Precision QCD

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1st Lecture

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Today’s high energy colliders

Today’s high energy physics program relies mainly on results from

Collider	Process	status
LEP/LEP2	e^+e^-	1989-2000
Hera	$e^{\pm}p$	1992-2007
Tevatron	pp	1983-2011
LHC- Run I	pp	2010-2012
LHC- Run II	pp	started 2015

- LEP high precision measurements of masses, couplings, EW parameters ...
- Hera: mainly measurements of parton densities and diffraction
- Tevatron: mainly discovery of top and many QCD measurements
- LHC designed to
  - discover the Higgs [done in RunI]
  - unravel possible BSM physics [elusive up to now]

Future high-energy colliders ?

- Future colliders are of course already under discussion: ILC (international linear collider,) CLIC, FCC (Future Circular Collider)...
- However no decision has been taken yet (collider type, beams, energy, location ...)
- The typical time-scale to build a collider is about 30 years. Still, given the huge scale of such a project decisions will happen only after LHC results from Run II

No matter what happens, for the next twenty years collider precision phenomenology will be LHC phenomenology

QCD interactions (the strong force) determine the structure of all background processes to New Physics searches, hence it is crucial to understand some basic properties of QCD

QCD

Satisfactory model for strong interactions: non-abelian gauge theory SU(3)

$$U^\dagger U = U U^\dagger = 1 \quad \det(U) = 1$$

Hadron spectrum fully classified with the following assumptions

- hadrons (baryons,mesons): made of spin 1/2 quarks
- each quark of a given flavour comes in N_c=3 colors
- SU(3) is an exact symmetry
- hadrons are colour neutral, i.e. colour singlet under SU(3)
- observed hadrons are colour neutral ⇒ hadrons have integer charge

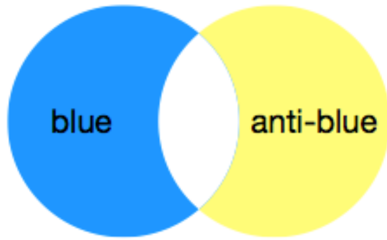
$$\begin{pmatrix} \psi_r \\ \psi_g \\ \psi_b \end{pmatrix}$$

Color singlet hadrons

Quarks can be combined in 2 elementary ways into color singlets of the $SU_c(3)$ group

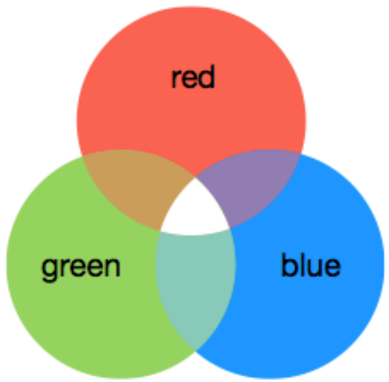
<u>Mesons</u> (bosons, e.g. pion ...)

$$\sum_i \psi_i^* \psi_i \rightarrow \sum_{ijk} U_{ij}^* U_{ik} \psi_j \psi_k = \sum_k \psi_k^* \psi_k$$



<u>Baryons</u> (fermions, e.g. proton, neutrons ...)

$$\sum_{ijk} \epsilon_{ijk} \psi_i \psi_j \psi_k \rightarrow \sum_{ii'jj'kk'} \epsilon_{ijk} U_{ii'} U_{jj'} U_{kk'} \psi_{i'} \psi_{j'} \psi_{k'} = \sum_{i'j'k'} \epsilon_{i'j'k'} \det(U) \psi_{i'} \psi_{j'} \psi_{k'}$$



First experimental evidence for colour

1. Existence of Δ^{++} particle: particle with three up quarks of the same spin and with symmetric spacial wave function. Without an additional quantum number Pauli's principle would be violated
$\Rightarrow$ color quantum number

