Sigma models for genuinely non-geometric backgrounds

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arXiv:1311.4878[hep-th], arXiv:1505.05457[hep-th]

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Quantum Structure of Spacetime

Noncommutative Field Theory and Gravity,
September 22nd, 2015, Corfu

Motivation

- Quantum nature of space-time?
- New symmetries - dualities in string theory, non-commutative gauge theory
- Emergent standard symmetries, emergent space-time

Recent ideas

$\rightsquigarrow$ Non-commutative geometry, matrix models

$\rightsquigarrow$ Holography

$\rightsquigarrow$ Generalized geometry, Double field theory

In this talk

- T-duality, as an exact symmetry of perturbative string theory, uncovered existence of "non-geometric" backgrounds.

  generalized T-duality:

$$H^{(0,3)} \xleftrightarrow{T} f^{(1,2)} \xleftrightarrow{T} Q^{(2,1)} \xleftrightarrow{T?} R^{(3,0)}$$

- Appropriate mathematical setting for describing these new geometries is that of Lie and Courant algebroids (Hitchin's generalized geometry).

> **The main question**
>
> Can we construct backgrounds that are NOT related by dualities to geometric ones?

Overview

- Courant algebroid - very brief intro
- Membrane sigma models
- One step beyond
- Outlook

Generalized geometry & Courant algebroid

GG allows organization and extension of the geometric transformations of the fields of a theory, i.e. diffeomorphisms and gauge transformations, to $\mathrm{O}(\mathrm{d},\mathrm{d})$ transformations.

- Extend the standard tangent bundle to $\mathcal{T}\mathsf{M} = \mathsf{TM} \oplus \mathsf{T}^\star\mathsf{M}$
- Sections: $\mathfrak{X} \in \Gamma(\mathcal{T}\mathsf{M})$: $\mathfrak{X} = X + \eta$,
- Courant bracket:
  $[\mathfrak{X}_1, \mathfrak{X}_2]_{\mathcal{T}\mathsf{M}} = [X_1, X_2]_{\mathsf{Lie}} + \mathcal{L}_{X_1}\eta_2 - \mathcal{L}_{X_2}\eta_1 - \frac{1}{2}\mathrm{d}\big(X_1(\eta_2) - X_2(\eta_1)\big)$.
- Pairing: $\langle \mathfrak{X}_1, \mathfrak{X}_2\rangle = \frac{1}{2}\big(X_1(\eta_2) + X_2(\eta_1)\big)$.
- Anchor: $\rho(\mathfrak{X}) = X$.

Plus some compatibility condition between structures. Liu, Weinstein, Xu '97

Generically (Roytenberg '99) take a vector bundle L and its dual $L^\star$ with closed bracket and anchor map on each of them and a pair of generalized 3-forms $\phi \in \Gamma(\wedge^3 L^\star)$ and $\psi \in \Gamma(\wedge^3 L)$. s.t.

- $[X, fY]_L = f[X, Y]_L + (\rho(X)f)Y$, $f \in C^\infty(M)$,
- $\rho([X, Y]_L) = [\rho(X), \rho(Y)]_{\text{Lie}} + \rho_\star\phi(X, Y, \cdot)$,
- $\text{Jac}[X, Y, Z]_L =$
  $\mathrm{d}_{L^\star}\phi(X, Y, Z) + \phi(\mathrm{d}_{L^\star}X, Y, Z) + \phi(X, \mathrm{d}_{L^\star}Y, Z) + \phi(X, Y, \mathrm{d}_{L^\star}Z)$,
- $\mathrm{d}_L\phi = 0$ and $\mathrm{d}_{L^\star}\psi = 0$,

and similarly for $L^\star$.

The bracket of the CA is

$$\begin{aligned}[X+\eta, Y+\xi]_E &= [X, Y]_L + \mathcal{L}_X\xi - \mathcal{L}_Y\eta - \tfrac{1}{2}\mathrm{d}_L(X(\xi) - Y(\eta)) \\ &\quad + [\eta, \xi]_{L^\star} + \mathcal{L}_\eta Y - \mathcal{L}_\xi X + \tfrac{1}{2}\mathrm{d}_{L^\star}(X(\xi) - Y(\eta)) - \\ &\quad - \phi(X, Y, \cdot) - \psi(\eta, \xi, \cdot) \ .\end{aligned}$$

A class of Courant algebroids

Consider as M a nilmanifold (geometric flux built-in).

