\sim H^{-1}}$$

How do we calculate ϕ_ℓ = the value of the inflaton field when the "comoving" scale ℓ crossed outside the event horizon?

A "comoving" (present) scale $\ell \xrightarrow{T_r} \ell \dfrac{a(t_d)}{a(t_0)} = \ell \dfrac{T_0}{T_r}$

$$\xrightarrow{t_f} \ell \frac{T_0}{T_r} \frac{a(t_f)}{a(t_d)} = \ell \frac{T_0}{T_r} \left(\frac{t_f}{t_d}\right)^{\frac{\nu+2}{3\nu}} = \ell_{phys.}(t_f)$$

But at horizon crossing it was $H^{-1}(\phi_\ell) \Longrightarrow$

$$H^{-1}(\phi_\ell)\, e^{N(\phi_\ell)} = \ell_{phys.}(t_f)$$

$\Longrightarrow$ Solving this, we calculate ϕ_ℓ and $N(\phi_\ell)$ = # of e-foldings the scale ℓ suffered during inflation.

In particular, our present horizon $\ell = 2H_0^{-1} \sim 10^4$ Mpc

$$\Longrightarrow N_{H_0} \sim 50-60$$

Now, taking $V(\phi) = \lambda \phi^\nu$ $(\nu = 4)$

$$\left(\frac{\delta\rho}{\rho}\right)_\ell = \frac{4\sqrt{6\pi}}{5} \lambda^{1/2} \left(\frac{\phi_\ell}{M_P}\right)^3 = \frac{4\sqrt{6\pi}}{5} \lambda^{1/2} \left(\frac{N_\ell}{\pi}\right)^{3/2}$$

COBE $\longrightarrow \left(\dfrac{\delta\rho}{\rho}\right)_{H_0} \simeq 6 \times 10^{-5}$

For $N_{H_0} \approx 55 \Longrightarrow \lambda \approx 6 \times 10^{-14}$

INFLATON MUST BE VERY WEAKLY COUPLED

In non-SUSY GUTs must be necessarily a gauge singlet since otherwise radiative corrections will make it strongly coupled (NOT A NICE THING!!)

In SUSY GUTs, however, it could be a conjugate pair of gauge non-singlet fields $\phi, \bar{\phi}$ with mutual cancellation of D-terms

Spectrum of fluct.: For $V(\phi) = \lambda \phi^\nu \longrightarrow \left(\frac{\delta\rho}{\rho}\right)_\ell \propto \phi_\ell^{\frac{\nu+2}{2}}$

Then $N(\phi_\ell) \propto \phi_\ell^2 \Longrightarrow \left(\frac{\delta\rho}{\rho}\right)_\ell \propto N_\ell^{\frac{\nu+2}{4}}$

$$\Longrightarrow \left(\frac{\delta\rho}{\rho}\right)_\ell = \left(\frac{\delta\rho}{\rho}\right)_{H_0} \left(\frac{N_\ell}{N_{H_0}}\right)^{\frac{\nu+2}{4}}$$

But $\frac{\ell(\mathrm{Mpc})}{10^4} = e^{N_\ell - N_{H_0}} \Longrightarrow N_\ell = N_{H_0} + \ln(\ell/10^4\,\mathrm{Mpc})$

$$\Longrightarrow \left(\frac{N_\ell}{N_{H_0}}\right)^{\frac{\nu+2}{4}} = \left(1 + \ln(\ell/10^4\,\mathrm{Mpc})^{1/N_{H_0}}\right)^{\frac{\nu+2}{4}} \approx \left(\frac{\ell}{10^4\,\mathrm{Mpc}}\right)^{\frac{\nu+2}{4N_{H_0}}}$$

So $\left(\frac{\delta\rho}{\rho}\right)_\ell = \left(\frac{\delta\rho}{\rho}\right)_{H_0} \left(\frac{\ell}{10^4\,\mathrm{Mpc}}\right)^{\frac{\nu+2}{4N_{H_0}} \equiv \alpha_s}$ "Almost scale independ."

