Non-associative geometry in the representation category of a quasi-Hopf algebra

Gwendolyn E. Barnes

HERIOT WATT UNIVERSITY

MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES

COMMONWEALTH SCHOLARSHIPS

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Overview

Physical origins

(Blumenhagen et. al., 2010, 2011; Lüst et. al., 2010, 2012): closed strings which wind and propagate in certain spacetime backgrounds probe a noncommutative and nonassociative deformation of the spacetime geometry.

(Mylonas, Schupp & Szabo, 2012) obtained explicit star product realisations of the nonassociative gravity using Kontsevich's deformation quantisation of twisted Poisson manifolds.

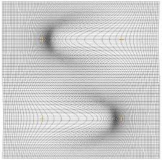
$$f \star g = \cdot \Big(e^{\big(-\frac{\mathrm{i}\,\hbar}{2}\, \big(\frac{1}{4}\, \mathrm{R}^{ijk}\, (m_{ij} \otimes t_k - t_i \otimes m_{jk}) + t_i \otimes \tilde{t}^{\,i} - \tilde{t}^{\,i} \otimes t_i \big) \big)} (f \otimes g) \Big) \ .$$

In 2014 they observed that this star product could be obtained via a particular cochain twisting of the Hopf algebra which is the universal enveloping algebra of a certain Lie algebra to a quasi-Hopf algebra.

$$f \star g = \cdot \circ F^{-1}(f \otimes g) \ , \quad F \in U\mathfrak{g} \otimes U\mathfrak{g}$$

Our aim

- systematically develop a formalism for differential geometry on noncommutative and nonassociative spaces,
- with particular focus on developing notions of differential calculi, connections on bimodules (together with their tensor product structure and lift to dual objects) and the curvature of bimodule connections.

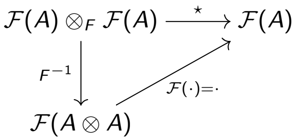


Category theory as a framework of guiding principles

$$f \star g = \cdot \circ F^{-1}(f \otimes g) \ , \quad F \in U\mathfrak{g} \otimes U\mathfrak{g}$$

F twists $U\mathfrak{g}$ to a quasi Hopf algebra H_F with coproduct $\Delta_F(\cdot) = F\,\Delta(\cdot)\,F^{-1}$. $\star$ is defined in such a way that it is H_F-equivariant.

- if we depict this information diagrammatically ($f, g \in C^\infty(M) =: A$)



- we see there is an isomorphism (since F invertible): $\mathcal{F}(A) \otimes_F \mathcal{F}(A) \overset{F^{-1}}{\to} \mathcal{F}(A \otimes A)$.
- Insight: all bilinear operations which are required to be H_F-equivariant are of this form (usual operations modified by precomposition with F^{-1}). This points towards the existence of a map between categories with a monoidal structure which the map preserves (a monoidal functor).
- This observation and similar like it point towards a bigger formal structure which can serve as a framework of guiding principles for developing descriptions of geometry in an algebraic setting- category theory.

Spaces and algebras

In the spirit of noncommutative geometry: interpret the deformed algebra $(C^\infty(M), \star)$ as the noncommutative and nonassociative space we are interested in studying the geometry of. $\star$ is H_F-equivariant (so view H_F as the algebra of symmetries of the "space" $(C^\infty(M), \star)$).

- Since we are interested in a differential graded algebra as a basis for a differential calculus, view $(C^\infty(M), \star)$ as the component in degree zero of an exterior algebra of differential forms A (with wedge product compatible with the action of H_F).
- Crucial observation: A is a commutative and associative algebra object in the braided monoidal category of H_F-modules ${}^{H_F}\mathscr{M}$ (axioms hold with explicit R-matrix and associator insertions whenever we flip or rebracket).

$$f \star g = \star \circ R\,(g \otimes f), \quad R = F_{21} F^{-1}$$

$$(f \star g) \star h = \star \circ (\mathrm{id} \otimes \star)\phi\,(f \otimes g \otimes h), \quad \phi = (1 \otimes F)\cdot(\mathrm{id} \otimes \Delta)(F)\cdot(\Delta \otimes \mathrm{id})(F^{-1})\cdot(F^{-1} \otimes 1)$$

Our approach to geometry

The monoidal structure (the tensor product) $\otimes$ in ${}^{H_F}\mathscr{M}$ admits

internal homomorphisms: k-linear maps which do not respect the H_F-action but are themselves transformed under it, together with (complicated) evaluation, composition and tensor product operations.