$$\begin{aligned} \mathrm{TM} &= \mathrm{span}\{\theta_i = e_i^a(x)\partial_a = \delta_i^a(\partial_a + f_{ba}^c x^b \partial_c)\}\ , \\ \mathrm{T^\star M} &= \mathrm{span}\{e^i = e_a^i(x)\mathrm{d}x^a = \delta_a^i(\mathrm{d}x^a - f_{bc}^a x^b \mathrm{d}x^c)\}\ . \end{aligned}$$

Take (non-closed) 2-form and (non-Poisson) 2-vector fields:

$$B = \tfrac{1}{2}B_{ij}e^i \wedge e^j\ , \quad \beta = \tfrac{1}{2}\beta^{ij}\theta_i \wedge \theta_j\ ,$$

and deform the bundles with the element $e^B e^\beta$:

$$\begin{aligned} L &= e^B e^\beta \mathrm{TM} = \mathrm{span}\{\Theta_i = \theta_i + B_{ij}e^j\}\ , \\ L^\star &= e^B e^\beta \mathrm{T^\star M} = \mathrm{span}\{E^i = e^i + \beta^{ik}B_{kj}e^j + \beta^{ij}\theta_j\}\ . \end{aligned}$$

Brackets, anchors, generalized 3-forms, via $e^B e^\beta$ deformations.

The twists

A generic Courant algebroid includes twists $\phi \in \Gamma(\wedge^3 L^\star)$ and $\psi \in \Gamma(\wedge^3 L)$.

Consider the expansions:

$$
\begin{aligned}
\phi &= \tfrac{1}{6}\phi_{ijk} E^i \wedge E^j \wedge E^k \\
&= \tfrac{1}{6}\big((1+\beta B)^i_\rho (1+\beta B)^j_\sigma (1+\beta B)^k_\tau \phi_{ijk} e^\rho \wedge e^\sigma \wedge e^\tau \\
&\quad +3(1+\beta B)^i_\rho (1+\beta B)^j_\sigma \beta^{kl} \phi_{ijk} e^\rho \wedge e^\sigma \wedge \theta_l \\
&\quad +3(1+\beta B)^i_\rho \beta^{jl}\beta^{km}\phi_{ijk} e^\rho \wedge \theta_l \wedge \theta_m \\
&\quad +\beta^{il}\beta^{jm}\beta^{kn}\phi_{ijk}\theta_l \wedge \theta_m \wedge \theta_n\big) \ , \\
\psi &= \tfrac{1}{6}\psi^{ijk}\Theta_i \wedge \Theta_j \wedge \Theta_k \\
&= \tfrac{1}{6}\big(\psi^{ijk}\theta_i \wedge \theta_j \wedge \theta_k \\
&\quad +3B_{kn}\psi^{ijk}\theta_i \wedge \theta_j \wedge e^n \\
&\quad +3B_{jm}B_{kn}\psi^{ijk}\theta_i \wedge e^m \wedge e^n \\
&\quad +B_{il}B_{jm}B_{kn}\psi^{ijk} e^l \wedge e^m \wedge e^n\big) \ .
\end{aligned}
$$

$\rightsquigarrow$ From the twisted torus viewpoint *all* types of twists T_{ijk}, T_{ij}^k, T_i^{jk}, T^{ijk} are present.

From Courant algebroids to AKSZ σ-models

Every Courant algebroid has an associated 3D AKSZ sigma model. Roytenberg '06

AKSZ: Unification of a large class of TFTs in the spirit of BV quantization.
Alexandrov, Kontsevich, A. Schwarz, Zaboronsky '95

Turn out to be relevant in the description of non-geometric backgrounds.
Halmagyi '09, Mylonas, Schupp, Szabo '12, Chatzistavrakidis, L.J., Lechtenfeld '15

Membrane σ-models

Membrane action (3D, bosonic fields):

$$S_{\Sigma_3} = \int_{\Sigma_3} \left(F_a \wedge \mathrm{d}X^a + \tfrac{1}{2}\eta_{IJ} A^I \wedge \mathrm{d}A^J - \rho^a_I A^I \wedge F_a + \tfrac{1}{6} T_{IJK} A^I \wedge A^J \wedge A^K \right) .$$

$X^a : \Sigma_3 \to M$ worldvolume scalars. F_a: auxiliary worldvolume 2-form. ρ: the anchor A^I: generalized 1-form, $I = 1, \ldots, 2d$. η_{IJ}: $O(d,d)$-invariant metric. T: generalized 3-form.