$\alpha_s(\nu=4) \approx 0.03$ $\qquad n = 1 - 2\alpha_s$

Density fluctuations in "matter":

Introduce "conformal" time η

RW metric: $ds^2 = -dt^2 + a^2(t)\,d\bar{r}^2 \equiv a^2(\eta)(-d\eta^2 + d\bar{r}^2)$

"conformally expanding Minkowski space"

$H = \frac{\dot{a}(t)}{a(t)} = \frac{a'(\eta)}{a^2(\eta)}$, Friedmann equ: $\frac{1}{a^2}\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\rho$

Continuity equ: $\rho' = -3\tilde{H}(\rho + p)$, $\tilde{H} = a'/a$

"matter" $\rho \propto a^{-3} \longrightarrow a = \left(\frac{\eta}{\eta_0}\right)^2 \qquad \frac{a'}{a} = \frac{2}{\eta}$

"Newtonian equation" : $\delta_{\bar{k}}''(\eta) + \frac{a'}{a}\delta_{\bar{k}}'(\eta) - 4\pi G\rho a^2 \delta_{\bar{k}}(\eta) = 0$

↑ 2 is missing   ↑ a^2 present

Growing mode : $\delta_{\bar{k}}(\eta) \propto \eta^2 \propto a(\eta)$

$$\Longrightarrow \delta_{\bar{k}}(\eta) = \epsilon_H \left(\frac{k\eta}{2}\right)^2 \hat{s}(\bar{k})$$

"Gaussian random variable"

↑ amplitude at "horizon crossing"

A physical scale $\ell_{phys.}$ crosses inside the horizon

iff $\frac{\ell_{phys.}}{2\pi} = \frac{1}{H(\eta_H)} \Rightarrow \frac{a\ell}{2\pi} = \frac{1}{H} = \frac{a^2}{a'} \Longrightarrow$

$$\frac{\ell}{2\pi} \equiv \frac{1}{k} = \frac{a}{a'} = \frac{\eta_H}{2} \Longrightarrow \frac{k\eta_H}{2} = 1$$

So, at horizon crossing : $\delta_{\bar{k}}(\eta_H) = \epsilon_H \hat{s}(\bar{k})$

For scale invariant perturbations : $\epsilon_H = \text{const.}$

Now perturbations to the "scalar gravitational potential":

$$\Phi = -4\pi G \frac{a^2}{k^2}\rho\, \delta_{\bar{k}}(\eta) \qquad (\text{POISSON'S EQU.})$$

Friedmann eq. $\longrightarrow \Phi = -\frac{3}{2}\left(\frac{a'}{a}\right)^2 \frac{1}{k^2}\delta_{\bar{k}}(\eta) = \frac{3}{2}\left(\frac{2}{\eta k}\right)^2 \delta_{\bar{k}}(\eta)$

$$\Longrightarrow \quad \Phi = -\frac{3}{2}\epsilon_H \, \hat{s}(\bar{k}) \quad \text{always}$$

Gaussian Random variable $\hat{s}(\bar{k})$:

$$\langle \hat{s}(\bar{k}) \rangle = 0, \quad \langle \hat{s}(\bar{k})\, \hat{s}(\bar{k}') \rangle = \frac{1}{k^3}\, \delta(\bar{k}-\bar{k}')$$

density fluctuations are dimensionless in both $\bar{x}$, $\bar{k}$-space.

Power spectra:

$$\tilde{\delta}(\vec{x},\eta) = \int d^3k \, \delta_{\bar{k}}(\eta)\, e^{i\bar{k}\cdot\bar{x}}$$

"Correlation function" $\xi(r) \equiv \langle \tilde{\delta}^*(\bar{x},\eta)\, \tilde{\delta}(\bar{x}+\bar{r},\eta) \rangle$

$$= \int d^3k\, d^3k'\, e^{-i\bar{k}\cdot\bar{r}}\, \epsilon_H^2 \left(\frac{k\eta}{2}\right)^2 \left(\frac{k'\eta}{2}\right)^2 \langle \hat{s}(\bar{k})\, \hat{s}(\bar{k}') \rangle$$

$$= \int d^3k \, e^{-i\bar{k}\cdot\bar{r}}\, \epsilon_H^2 \left(\frac{k\eta}{2}\right)^4 \frac{1}{k^3}$$