We describe all notions of geometry as internal homomorphisms (with additional properties).

Using internal homomorphisms rather than morphisms (as in (Beggs & Majid, 2010)) leads to a much richer framework because the conditions for being a morphism are very restrictive. (e.g. does not allow for curved geometry on Moyal Weyl plane.)

Our internal homomorphism approach is inspired by the formalism of (Aschieri & Schenkel, 2014) and it clarifies these ideas and constructions in the framework of category theory.

Derivations

The idea: since A is a commutative and associative algebra object in ${}^{H_F}\mathscr{M}$, one can

describe its geometric properties exactly as for the commutative and associative case but taking care to insert R-matrices resp. associators whenever one flips resp. rebrackets expressions, and to use the appropriate operations for internal homomorphisms whenever they are involved.

$$\mathrm{der}(A) \longrightarrow \mathrm{end}(A) \underset{\zeta(\widehat{l}\circ \mathrm{ev})}{\overset{\zeta([\,\cdot\,,\widehat{l}(\cdot)\,])}{\rightrightarrows}} \mathrm{hom}(A,\mathrm{end}(A))$$

For commutative and associative spaces:

$$X(b\,a) - X(b)\,a = (-1)^{|X||b|}\,b\,X(a)\ .$$

For noncommutative and nonassociative spaces:

$$\mathrm{ev}\Big(\big(\phi^{(1)}\triangleright X\big)\otimes\big((\phi^{(2)}\triangleright a)\star(\phi^{(3)}\triangleright b\,)\big)\Big)$$

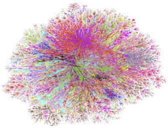
$$\text{- } (\text{-}1)^{|X|\,|a|}\ \big(\phi^{(1)}\,R^{(2)}\triangleright a\big)\star \mathrm{ev}\Big(\big(\phi^{(2)}\,R^{(1)}\triangleright X\big)\otimes\big(\phi^{(3)}\triangleright b\big)\Big) = \mathrm{ev}\big(X\otimes a\big)\star\, b\ ,$$

$$\mathrm{ev}(X\otimes a) = \phi^{(1)}\triangleright X\big(S(\phi^{(2)})\,\alpha\,\phi^{(3)}\triangleright a\big)\ ,\ \ h\triangleright X = (h_{(1)}\triangleright\,\cdot\,)\circ X\circ(S(h_{(2)})\triangleright\,\cdot\,)\ ,$$
$$\phi = (1\otimes F)\cdot(\mathrm{id}_H\otimes\Delta)(F)\cdot(\Delta\otimes\mathrm{id}_H)(F^{-1})\cdot(F^{-1}\otimes 1)\ ,$$
$$\alpha = S(F^{(-1)})\,F^{(-2)}\ ,\ \ R = F_{21}\,F^{-1}.$$

Vector bundles and bimodules

In the spirit of noncommutative geometry: interpret $(C^\infty(M),\star)$-bimodules (with left and right $(C^\infty(M),\star)$-action being H_F-equivariant) as noncommutative and nonassociative vector bundles over the noncommutative and nonassociative "space" $(C^\infty(M),\star)$.

- For a given vector bundle $E \xrightarrow{\pi} M$,
- Since it is desirable to view connections as endomorphisms, view the A-bimodule of sections $(\Gamma^\infty(E \xrightarrow{\pi} M), l_\star, r_\star)$ as the component in degree zero of the E-valued differential forms $(\Omega^\sharp(M,E), l_\star, r_\star) =: V$ (with left and right actions compatible with the action of H_F).
- Crucial observation: V is a commutative and associative A-bimodule object in the braided monoidal category of H_F-modules ${}^{H_F}\mathscr{M}$ (axioms hold with explicit R-matrix and associator insertions whenever we flip or rebracket).