The 3-form T systematically includes all types of 3-elements "H, f, Q, R".

Add general topological boundary term

$$S_{\partial\Sigma_3,\mathrm{top}} = \int_{\partial\Sigma_3} \tfrac{1}{2} \mathcal{B}_{IJ}(X) A^I \wedge A^J .$$

In the boundary 2D theory, add dynamics, e.g. $S_{\partial\Sigma_3,\mathrm{kin}} = \int_{\partial\Sigma_3} \frac{1}{2} g_{ab} \mathrm{d}X^a \wedge \star \mathrm{d}X^b$.

The sigma model for $E = L \oplus L^\star$

For our class of CAs,

- $\rho_i^a = e_i^a(X)$, $\rho^{ai} = \beta^{ij}(X) e_j^a(X)$.
- $A^I = (q^i, p_i)$.
- $T = f - \phi - \psi$, with $f = \frac{1}{2} f_{ij}^{k} q^i \wedge q^j \wedge p_k$.
- $\mathcal{B}_{ij} = B_{ij}(X)$, $\mathcal{B}^{ij} = \beta^{ij}(X)$, $\mathcal{B}_i^j = h_i^j(X)$.

The action is:

$$S = \int_{\Sigma_3} \Big(F_a \wedge \mathrm{d}X^a + q^i \wedge \mathrm{d}p_i + p_i \wedge \mathrm{d}q^i - (e_i^a q^i + \beta^{ij} e_j^a p_i) \wedge F_a + f - \phi - \psi \Big)$$
$$+ \int_{\Sigma_2} \Big(\tfrac{1}{2} B_{ij}(X) q^i \wedge q^j + \tfrac{1}{2} \beta^{ij}(X) p_i \wedge p_j + \tfrac{1}{2} h_j^i(X) q^j \wedge p_i \Big) .$$

For consistency, the boundary conditions should match the equations of motion on $\partial\Sigma_3$.

- ⇝ vary the action w.r.t. X^a, q^i, p_i and determine BCs such that the variations vanish.
- ⇝ ensure that 3D terms which did not reduce to the boundary vanish on it.

Boundary conditions and consistency

Bulk/boundary consistency conditions:

- for $\delta p_i|_{\Sigma_2} = 0$:

  $$\mathcal{H}_{ijk} - \mathcal{F}^n_{[ij}B_{\underline{p}k]}\chi'^p_n + \mathcal{Q}^{mn}_{[i}B_{\underline{p}j}B_{\underline{q}k]}\chi'^p_m\chi'^q_n - \mathcal{R}^{lmn}B_{pi}B_{qj}B_{rk}\chi'^p_l\chi'^q_m\chi'^r_n = 0 \ ,$$

  where $\chi' = 1 + \frac{1}{2}h$.

- for $\delta q^i|_{\Sigma_2} = 0$:

  $$\mathcal{R}^{ijk} - \mathcal{Q}^{[ij}_n\beta^{\underline{p}k]}\chi^n_p + \mathcal{F}^{[i}_{mn}\beta^{\underline{p}j}\beta^{\underline{q}k]}\chi^m_p\chi^n_q - \mathcal{H}_{lmn}\beta^{pi}\beta^{qj}\beta^{rk}\chi^l_p\chi^m_q\chi^n_r = 0 \ ,$$

  where $\chi = 1 - \frac{1}{2}h$.

$$\begin{aligned}
\mathcal{R}^{ijk} &= \psi^{ijk} - 3\beta^{[i\underline{l}}\theta_l\beta^{jk]} + \beta^{li}\beta^{mj}\beta^{nk}\phi_{lmn} \ , \\
\mathcal{Q}^{ij}_k &= -3\theta_k\beta^{ij} + 3\beta^{[i\underline{l}}\theta_l h^{j]}_k + 3B_{lk}\psi^{ijl} + 3(1+\beta B)^l_k\beta^{mi}\beta^{nj}\phi_{lmn} \ , \\
\mathcal{F}^i_{jk} &= -3\theta_{[j}h^i_{k]} - 3f^i_{jk} - 3\beta^{il}\theta_l B_{jk} + 3B_{lj}B_{mk}\psi^{lmi} + 3(1+\beta B)^l_j(1+\beta B)^m_k\beta^{ni}\phi_{lmn} \\
\mathcal{H}_{ijk} &= (1+\beta B)^l_i(1+\beta B)^m_j(1+\beta B)^n_k\phi_{lmn} - 3\theta_{[i}B_{jk]} + B_{li}B_{mj}B_{nk}\psi^{lmn} \ .
\end{aligned}$$

