Quantum Non-commutative Geometry

by John Barrett
University of Nottingham

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Overview

$$
\begin{tikzcd}
\text{NCG} \arrow[rr, "\text{functional integral}"] \arrow[dr, leftrightarrow, "\text{Diagram calculus}"'] & & \text{QNCG} \arrow[dl, "\text{Feynman diagrams}"] \\
& \text{TQFT with defects} &
\end{tikzcd}
$$

Dirac operator

- fundamental in geometry
- fundamental in physics

$$D_M = \gamma^a e_a^\mu \nabla_\mu \longleftrightarrow e_a^\mu \quad \text{metric of } M$$

$$D = D_M + A + yH + N \quad \text{in SM}$$

Functional integral

$$\int_{D \in \mathcal{G}} \dots \; dD$$

$\mathcal{G}$: space of all Dirac operators

e.g. $[[D,a],b] = 0$ for all $a,b \in \mathcal{A} = C^\infty(M)$

- Seems impossible

Planck scale

Minimum length $\longleftrightarrow$ Maximum eigenvalue for D

Operator cutoff vs Finite # modes

$\Downarrow$ large Λ

$\Downarrow$? no large Λ

NC geometry

$\mathcal{A}$ commutative, finite-dimensional

$$\Rightarrow \quad \mathcal{A} = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \cdots$$

lattice

$\mathcal{A}$ non-C, finite-dimensional

$$\Rightarrow \quad \mathcal{A} = M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots$$

e.g. SM internal space

Example: S^2

$$\mathcal{A}_{S^2} = C(S^2) = \bigoplus_j \mathrm{Harm}(j)$$

f spin j $\Rightarrow$ f^2 spin $0, 1, \dots, 2j$

- cutoff not compatible with product in $\mathcal{A}_{S^2}$

Fuzzy S^2

$$\mathcal{A} = M_n(\mathbb{C}) \cong \bigoplus_{j=0}^{n-1} \mathrm{Harm}(j)$$

- cutoff compatible with matrix multiplication
- $\approx$ commutative for $j \ll n-1$

Dirac for fuzzy S^2

$$\mathcal{A} = M_n(\mathbb{C})$$

$$\mathcal{H} = \mathbb{C}^4 \otimes M_n(\mathbb{C})$$

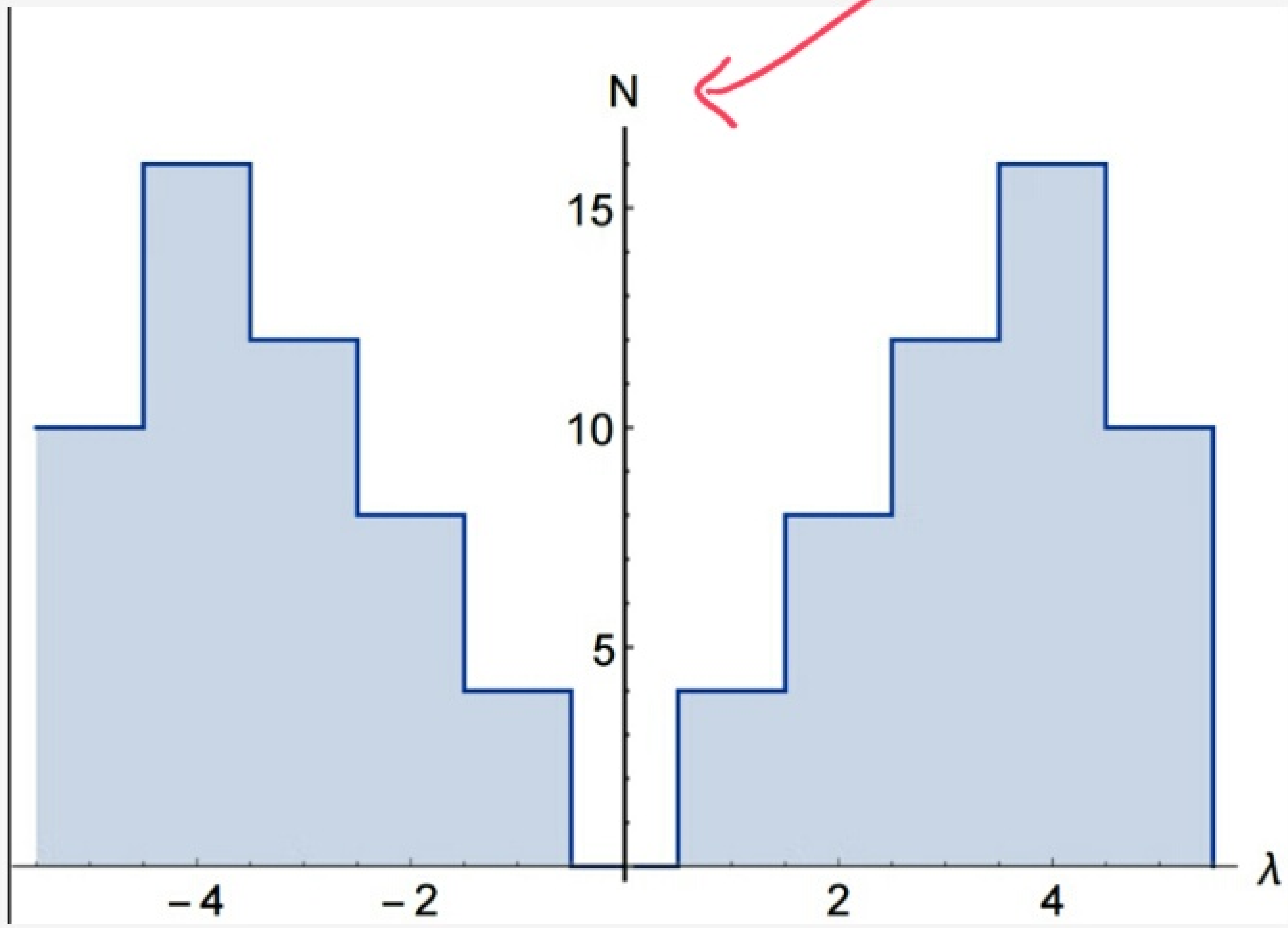
"real spectral triple"