"Spectral function" $P(k,\eta) = \epsilon_H^2 \dfrac{\eta^4}{16} k^{1=n}$

$n \equiv$ spectral index

In general: $P \propto k^n$, $n = 1 - 2\alpha_s$

SO FOR $V(\phi) = \lambda\phi^\nu$ $(\nu = 4)$

$n = 0.94$

TEMPERATURE FLUCTUATIONS

Density inhomogeneities $\longrightarrow$ temperature fluctuations in CBR

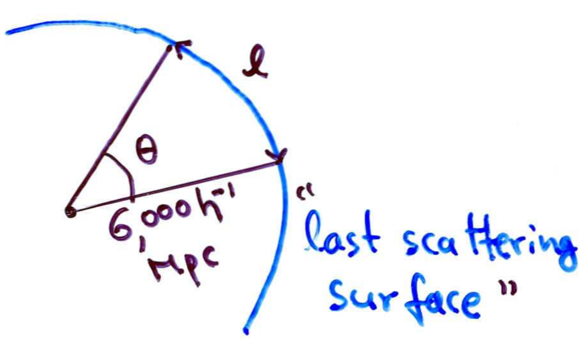
For $\theta \gtrsim 2^\circ$, the dominant effect is the "SCALAR SACHS-WOLFE" effect : Density perturbations on the "last scattering surface"
$\Longrightarrow$ scalar potential fluctations Φ
$\Longrightarrow$ temperature fluctuations (Regions with a deep gravitational potential will cause γ's to lose energy as they clim up the well $\longrightarrow$ appear cooler).

For $\theta \lesssim 2^\circ$ the dominant effects are:

(i) Motion of last scattering surface
$\longrightarrow$ Doppler shifts

(ii) Intrinsic fluctuation of T_γ (more difficult to calculate since it depends on microphysics: ionization history, γ streaming, ...)

Sachs-Wolfe effect :



$$\left(\frac{\delta T}{T}\right)_\theta = -\frac{1}{3}\,\Phi_\ell$$

$\ell =$ "comoving" scale that subtends the angle θ

$= 100\, h^{-1}$ Mpc $(\theta / \text{degrees})$

Since $\Phi = -\frac{3}{2}\epsilon_H \hat{s}(\bar{k})$ always

$$\Longrightarrow \quad \left(\frac{\delta T}{T}\right)_\theta = \frac{1}{2}\epsilon_H \hat{s}(\bar{k})$$

Also $$\left(\frac{\delta\rho}{\rho}\right)_{l=k^{-1}} \equiv \delta_{\bar{k}}(\eta_H) = \epsilon_H \hat{s}(\bar{k})$$

$$\Longrightarrow \quad \left(\frac{\delta T}{T}\right)_\theta = \frac{1}{2}\left(\frac{\delta\rho}{\rho}\right)_l$$

$\theta \approx 60^\circ \rightarrow l =$ COBE scale (present horizon)

$$\left(\frac{\delta T}{T}\right)_l \propto \left(\frac{\delta\rho}{\rho}\right)_l \propto \frac{V^{3/2}(\phi_l)}{M_p^3 V'(\phi_l)} \propto N_l^{\frac{\nu+2}{4}}$$

Analysis of temperature fluctuations in spherical harmonics $\Longrightarrow$

Quadrupole anisotropy due to the scalar Sachs-Wolfe effect:

$$\left(\frac{\delta T}{T}\right)_Q = \left(\frac{32\pi}{45}\right)^{1/2} \frac{V^{3/2}(\phi_l)}{M_p^3 V'(\phi_l)}$$

