Physical points (events) of spacetime

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The issue of point-event according to GR

- uneducated approach: point is x^μ
- educated: a manifold M, a point $m \in M$, coordinates

$$x^\mu : M \to \mathbb{R}$$

- dynamics of GR:

$$(M, \quad \text{geometry}, \quad \text{fields})/\text{Diff}$$

  no points
- Elie Cartan, the equivalence problem: use coordinates defined naturally from $R_{\mu\nu\alpha\beta}$, $\nabla_\delta R_{\mu\nu\alpha\beta}$, ...
- Kijowski, Kuchar, material points: choose 4 dynamical fields

$$\phi_1, ..., \phi_4 : M \to \mathbb{M}$$

  and use as natural coordinates on M
- coordinates defined by distances, geodesics, angles *Duch, JL, Swiezewski, Kaminski, Bodendorfer...*,

Observed spacetime

Observer:

- M - a 4-manifold
- $\mathbb{R} \ni \tau \mapsto \gamma(\tau) \in M$ - a curve

$\mathcal{G}$ is the set of the spacetime metric tensors g on M such that

$$\nabla_{\dot{\gamma}}\dot{\gamma} \ = \ 0, \qquad \dot{\gamma} := \frac{d}{d\tau}\gamma(\tau)$$

- $\mathcal{G} \ni g \mapsto e_0, e_1, e_2, e_3$ - a tangent frame along γ

such that

$$(e_1, e_2, e_3, e_4 := \dot{\gamma})$$

is orthonormal, and

$$\nabla_{\dot{\gamma}} e_\mu = 0$$

Metric depending coordinates

For every metric tensor

$$g \in \mathcal{G}$$

we define in a neighborhood of γ (BLACKBOARD):

- cylindrical coordinates singular at γ

$$(r, \theta, \phi, \tau)$$

- and non-singular Cartesian coordinates

$$(x, y, z, \tau)$$

$$z = r\cos\theta, \quad x = \cos\varphi \sin\theta, \quad y = \sin\varphi \sin\theta$$

Each coordinate is defined in a way invaruiant the diffeomorhisms.

Spacetime according to the observer

We have defined a metric g dependent map

$$O : \mathbb{R}^4 \ni \ (x, y, z, \tau) \ \mapsto m(g; x, y, z, \tau) \in M$$

It endowes $\mathbb{R}^4$ with

$$G \ := \ O^* g,$$

G is a metric tensor in a neighborhood of the line $(0, 0, 0, *)$,

$$G_{rr} \ = \ 1, \qquad G_{r\theta} \ = \ 0 \ = \ G_{r\phi} \tag{1}$$

$$G_{\mu\nu}|_\gamma \ = \ \eta_{\mu\nu} \tag{2}$$

$$G_{\mu\nu,\alpha}|_\gamma \ = \ 0. \tag{3}$$

$$K_{rr} \ = \ 0 \tag{4}$$

where $K_{ij}dx^i dx^j$ is the extrinsic curvature of $\tau \ = \ \mathrm{const}$ surfaces.
The conditions (1-4) are sufficient for the convers construction

Gauge and, respectively, active diffeomorphisms

Now, on the observer's $\mathbb{R}^4$ we have the pullback of g, as well as the pullback of every (covariant) field ψ defined on M,

$$G \;=\; O^* g, \quad \Psi \;=\; O^* \psi.$$

They are all invariant with respect to all the diffeomorphisms of M preserving the observer ($\mathrm{Diff}_{\mathrm{gauge}}$).
Active diffeomorphisms ($\mathrm{Diff}_{\mathrm{act}}(g)$) may be defined by considering

$$\tilde{f} : \mathbb{R}^4 \to \mathbb{R}^4$$

such that the conditions (1-4) are satisfied by the metric $\tilde{f}^* G$ (everything locally). That property leads to the following definition: Given a metric tensor g on M, a diffeomorphism f of M is active if the Cartesian coordinates of $f^* g$ coincide with those of g in a neighborhood of γ;

$$(x, y, z, \tau)(g) \;=\; (x, y, z, \tau)(f^* g).$$

The group of the active diffeomorphisms

$\mathrm{Diff}_{\rm gauge}$ and $\mathrm{Diff}_{\rm act}(g)$ for arbitrarily fixed $g \in \mathcal{G}$, generate all the Diff. While $\mathrm{Diff}_{\rm gauge}$ are a subgroup of Diff, this is not true for $\mathrm{Diff}_{\rm act}(g)$. Nonetheless they define the group of movements $\mathcal{G} \to \mathcal{G}$, in the following way:

$$\mathcal{G} \ni g \ \mapsto \ f_g \in \mathrm{Diff}_{\rm act}(g)$$

acts in $\mathcal{G}$ naturally

$$g \mapsto (f_g)^* g .$$

Given another

$$\mathcal{G} \ni g \ \mapsto \ f'_g \in \mathrm{Diff}_{\rm act}(g)$$

the composition of the actions coincides with the action of

$$\mathcal{G} \ni g \ \mapsto \ f"_g := f_g \circ f'_{f*g} \in \mathrm{Diff}_{\rm act}(g)$$