$$D = \gamma^0 + \gamma^0\gamma^1\gamma^2 \otimes [L_{12}, \cdot\,] + \gamma^0\gamma^1\gamma^3 \otimes [L_{13}, \cdot\,] + \gamma^0\gamma^2\gamma^3 \otimes [L_{23}, \cdot\,]$$

L_{ij} $su(2)$ generators

γ^μ type $(1,3)$ gamma matrices

Spectrum of D $(n=5)$



multiplicity

L. Glaser

- $\dim \mathcal{H} = 4 \cdot 5^2 = 100$
- same spectrum as $2\times$ fermion on S^2

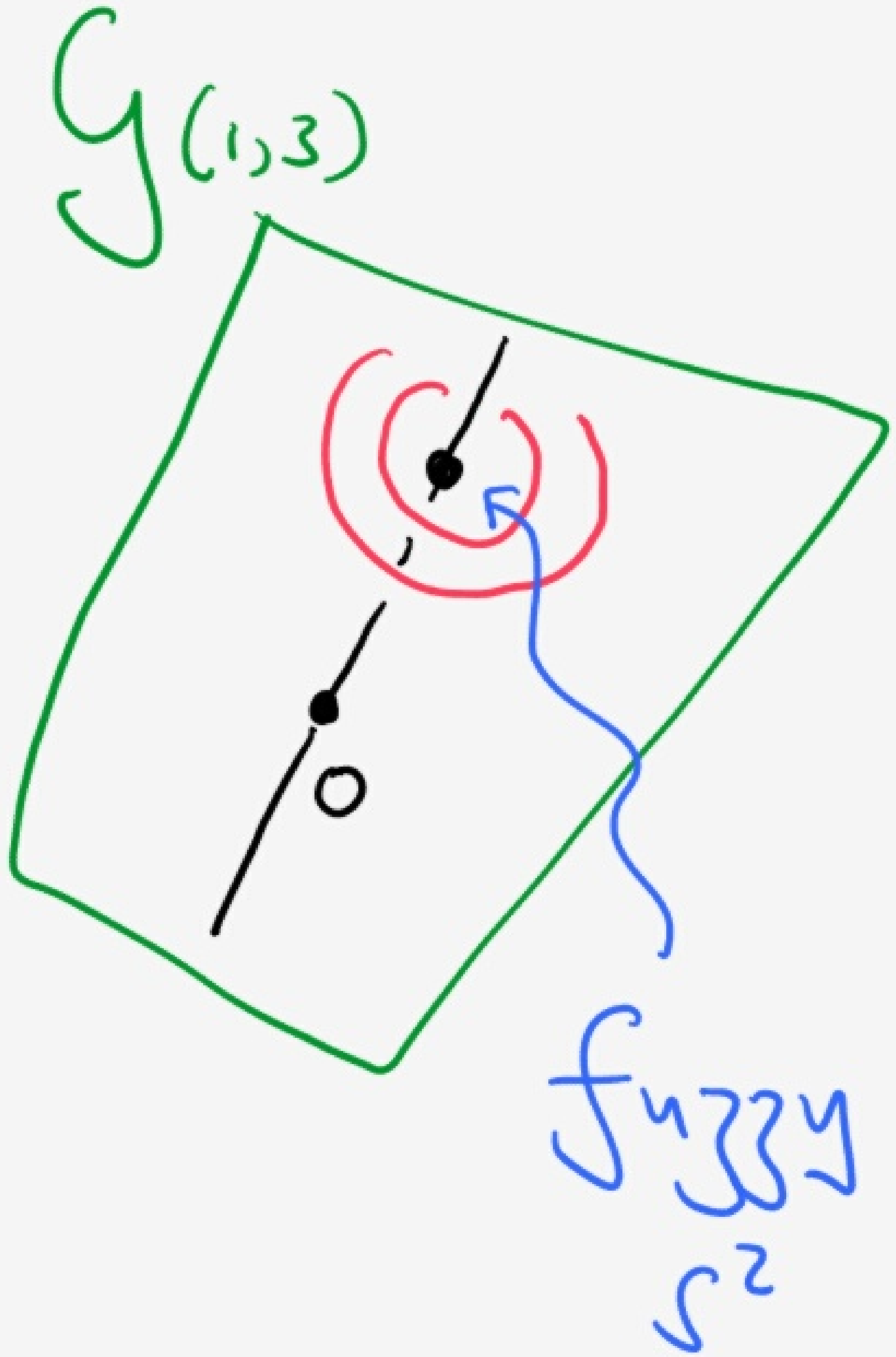
Classical limit

$$x_1 = \frac{1}{n^2}\left\{ L_{23}, \cdot \right\} \qquad \text{etc.}$$

$$[x_i, x_j] \sim \frac{1}{n^2} \longrightarrow 0 \quad \text{as } n \to \infty$$

$$x_1^2 + x_2^2 + x_3^2 \longrightarrow 1$$

Fuzzy spaces



$$\mathcal{A} = M_n(\mathbb{C})$$

γ^μ type (p,q) acting in $V = \mathbb{C}^k$

$$\mathcal{H} = V \otimes M_n(\mathbb{C})$$

$D \in \mathcal{G}$, vector space

Fuzzy space Dirac operator

$$D(v \otimes m) = \sum_i (\gamma^i v) \otimes [L_i, m]_\pm$$
$$+ \sum_{ijk} (\gamma^i \gamma^j \gamma^k v) \otimes [L_{ijk}, m]_\pm$$
$$+ \ldots$$

using
- $[L, m]_\pm = Lm \pm mL$