For $V(\phi) = \lambda\phi^\nu$:

$$\left(\frac{\delta T}{T}\right)_Q = \left(\frac{32\pi}{45}\right)^{1/2} \frac{\lambda^{1/2}\phi_l^{\frac{\nu+2}{2}}}{\nu M_p^3} = \left(\frac{32\pi}{45}\right)^{1/2} \frac{\lambda^{1/2}}{\nu M_p^3}\left(\frac{\nu M_p^2}{4\pi}\right)^{\frac{\nu+2}{2}} N_l^{\frac{\nu+2}{2}}$$

COBE $\rightarrow \left(\frac{\delta T}{T}\right)_Q \approx 6.6\times 10^{-6} \Longrightarrow \lambda \approx 6\times 10^{-14}$ (for $\nu = 4$)

$n \approx 1.1 \pm 0.2$, $\epsilon_H \approx 6\times 10^{-5}$

There are also "tensor" (or gravitational wave) fluctations in the temperature of CBR:

The quadrupole tensor anisotropy is

$$\left(\frac{\delta T}{T}\right)_{Q-T} \approx 0.77 \frac{V^{1/2}(\phi_e)}{M_p^2}$$

The total quadrupole anisotropy is given by

$$\left(\frac{\delta T}{T}\right)_{Q} = \left[\left(\frac{\delta T}{T}\right)_{Q-S}^2 + \left(\frac{\delta T}{T}\right)_{Q-T}^2\right]^{1/2}$$

Define $$r = \frac{(\delta T/T)_{Q-T}^2}{(\delta T/T)_{Q-S}^2} \approx 0.27 \frac{M_p^2 V'^2(\phi_e)}{V^2(\phi_e)}$$

for $V(\phi) = \lambda \phi^\nu \longrightarrow \quad r \approx \frac{3.4\nu}{N_H} \ll 1$

Non-SUSY hybrid Inflation

"New" and "chaotic" inflation have two important and related disadvantages: They require very small coupling constants ($\sim 10^{-14}$) to reproduce $\delta T/T$ from COBE and the inflaton is necessarily a gauge singlet (non-SUSY) and, thus, not directly related to the GUT breaking.

Hybrid Inflation (LINDE): Two real scalar fields σ, χ

σ = the "inflaton" (slowly varying field)

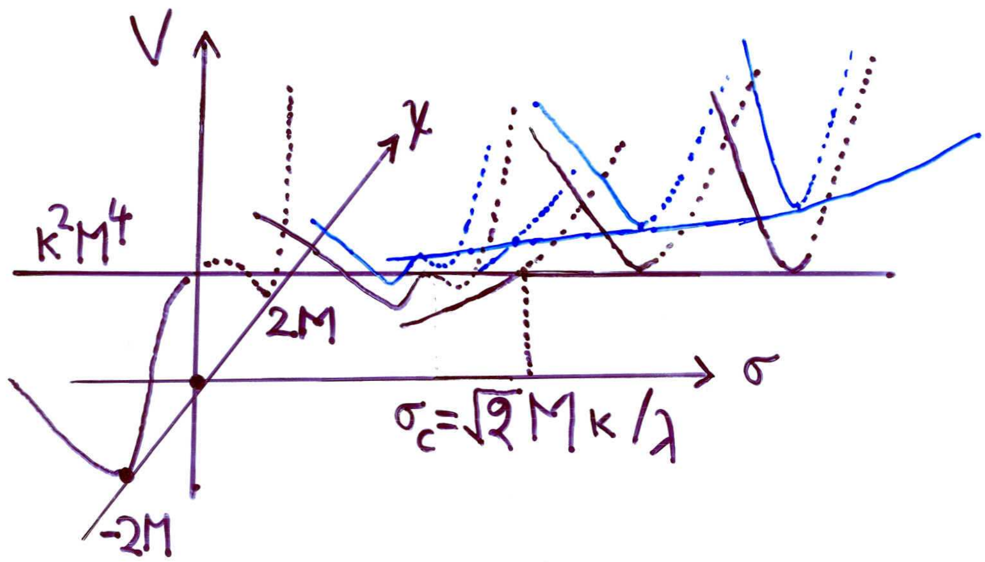
χ = provides the vacuum energy for inflation (gauge non-singlet)

