Renormalization in Tensorial Group Field Theories

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Open Problems in Theoretical Physics: the Issue of Quantum Space-Time

Corfu Summer Institute '15
Corfu, Greece

Alexander von Humboldt Stiftung/Foundation

cost Action MP 1405
Quantum Structure of Spacetime

Outline

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Quantizing Gravity

• "Ordinary" way $\rightsquigarrow$ Several issues. The Einstein theory of gravity is perturbatively divergent and nonrenormalizable [DeWitt PR '67, Goroff & Sagnotti, NPB '86].

• Alternative scenarios (coupling gravity to other fields, Asymptotic Safety, . . .) and more "daring" ones (extra-dimensions, susy, background field independent methods, . . .).

• Mid 80's: *Matrix Models* [Di Francesco et al.., PR '95] prove to be a solvable framework and concrete realization of an "emergent gravity" scenario.

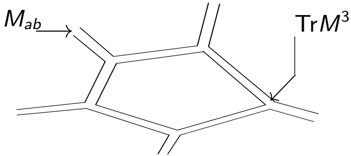
• Matrix Models and Random 2D geometry: To "replace" the sum over topologies and geometries of 2D surfaces in a path integral, by a sum over random triangulations of surfaces.

Quantizing Gravity: From Matrices ...

- Probability measures for matrices M of *large size* N and describe 2D gravity. Archetype:

$$Z_{\text{matrix}} = \int dM \, e^{-\frac{1}{2}\mathrm{Tr}M^2 + \frac{\lambda}{\sqrt{N}}\mathrm{Tr}M^3}$$

• A triangulated surface ≡ A Feynman ribbon graph:



• 't Hooft's Large N limit: Planar graphs' sector ≡ surfaces of genus 0.
• Stat. Mech.: $\exists$ phase transition ($N \to \infty; \lambda \to \lambda_c$) $\rightsquigarrow$ a continuum limit (infinitely refined Riemann surfaces) as a 2D theory of gravity (Liouville gravity (gravity +CF matter)).

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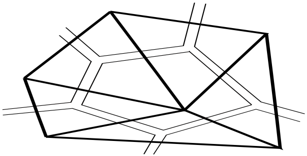

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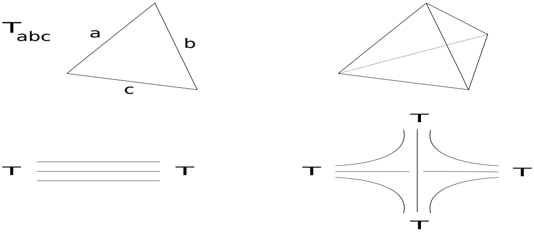
... to Tensors ...

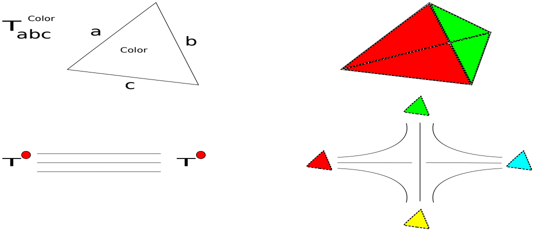
• Tensor Models (TM) of rank D: Tool for randomizing geometry in dimension D
Basic building blocks $(D-1)$-simplexes & Interaction forms a D-simplex;
For e.g. in 3D:



• Some results [Ambjorn et al. '91, Sasakura '91, Gross '92, Boulatov-Ooguri '92]
Ambjorn et al., 91', phase transition (numerics).
• $1/N$ expansion missing $\Rightarrow$ several exact results of MM difficult to extend to TM.
• TM need improvement(s): C + DT's, Boulatov-Ooguri models, Group Field Theory [Oriti, '05–].