- $L_i, L_{ijk}, \ldots$ Hermitian/anti-H as $+/-$.

free data

... solving axioms

$$D = D^*$$

$$[[D, a], Jb^*J^{-1}] = 0 \qquad \forall a, b \in A$$

$$DJ = \pm JD$$

$$D\Gamma = \pm' \Gamma D$$

Random NC geometry (= Euclidean QNCG)

Fix p, q, $\mathcal{A} = M_n(\mathbb{C})$

$$Z = \int_{D \in \mathcal{G}} dD \, e^{-S(D)}$$

$S \geq 0$, $S \to \infty$ as $D \to \infty$

Numerical results: see talk by Lisa Glaser

Spectral action

$$S(D) = \mathrm{tr}\, V(D) = \sum_{\lambda} V(\lambda)$$

$$\Rightarrow \quad S(UDU^{-1}) = S(D)$$

gauge & diffeo symmetry

Examples

- $V(D) = D^2$
- $V(D) = D^4 + g D^2$

Need $V(\lambda) \to \infty$ as $\lambda \to \infty$ (i.e., not C-C)

Euclidean QG + matter

$q - p = 2 \mod 8$

$$Z = \int dD \, e^{-S(D)} \int d\psi \, e^{\frac{1}{2}\langle J\psi, D\psi\rangle}$$

$\psi \in \mathcal{H}_+ : \quad \Gamma\psi = \psi$

J antilinear, $J^2 = -1$

$\Gamma^2 = 1$ "chirality"

$\int d\psi$ Berezin $\rightsquigarrow \operatorname{Pf} D = (\det D)^{1/2}$ No doubling!