Main Advantage : produces $\frac{\delta T}{T}$ with "natural" values of parameters

Potential: $V(\chi,\sigma) = \kappa^2\left(M^2 - \frac{\chi^2}{4}\right)^2 + \frac{\lambda^2\chi^2\sigma^2}{4} + \frac{1}{2}m^2\sigma^2$

Vacua : $\langle\chi\rangle = \pm 2M \quad , \quad \langle\sigma\rangle = 0$

For $m=0 \longrightarrow$ Flat direction at $\chi=0, \forall\sigma$ with $V=\kappa^2 M^4$



$$m_\chi^2(\chi=0) = -\kappa^2 M^2 + \frac{1}{2}\lambda^2\sigma^2 \geqslant 0 \quad \text{for } \sigma \geqslant \sigma_c = \frac{\sqrt{2}\,\kappa M}{\lambda}$$

Reintroducing $m \neq 0 \longrightarrow$ Flat direction acquires a slope and the system can inflate along the "valley of minima"

It is called "HYBRID" since: (i) Vacuum energy $=\kappa^2 M^4 \longleftrightarrow \chi$

(ii) Inflaton (slowly rolling down) $\longleftrightarrow \sigma$

ε, η criteria $\longrightarrow$ Inflation continues till $\sigma=\sigma_c$ (for the parameters below)

At $\sigma=\sigma_c$, INFLATION TERMINATES ABRUPTLY

$\Longleftrightarrow$ "WATERFALL" $\longrightarrow$ copious production of topological defects !!

So we have to make sure that the underlying symmetry breaking does not predict the existence of cosmologically disastrous topological defects like "Magnetic Monopoles" or "Domain Walls".

$$\text{Now}\quad \left(\frac{\delta T}{T}\right)_Q \approx \left(\frac{32\pi}{45}\right)^{1/2} \frac{V^{3/2}}{M_p^3 V'(\sigma)} \approx \left(\frac{16\pi}{45}\right)^{1/2} \frac{\lambda \kappa^2 M^4}{M_p^3 m^2} \approx 6.6 \times 10^{-6}$$

$$\Longrightarrow \quad m = \kappa\sqrt{\lambda} \cdot 1.3 \times 10^{15}\,\text{GeV} \sim 10^{12}\,\text{GeV} \quad (\kappa, \lambda \sim 10^{-2})$$

$$\text{for} \quad M = 2.86 \times 10^{16}\,\text{GeV} \; \left(\text{MSSM}: \; M_X \approx 2 \times 10^{16}\,\text{GeV}, \, g \approx 0.7\right)$$

SUSY Hybrid Inflation

Hybrid Inflation is 'talor-made' for SUSY GUTs.

Consider the globally supersymmetric renormalizable superpotential

$$W = \kappa S(\bar{\phi}\phi - M^2) \qquad (\kappa > 0, \; M > 0)$$

S is a gauge singlet LH superfield

$\bar{\phi}, \phi$ are the SM singlet components of a conjugate pair of $SU(2)_R \times U(1)_{B-L}$ doublet LH superfields:

$$\begin{matrix} SO(10) \\ SU(3)^3 \\ \vdots \end{matrix} \supseteq \left(SU(3)_c \times SU(2)_L \times\right) SU(2)_R \times U(1)_{B-L} \xrightarrow[M]{\langle\bar{\phi}\rangle, \langle\phi\rangle} U(1)_Y$$

W is the most general renormalizable superpotential allowed by the $U(1)$ R-symmetry:

$$S \longrightarrow e^{i\alpha} S, \quad W \longrightarrow e^{i\alpha} W, \quad \bar{\phi}\phi \longrightarrow \bar{\phi}\phi$$

W gives rise to the potential

$$V(\bar{\phi}, \phi, S) = \kappa^2 |S|^2 \left(|\bar{\phi}|^2 + |\phi|^2\right) + \kappa^2 |\bar{\phi}\phi - M^2|^2 + \text{D-terms}$$

D-terms vanish along the 'D-flat' direction $\bar{\phi}^* = \phi$ which contains the SUSY vacua.