. . . and Colored Tensor Models

• '10 Gurau's $1/N$ expansion for colored TM [Gurau, AHP, '11]
3D:



- triangulate better objects (pseudo-manifolds) [Gurau, CMP '11]
- $\exists\, 1/N$ and Leading graphs triangulate only spheres in any D [Gurau, AHP '11; Bonzom, Gurau, Rivasseau '15]
- have computable phase transition, and critical exponent [Bonzom, Gurau, Riello, Rivasseau, NPB, '11];
[Benedetti, Bonzom, Carrozza, Dartois, Delpouve, Gurau, Lionni, Rivasseau, Ryan];
- Define, renormalizable field theories called TGFTs [BG & Rivasseau, '11; Carrozza, Oriti, Rivasseau '12; Samary & Vignes-Tourneret '12; BG, '13—; Avohou, Benedetti, Lahoche, Krajewski, Martini, Toriumi].

Outline

Tensor Models [Gurau, '10, '11]

• Study of probability measures of random tensor spaces +
Geometric/Topological/Physical inputs.

• A covariant complex tensor $T_{p_1,\dots,p_d}$ with transformation rule

$$T^U_{p_1,\dots,p_d} = \sum_{q_k} U^{(1)}_{p_1 q_1} \dots U^{(d)}_{p_d q_d} T_{q_1,\dots,q_d} \,, \qquad U^{(a)} \in U(N_a) \tag{1}$$

• G/T/Physics input: T is viewed as a $(d-1)$-simplex.

• Tensor Invariance for defining the interactions

$$S^{\rm int}_{\mathbf{b}}(T,\bar T) = S^{\rm int}_{\mathbf{b}}(T^U,\bar T^U) = \mathrm{Tr}_{\mathbf{b}}(\bar T \cdot T \dots \bar T \cdot T) \tag{2}$$

$\mathbf{b}$ a colored graph encoding the contraction pattern; $S^{\rm int}_{\mathbf{b}}$ "is" a gluing of simplexes and represents a d-simplex. Ex:

• Counting TM invariants is counting the number of branched covers of S^2! (JBG & S. Ramgoolam [AIHP D, '14])

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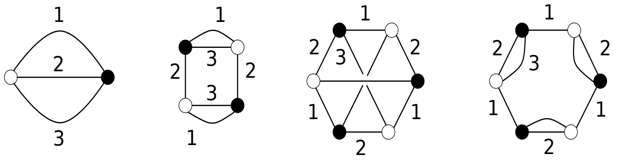

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Action/Partition function

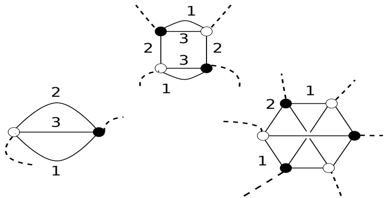
- Invariant action:

$$S[T, \bar{T}, \{\lambda_{\mathbf{b}}\}_{\mathbf{b}}] = \sum_{\mathbf{b}} \lambda_{\mathbf{b}} S^{\text{int}}_{\mathbf{b}}(T, \bar{T}), \tag{3}$$

- Partition function

$$Z[\{\lambda_{\mathbf{b}}\}_{\mathbf{b}}] = \int \prod_{p_i} [dT_{p_1,\dots,p_d} d\bar{T}_{p_1,\dots,p_d}] e^{-S[T,\bar{T},\{\lambda_{\mathbf{b}}\}_{\mathbf{b}}]} \tag{4}$$

- Expanding the partition function: colored Feynman graphs



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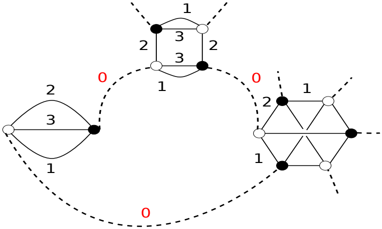

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Foundation questions:

• Tensor: Only a discretization tool or a "real" quanta of some "thing" ?

Big bang is identified with geometrogenesis, i.e. emergence of classical space-time. Pre-space (analog of space-time before condensation) is treated as a physical transplanckian early phase of the universe (not just as a mathematical artefact) [Konopka '06, Oriti, '08].

• Divergences $\Rightarrow$ Need of Renormalization. [Rivasseau: The Tensor Track '11, '12, '13] Renormalization group is a guiding/selecting thread in theory space.