Feynman diagrams

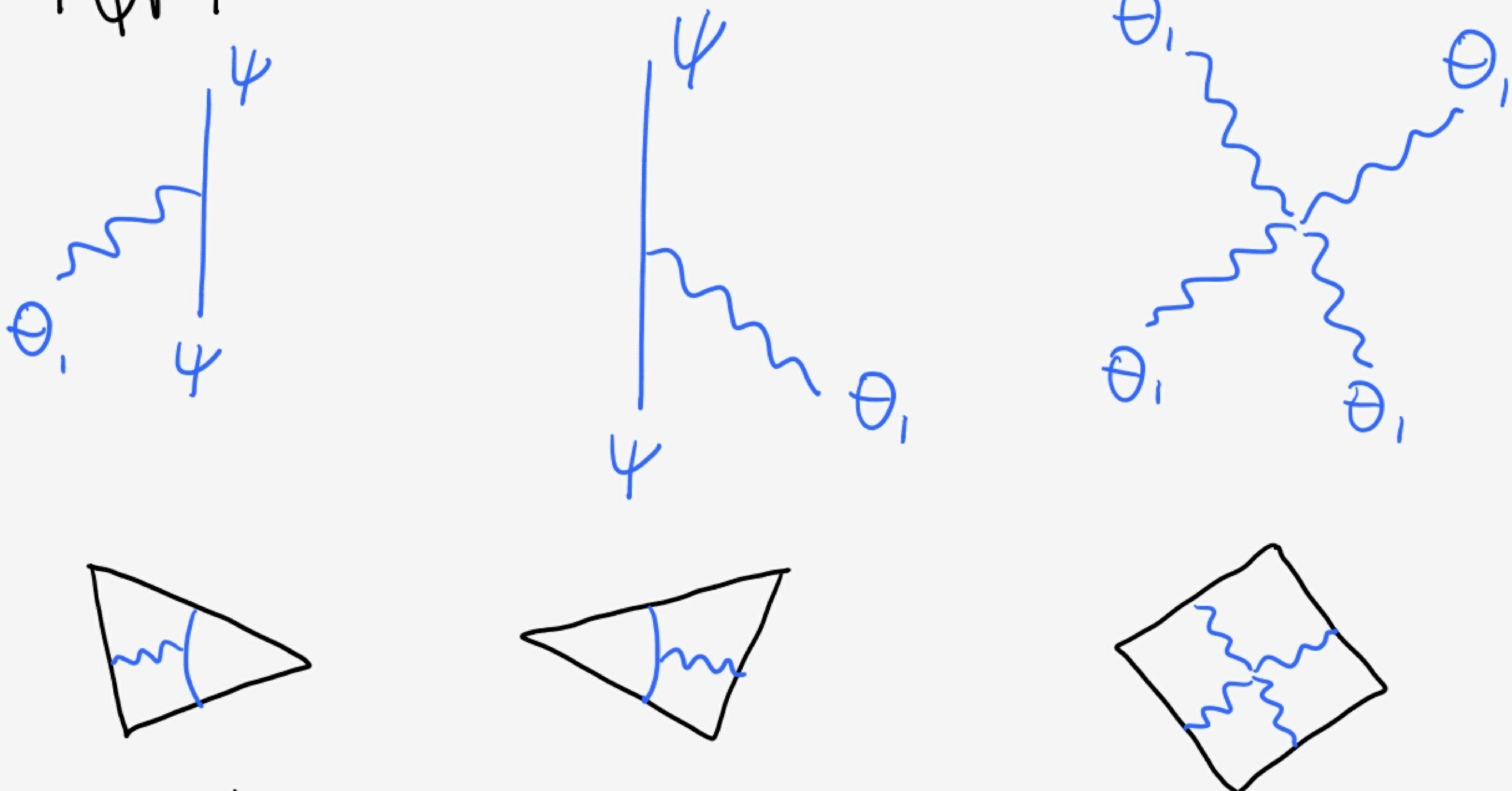
$$D = D_0 + \Theta_1 + J\Theta_1 J^{-1}$$

Θ_1 : perturbation

$\mathrm{tr}\,\Theta_1^2$ $\quad\langle J\psi, D_0\psi\rangle$ $\quad\langle J\psi, \Theta_1\psi\rangle$ $\quad\langle J\psi, J\Theta_1 J^{-1}\psi\rangle$

also  etc

2d TQFT



$\longrightarrow$ oriented surface with graph $\rightarrow$ TQFT with defects

References

JWB & S. Tavares, Two-dimensional state sum models and spin structures CMP 2014

JWB, Matrix geometries and fuzzy spaces as finite spectral triples JMP 2015

JWB & L. Glaser, Monte Carlo simulations of random non-commutative geometries arXiv, very soon

plus Como lectures 2014 on my website.