By appropriate gauge and R transformations $\longrightarrow$

$$S = \frac{\sigma}{\sqrt{2}} \quad , \quad \bar{\phi} = \phi = \frac{\chi}{2} \quad (\sigma, \chi = \text{normalized real scalars})$$

V then takes Linde's form but without the mass term of σ (SUSY breaking allows such a term but with very small magnitude $\sim m_{3/2} \sim 1$ TeV). Also $\kappa = \lambda$ from SUSY.

The necessary slope along the "inflationary trajectory" is produced in this case by 'Radiative Corrections'.

Observe that on the 'flat valley' SUSY is broken $(V \neq 0)$

$\longrightarrow$ fermion-boson mass splitting:

$\bar{\phi}, \phi$ fermions: $m = \kappa S$

$\bar{\phi}, \phi$ bosons: $m^2 = \kappa^2 S^2 \pm \kappa^2 M^2$

$\Longrightarrow$ Non-trivial 'Radiative Corrections' (to one loop)

$$V_{\text{eff}}(S) = \kappa^2 M^4 \left[1 + \frac{\kappa^2}{16\pi^2} \left(\ln\left(\frac{\kappa^2 S^2}{\Lambda^2}\right) + \frac{3}{2} - \frac{S_c^4}{12 S^4} + \cdots\right)\right], \quad S_c = M$$

providing a logarithmic slope driving the 'inflaton' towards the SUSY vacua.

The 'quadrupole anisotropy' of CBR $\longrightarrow$

$$\left(\frac{\delta T}{T}\right)_Q \approx 8\pi \left(\frac{N_Q}{45}\right)^{1/2} \frac{x_Q}{y_Q} \left(\frac{M}{M_P}\right)^2$$

$N_Q \approx 56$ is the # of 'e-foldings' of the present horizon size during inflation

$$y_Q = x_Q \left(1 - \frac{7}{12 x_Q^2} + \cdots\right)$$

$x_Q = \dfrac{S_Q}{M}$ with S_Q = value of S when the present horizon crossed outside the 'inflationary' horizon.

Also $\kappa = \dfrac{8\pi^{3/2}}{\sqrt{N_Q}}\, y_Q \dfrac{M}{M_P}$ $(\sim 4.5 \times 10^{-3})$

Now $1 \leqslant x_Q \lesssim 2.6$ (as it turns out) $\longrightarrow$ Take $x_Q \approx 2$ for definiteness

COBE $\left(\left(\frac{\delta T}{T}\right)_Q \approx 6.6 \times 10^{-6}\right) \Longrightarrow M \approx 5.5 \times 10^{15}\,\mathrm{GeV}$

Inflation ends as $S = S_c$ (ε, η-criteria) and the transition $SU(2)_R \times U(1)_{B-L} \to U(1)_Y$ takes place (no topological defects). The system falls towards the SUSY minima oscillates about them and eventually decays 'reheating' the universe. The "oscillating system" consists of two complex scalar fields S and $\theta = \frac{1}{\sqrt{2}}(\delta\bar{\phi} + \delta\phi)$ $(\delta\bar{\phi} = \bar{\phi} - M,\ \delta\phi = \phi - M)$ with common mass $m_{infl} = \sqrt{2}\,\kappa M$.

SUSY hybrid inflation is "natural":

(a) No need of 'tiny' coupling constants

(b) W has the most general renormalizable form allowed by gauge and R-symmetries

(c) SUSY $\to$ Radiative corrections do not invalidate inflation, but rather provide a slope along the 'inflationary' trajectory driving the inflaton.

(d) "SUGRA" corrections are under control $(S \ll M_P)$ so as to leave inflation intact

BAU can be produced via a primordial "Leptogenesis" taking place during the "decay" of the "inflaton system".