Connections and curvature

Since noncommutative and nonassociative "vector bundles" V are commutative and associative when viewed in ${}^{H_F}\mathscr{M}$ we can describe their geometric properties exactly as for the commutative and associative case (with $R, \phi, \alpha, \cdots$ inserted explicitly whenever appropriate).

$$\mathrm{con}(V) \longrightarrow \mathrm{end}(V) \times k[1] \underset{\zeta(\widehat{l}\circ \mathrm{ev}\circ(\mathrm{d}\,\otimes\mathrm{id})\circ(\mathrm{pr}_2\otimes\mathrm{id}))}{\overset{\zeta([\,\cdot\,,\,\cdot\,]\circ(\mathrm{pr}_1\otimes\mathrm{id}))}{\rightrightarrows}} \mathrm{hom}(A, \mathrm{end}(V))$$

For commutative and associative vector bundles:

$$\nabla(v\,a) - \nabla(v)\,a = (-1)^{|v|}\,v\,\mathrm{d}\,(1)a\ .$$

For noncommutative and nonassociative vector bundles:

$$\mathrm{ev}\Big(\big(\phi^{(1)} \triangleright \nabla\big) \otimes \big(\big(\phi^{(2)} \triangleright a\big) \star \big(\phi^{(3)} \triangleright v\,\big)\big)\Big)$$

$$- \ (\text{-}1)^{|a|}\ \big(\phi^{(1)}\,R^{(2)} \triangleright a\big) \star \mathrm{ev}\Big(\big(\phi^{(2)}\,R^{(1)} \triangleright \nabla\big) \otimes \big(\phi^{(3)} \triangleright v\big)\Big) = \mathrm{ev}\big(\mathrm{d}\,(1) \otimes a\big) \star\ v\ .$$

Curvature: $\mathrm{Curv}(\nabla) := [\nabla, \nabla]$ (since the assignment of curvature is quadratic in the connection it cannot be defined in categorical terms but only elementwise).

Connections on tensor products

$$\boxed{\nabla_V \boxplus \nabla_W = \nabla_V \circledast 1 + 1 \circledast \nabla_W \quad \in \mathrm{con}(V \otimes_A W)}$$

where $\circledast$ is the tensor product operation for internal homomorphisms.

- Defined for an arbitrary (finite) number of A-bimodules.
- Useful for constructing tensor fields in noncommutative and nonassociative geometry.
- Curvature behaves additively on the tensor product of connections: $\mathrm{Curv}(\nabla_V \boxplus \nabla_W) = \mathrm{Curv}(\nabla_V) \circledast 1 + 1 \circledast \mathrm{Curv}(\nabla_W)$.

For commutative and associative vector bundles:

$$\big(\nabla_V \circledast 1 + 1 \circledast \nabla_W\big)(v \otimes_A w) = \nabla_V(v) \otimes_A w + (-)^{|v|}\, v \otimes_A \nabla_W(w)\ .$$

For noncommutative and nonassociative vector bundles:

$$\mathrm{ev}\big((\nabla_V \circledast 1 + 1 \circledast \nabla_W) \otimes (v \otimes_A w)\big) = \mathrm{ev}\big((\phi^{(-1)} \triangleright \nabla_V) \otimes (\phi^{(-2)} \triangleright v)\big) \otimes_A (\phi^{(-3)} \triangleright w) +$$

$$(\text{-})^{|v|}\big(\widetilde{\phi}^{(1)}\, R^{(2)}\, \phi^{(-2)} \triangleright v\big) \otimes_A \mathrm{ev}\big((\widetilde{\phi}^{(2)}\, R^{(1)}\, \phi^{(-1)} \triangleright \nabla_W) \otimes (\widetilde{\phi}^{(3)}\, \phi^{(-3)} \triangleright w)\big)\ .$$

$\otimes_A$ denotes equivalence classes $(v\, a) \otimes_A w \sim (\phi^{(1)} \triangleright v) \otimes_A \big((\phi^{(2)} \triangleright a)\, (\phi^{(3)} \triangleright w)\big)$.

Connections on internal homomorphisms

$$\boxed{\mathrm{ad}_\bullet(\nabla_W,\nabla_V)=\mathscr{L}(\nabla_W)-\mathscr{R}(\nabla_V)\quad\in\mathrm{con}(\mathrm{hom}_A(V,W))}$$

where $\mathscr{L}:\mathrm{end}(W)\to\mathrm{end}(\mathrm{hom}(V,W)), \mathscr{R}:\mathrm{end}(V)\to\mathrm{end}(\mathrm{hom}(V,W))$.