• Does it work for matrices ? The Grosse-Wulkenhaar model [Grosse-Wulkenhaar, '02–'03]: Renormalizable and Asymptotically Safe model (without any extra-symmetry required) and induces important NEW developments in Field Theory.

Defining Tensor-like QFTs

BG & Rivasseau, Commun. Math. Phys. '12; BG, Commun. Math. Phys. '14

• Simple TM: A complex tensor $T_{p_1,\dots,p_d}$ with "Tensor Invariance" for defining the interactions and kinetic term

$$S^{\rm kin}(T,\bar T) = \sum_{p_s} \bar T_{p_1,\dots,p_d} \Big(\sum_s (p_s)^2 + \mu\Big) T_{p_1,\dots,p_d} \qquad (5)$$

$p_s^2 \equiv \Delta$. More generally: $(p_s)^{2a}$, $a > 0$.

• Gauge invariant models (gi) [Gurau, Bonzom, Carrozza, BG, Girelli, Livine, Lahoche, Oriti, Ousmane Samary, Rivasseau, Vignes-Tourneret] Multi-orientable models: [Tanasa, Dartois, Rivasseau]

• Summing over arbitrary high momenta may imply divergent amplitudes.

Results: Several Renormalizable Models and UV-Asymptotically Free [BG, 12'– '13]

Multi-scale Renormalizable TGFTs: $T : U(1)^d, SU(2)^d \to \mathbb{C}$

TGFT (type)	G_D	$\Phi^{k_{\max}}$	d	Renormalizability	UV behavior
	$U(1)$	Φ^6	4	Just-	AS (WiP with T. Koslowski)
	$U(1)$	Φ^4	5	Just-	AF
	$U(1)^2$	Φ^4	4	Just-	AF
	$U(1)$	Φ^{2k}	3	Super-	-
gi-	$U(1)$	Φ^4	6	Just-	AF
gi-	$U(1)$	Φ^6	5	Just-	-
gi-	$SU(2)^3$	Φ^6	3	Just-	AS ??
gi-	$U(1)$	Φ^{2k}	4	Super-	-
gi-	$U(1)$	Φ^4	5	Super-	-

Table: Updated list of tensorial renormalizable models and their features (AF $\equiv$ asymptotically free).

Beyond perturbation: the Functional Method

• A goal for TGFT: Achieve a universal scenario for an emergent spacetime. One of the most robust mechanism for this is to see this emergence through one or several phase transitions, running towards the IR.

• But probing the IR, seeking and including nonperturbative effects & Deliver phase diagrams: FRG prominent tool.

• Today's goal: (1) Show that the FRG methods can be defined on TGFT models. (2) Applied on a rank 3 model, results suggest a phase transition but difficult to ascertain.

Today: YES to the 2 first questions !

☺

• Already some results: [Krajewski & Toriumi, '14], [Benedetti, BG, Oriti JHEP '15], [Martini, Master Thesis '15], [BG, Martini, Oriti, '15], [Benedetti, Lahoche, '15], [BG, Koslowski, 1?].

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Wetterich-Morris formalism for TGFT

[Benedetti, BG, Oriti, JHEP '14]

• Rank d model: $\phi_{\mathbf{P}}$, $\mathbf{P}$ a generic multi-index and an action

$$S = \sum_{\mathbf{P},\mathbf{P}'} \frac{Z}{2} \phi_{\mathbf{P}} \, K(\mathbf{P};\mathbf{P}') \, \phi_{\mathbf{P}'} + \sum_{\mathrm{b}} \lambda_{\mathrm{b}} \mathrm{Tr}_{\mathrm{b}}(\mathcal{V}\phi^{\mathbf{n}_{\mathrm{b}}}) \,. \tag{6}$$

• Cut-offed partition function $Z_\Lambda[J] = \int d\mu_\Lambda(\phi) \, e^{-S[\phi]+\mathrm{Tr}_2(J\cdot\phi)} = e^{W_\Lambda[J]}$,