Given a metric tensor g, the observer can be characterized by a point $m_0 = \gamma(0)$, and the frame $e_1, ..., e_4$ at m (it determines the geodesic γ, and the frame e_μ along γ) Consider another another point $m_0' \in M$ and an orthonormal frame $e_1', ..., e_4'$. They define another observer and coordinate chard $x'^1, ..., x'^4$. And can be used to define a local diffeomorphism f, such that

$$x'^\mu(f(m)) \; = \; x^\mu(m).$$

It is easy to check that

$$f \in \mathrm{Diff}_{\mathrm{act}}(g)$$

and actually, every element of $\mathrm{Diff}_{\mathrm{act}}(g)$ can be characterized in that way. So $\mathrm{Diff}_{\mathrm{act}}(g)$ is 10 dimensional.

Infinitesimal (local) characterisation of $\mathrm{Diff}_{\mathrm{act}}(g)$

In observer's $\mathbb{R}^4$ fix a pair

$$(L, T)$$

a Lorenz transformation and, respectively, a vector. They uniquely determine a vector field on $\mathcal{G}$

$$X_{L,T} = \int d^4x X_{L,T}(g,x)\frac{\delta}{\delta g_{\mu\nu}}$$

We have: $L = R_x, R_y, R_z, B_x, B_y, B_z$, $T = T_x, T_y, T_z$. A general vector field on $\mathcal{G}$, at g, generated by the active diffeomorphisms is

$$\begin{aligned} X(g) &= \omega^i(g)X_{R_i,0} + a^i(g)X_{B_i,0} + l^i(g)X_{0,T_i} \\ [X_{L_1,0,}, X_{L_2,0}] &= X_{[L_1,L_2],0} \\ [X_{L,0}, X_{0,T}] &= X_{[(L,0),(0,T)]} \\ [X_{0,T}, X_{0,T'}] &= X_{\mathrm{Riem}(T,T'),0} \end{aligned}$$

where

$$\mathrm{Riem}(T,T')^\alpha_\beta = R^\alpha{}_{\beta\mu\nu}(g,m_0)T^\mu T'\nu$$

Observables

Suppose our system consists of: the metric g and a scalar a matter field ψ. The corresponding observables have been defined:

$$(g,\psi) \mapsto \Psi(r,\theta,\phi,\tau;g,\psi) \;=\; \psi(m(g;r,\theta,\phi,\tau))$$

$$(g,\psi) \mapsto G_{\theta\theta}(r,\theta,\phi,\tau;g,\psi) \;=\; g_{\mu\nu}(m(g;r,\theta,\phi,\tau))\frac{\partial m^\mu}{\partial\theta}\frac{\partial m^\nu}{\partial\theta}$$

$$G_{\theta\phi},\, G_{\phi\phi},\, G_{\tau\theta},\, G_{\tau\phi},\, G_{\tau r} \;=\; \ldots$$

The Poisson bracket:

$$\{\Psi(r,\theta,\phi,\tau), \Psi(r',\theta',\phi',\tau')\} \;= ?$$

Canonical structure

$$M = \Sigma \times \mathbb{R}$$

$$g, \psi \ \text{ on } M \ \leftrightarrow q, p, \psi, \pi, N, N^a \quad \text{on} \quad \Sigma$$

$$\{q_{ab}(\sigma), p^{cd}(\sigma')\} \ = \ \delta^c_{(a}\delta^{d)}_b \delta(\sigma, \sigma'), \qquad \{\psi(\sigma), \pi(\sigma')\} \ = \ \delta(\sigma, \sigma')$$

An embedding adjusted to our variables r, θ, ϕ, τ, $((\theta^A) = (\theta, \phi))$:

$$\tau_\Sigma \ = \ \text{const}, \quad q_{rr} = 1, \quad q_{rA} \ = \ 0, \quad {p^r}_r - {p^A}_A \ = \ 0$$

Given fixed coordinates (r, θ^A) on Σ, our construction is equivalent to introducing the above conditions as gauge conditions. For example

$$\Psi(r, \theta^A, \tau = 0; q, \pi, \psi) \ = \ \psi(r, \theta^A)$$

Now,

$$\{\Psi(r, \theta^A, 0), \Psi(r', \theta'^A, 0)\} \ = \ \{\psi(r, \theta^A), \psi(r', \theta'^A)\}_{\mathrm{D}} \neq 0$$