- Useful for constructing connections on dual modules and also a Bianchi tensor $\mathrm{Bianchi}(\nabla):=\mathrm{ev}\big(\mathrm{ad}_\bullet(\nabla,\nabla)\otimes\mathrm{Curv}(\nabla)\big)\in\mathrm{end}_A(V)$.
- Non-vanishing of the Bianchi tensor can be seen as a measure of the noncommutativity and nonassociativity of the space.

For commutative and associative vector bundles (T a right A-linear tensor field):

$$(\mathscr{L}(\nabla_W)-\mathscr{R}(\nabla_V))(T)=\nabla_W\circ T-(-1)^{|T|}\,T\circ\nabla_V\ .$$

For noncommutative and nonassociative vector bundles ($T\in\mathrm{hom}_A(V,W)$):

$$\mathrm{ev}\big(\mathscr{L}(\nabla_W)-\mathscr{R}(\nabla_V)\otimes T\big)=\nabla_W\bullet T-(-1)^{|T|}(R^{(2)}\triangleright T)\bullet(R^{(1)}\triangleright\nabla_V)\ .$$

where $\nabla_V\bullet\nabla_W=\mathrm{ev}\Big(\big(\phi^{(-1)}\triangleright\nabla_V\big)\otimes\phi^{(-2)}\triangleright\nabla_W\big(S\big(\phi^{(-3)}\big)\triangleright(\,\cdot\,)\big)\Big)$.

Reflections on cochain twist quantisation

Cochain twist quantisation defines a closed braided monoidal functor which is an equivalence of closed braided monoidal categories preserving also all limits and colimits (and in particular equalisers).

This means that

- commutative and associative "spaces", "vector bundles", tensor fields, derivations and connections are twisted to braided commutative and quasi-associative "spaces", "vector bundles", derivations and connections.

This does not mean that the deformation of theories of physics (e.g. gravity and Yang-Mills theory) by cochain twists is trivial:

- Structurally things are isomorphic (configuration spaces of the deformed and undeformed theory are in bijective correspondence) but
- the choice of natural Lagrangians differs in these cases. The selection criteria for which quantities are realized in nature (e.g. as a critical point of an action) differs in the deformed and the undeformed case.

We have moreover that

- curvature is not preserved in general: $\mathrm{Curv}_F(\nabla_F) = \gamma^{-1}\big(\big[F^{(-1)} \triangleright \nabla, F^{(-2)} \triangleright \nabla\big]\big) \neq \gamma^{-1}\big(\big[\nabla, \nabla\big]\big) = \big(\mathrm{Curv}(\nabla)\big)_F.$
- In particular flat connections are in general not twisted to flat connections.

A summary of ideas

- nonassociative and noncommutative algebras ("spaces") and bimodules ("vector bundles") live as commutative and associative algebra and bimodule objects inside a braided monoidal category ${}^{H_F}\mathscr{M}$ with monoidal structure given by the tensor product $\otimes$.
- $\otimes$ permits internal homomorphisms in terms of which we have given descriptions of derivations and connections and their lifts to tensor products and tensor fields (in order to allow for richer geometries).
- We "think commutatively and associatively but are careful to put in the R-matrices and associators and to use the appropriate operations for internal homomorphisms whenever we flip, rebracket or make use of internal homomorphisms resp." (category theory tells us where to put $\phi, R, \alpha, \cdots$).
- Even though cochain twist quantisation is an equivalence of closed braided monoidal categories it does not lead to trivial deformations of theories of physics.
- Curvature is assigned quadratically on connections and so cannot be described using (the present) categorical techniques but only elementwise. Flat connections are in general twisted into non-flat connections.

Conclusion and summary

- We have provided descriptions of notions of noncommutative and nonassociative differential geometry, in particular of derivations, connections (and their lifts to tensor products and internal homomorphisms) and of curvature
- comparing the noncommutative and nonassociative descriptions to the commutative and associative ones
- thereby motivating that category theory is a powerful tool or framework to serve as a guiding principle for describing notions of differential geometry. It eliminates the guess work which can be involved in finding formula for noncommutative and nonassociative notions of geometry. Also since properties are given a priori to notions of geometry this eliminates the need to make checks afterwards that these properties are indeed satisfied.

Thank you

questions and comments