• One-parameter family ΔS_N of cut-off functions suppressing low energy modes up to N

$$\Delta S_N = \sum_{\mathbf{P},\mathbf{P}'} \phi_{\mathbf{P}} \cdot R_N(\mathbf{P};\mathbf{P}') \cdot \phi_{\mathbf{P}'} \,,$$

$$\mathcal{Z}_N[J] = e^{W_N[J]} = \int_N d\phi \, e^{-S(\phi)-\Delta S_N(\phi)+\mathrm{Tr}_2(J\cdot\phi)} \,. \tag{7}$$

• Effective action & Legendre transform $\Gamma_N = \sup_J \Big(\mathrm{Tr}_2(J\cdot\varphi) - W_N(J) \Big) - \Delta S_N(\varphi)$, $\varphi = \langle\phi\rangle$, leads to the Wetterich Equation

$$\boxed{\partial_t \Gamma_N(\varphi) = \frac{1}{2} \overline{\mathrm{Tr}} \left(\partial_t R_N \cdot [\Gamma_N^{(2)} + R_N]^{-1} \right)} \tag{8}$$

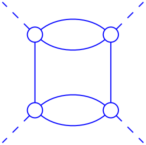
A rank 3 tensor model

- $\phi : U(1)^3 \to \mathbb{R}$, with Fourier components $\phi_{\mathbf{P}}$, $\mathbf{P} = (p_i)$, $p_i \in \mathbb{Z}$. UV & IR refer to $U(1)$.

- Action (a theory of spheres)

$$S(\phi) = \frac{Z}{2}\mathrm{Tr}_2(\phi \cdot K \cdot \phi) + \frac{m}{2}\mathrm{Tr}_2(\phi^2) + S^{\mathrm{int}} \,,$$

$$K(p_i; p_i) = \frac{1}{d}\sum_{i=1}^{d} |p_i| \boldsymbol{\delta}_{p_i, p_i'} \,, \quad S^{\mathrm{int}} = \frac{\lambda_N}{4}\Big[\quad \quad + \mathrm{Sym}\Big] \qquad (9)$$



- Truncation of the effective action:

$$\Gamma_N(\varphi) = \frac{Z_N}{2}\mathrm{Tr}_2(\varphi \cdot K \cdot \varphi) + \frac{m_N}{2}\mathrm{Tr}_2(\varphi^2) + S^{\mathrm{int}} \,,$$

$$R_N(p_i; p_i') = \boldsymbol{\delta}_{p_i, p_i'} \, Z_N \left(N - \sum_{i=1}^{d} |p_i|\right) \Theta(N - \sum_{i=1}^{3} |p_i|) \qquad (10)$$

β-functions of a rank 3 tensor model

$t = \ln N$, $\eta := \partial_t \ln Z_N$, $m_N = Z_N \mu_N$, $\lambda_N = Z_N^2 \bar{\lambda}_N$

$$\eta = \frac{54\lambda_N N(2N+1)}{3(N+\mu_N)^2 - 2\lambda_N(5+18N+27N^2)} \tag{11}$$

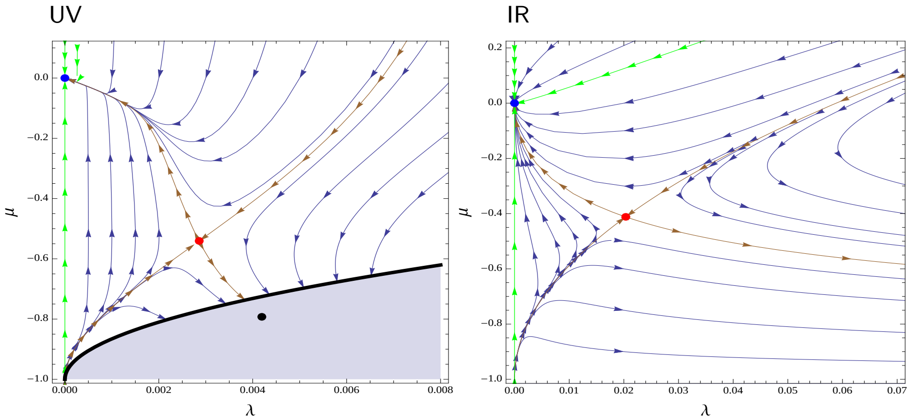
$$\partial_t \mu_N = \frac{-\lambda_N N}{(N+\mu_N)^2}\Big[\eta(18N^2+9N+4)+(54N^2+36N+9)\Big] - \eta\mu_N \tag{12}$$