Relation to Dirac structures and integrability

The dictionary between these σ-models and Courant algebroids is:

Courant algebroid	Sigma model
Bracket twist $[\cdot,\cdot]_T$	Bulk term $-\int_{\Sigma_3} T$
Dirac structure deformation $L_{\mathcal{B}}$	Boundary term $\int_{\partial\Sigma_3} \mathcal{B}$
Integrability condition for Dirac structure	Bulk/boundary consistency condition

Notably, the b/b conditions are also integrability conditions for Dirac structures.

This generalizes previous results. Severa, Weinstein '01

The corresponding 2D field theories belong to the class of Dirac sigma models.

However although they may contain e.g. 3-vector fluxes, they are not the ones that appear in string theory via generalized T-duality. Halmagyi '09

AKSZ for R flux

Mylonas, Schupp, Szabo '12

The AKSZ membrane action:

$$\int_{\Sigma_3}\Big(F_a\wedge \mathrm{d}X^a + q^i\wedge \mathrm{d}p_i - \delta_i^a q^i\wedge F_a + \tfrac{1}{6}R^{ijk}p_i\wedge p_j\wedge p_k\Big)\ .$$

Integrating out the auxiliary 2-form F_a, the EOM for X^i on $\partial\Sigma_3$ for $R=$ const. requires

$$\mathrm{d}p_a = 0 \quad \Rightarrow \quad p_a = \mathrm{d}P_a\ , \quad \text{locally}\ ,$$

with $P_a\in C^\infty(\Sigma_3, X^\star \mathrm{T}^\star \mathrm{M})$.

Very suggestive!

Generalized world volume coordinates:

$X^I=(X^a,P_a)$, or doubled formulation of closed string theory $X^I=(X^a,\tilde{X}_a)$.
Tseytlin '90

One step beyond

Chatzistavrakidis, L.J., Lechtenfeld '15

Embed the membrane theory in phase space of M from the beginning.

Minimal generalization of AKSZ action:

$$S=\int_{\Sigma_3}\Big(F_a\wedge \mathrm{d}X^a+\tilde{F}^a\wedge\mathrm{d}P_a+\tfrac{1}{2}\eta_{IJ}A^I\wedge\mathrm{d}A^J-\rho^a_I A^I\wedge F_a-\tilde{\rho}_{aI}A^I\wedge\tilde{F}^a+\tfrac{1}{6}T_{IJK}A^I\wedge A^J\wedge A^K\Big)$$

with the same boundary action as before, but $\mathcal{B}=\mathcal{B}(X,P)$.

$\tilde{F}^a$ is also an auxiliary world volume 2-form; the map $\tilde{\rho}: E\to \mathrm{T}^\star\mathrm{M}$.

In more compact notation, writing $F^I=(F_a,\tilde{F}^a)$ and $\rho^J_I=(\rho^a_I,\tilde{\rho}_{aI})$

$$S_{\Sigma_3}=\int_{\Sigma_3}\Big(\delta_{IJ}F^I\wedge\mathrm{d}X^J+\tfrac{1}{2}\eta_{IJ}A^I\wedge\mathrm{d}A^J-\delta_{JK}\rho^J_I A^I\wedge F^K+\tfrac{1}{6}T_{IJK}A^I\wedge A^J\wedge A^K\Big)\,.$$

A similar analysis as before, yields extended bulk/boundary consistency conditions.

We find an apparent similarity to the flux formulation of double field theory.

Geissbuhler '11, Aldazabal, Baron, Marques, Nunez '11; ...

Genuine non-geometry?

Can we go beyond the geometric orbit?

$$H^{(0,3)} \xleftrightarrow{T} f^{(1,2)} \xleftrightarrow{T} Q^{(2,1)} \xleftrightarrow{T?} R^{(3,0)}$$

We can construct models which:

- Satisfy the extended bulk/boundary consistency conditions for the new σ-model,
- Contain all types of fluxes,
- Cannot be reduced to standard 2D theories in any (standard) duality frame (only to theories with both X and P).

We need:

- Geometric interpretation of 3D sigma model- relation with work on topological T-duality. Bouwknegt, Evslin, Mathai '04, Cavalcanti, Gualtieri '11, Barmaz '13
- Physical analysis of boundary sigma model - Dirac sigma models.