$$\partial_t \lambda_N = \frac{2\lambda_N^2 N}{(N+\mu_N)^3}\Big[\tfrac{1}{3}\eta(18N^2+45N+25)+2(9N^2+18N+9)\Big] \tag{13}$$

• Explicit dependency in N, a non-autonomous system ! A implicit scale (the radius of the group manifold) and the combinatorics of the interaction.

• What replaces the usual notion of Canonical Dimension?
• In the UV $\Longrightarrow$ needs a certain scaling: $\mu_N = N\tilde{\mu}_N$, $\lambda_N = \tilde{\lambda}_N$
• In the IR $(N=0)$ $\Longrightarrow$ needs another scaling: $\eta \to 0$, $\mu_N = N\tilde{\mu}_N$, $\lambda_N = N^2\tilde{\lambda}_N$

Plots



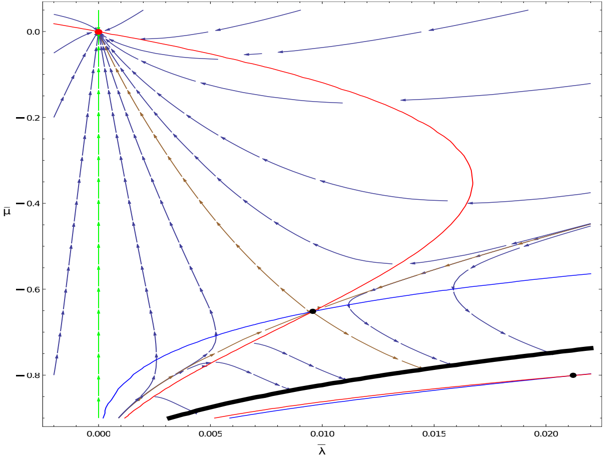
• The Gaussian FPs have a relevant and marginally relevant directions. Confirm the UV asymptotic freedom of the rank 3 model [BG and Samary, 2012].

• The NGaussian FPs have one relevant and one irrelevant directions (similar to the Wilson-Fisher fixed point of scalar field theory in dimension 3, but the models are radically $\neq$).

Flash: TGFT on noncompact $\phi : \mathbb{R}^3 \to \mathbb{C}$

[BG, Martini, Oriti, 1508.01855] (1st RG analysis of TGFT in the noncompact case ☺)

• Issue of IR divergences can be treated using discretization of the background manifold and then a thermodynamic limit;



• Rank 3 Gauge invariant TGFT has a NGFP in the IR.

Outline

Open questions/Next Challenges

- Tensor models generalize matrix models and Colored TMs is a rich and computable framework.

- Field Theory: There exist several models which are perturbatiely renormalizable, some AF and some AS.

  $\sim$ Open questions • Understand better the theory space; • There exist certainly different types of TGFT than the one present here and Renormalization analysis works as well: How can we understand the theory space?

  $\sim$ Challenge: • Hunting nonperturbative and universal effects (FRG methods, recovering phase diagrams); prove that there is phase transition; investigate precisely the possible phases; order of parameters, symmetry breaking mechanism;

- Connection with Field Theory on non-associative geometrical spaces [wip with Ousmane Samary]

$$\begin{array}{ccc} [x^\mu, x^\nu] = i\Theta^{\mu\nu} & \longrightarrow & [x^\mu, x^\nu, x^\rho] = \Theta^{\mu,\nu,\rho} \\ \text{Noncommutative geometry} & \longrightarrow & \text{Nonassociative geometry} \\ \downarrow & & \downarrow \\ \text{Matrix/GW models} & \longrightarrow & \text{Tensorial field models} \end{array} \tag{14}$$