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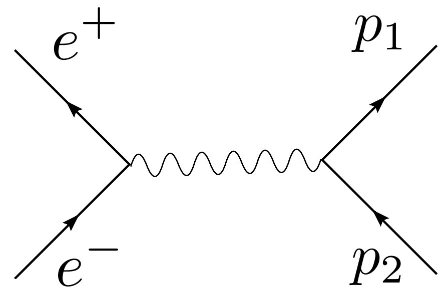
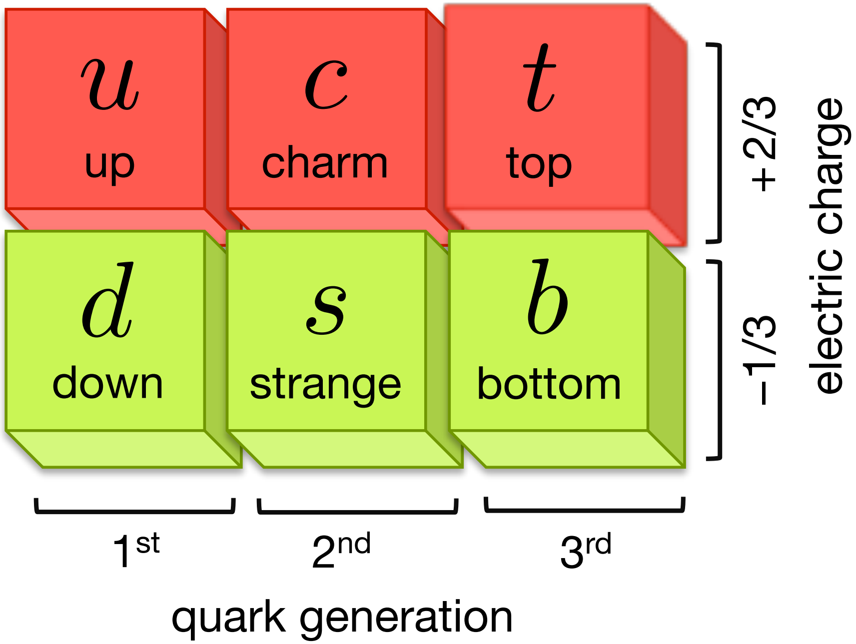
II. R-ratio: ratio of ($e^+e^- \rightarrow$ hadrons)/($e^+e^- \rightarrow \mu^+\mu^-$)

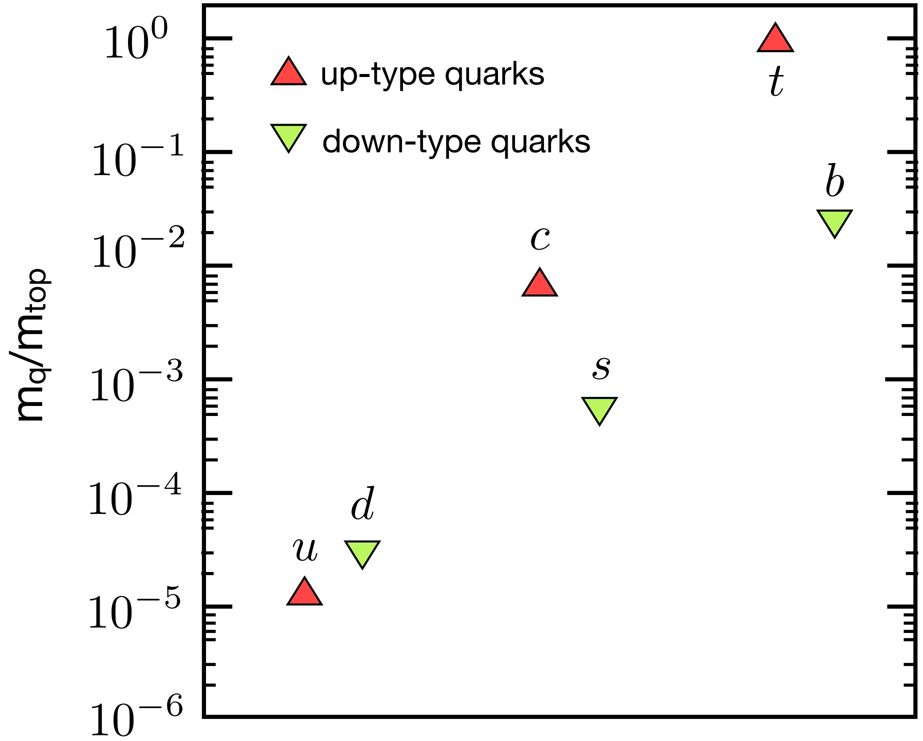
$$R \equiv \frac{e^+e^- \rightarrow \text{hadrons}}{e^+e^- \rightarrow \mu^+\mu^-} \propto N_c \sum_f Q_f^2$$



Data compatible with $N_c = 3$. Will come back to R later.

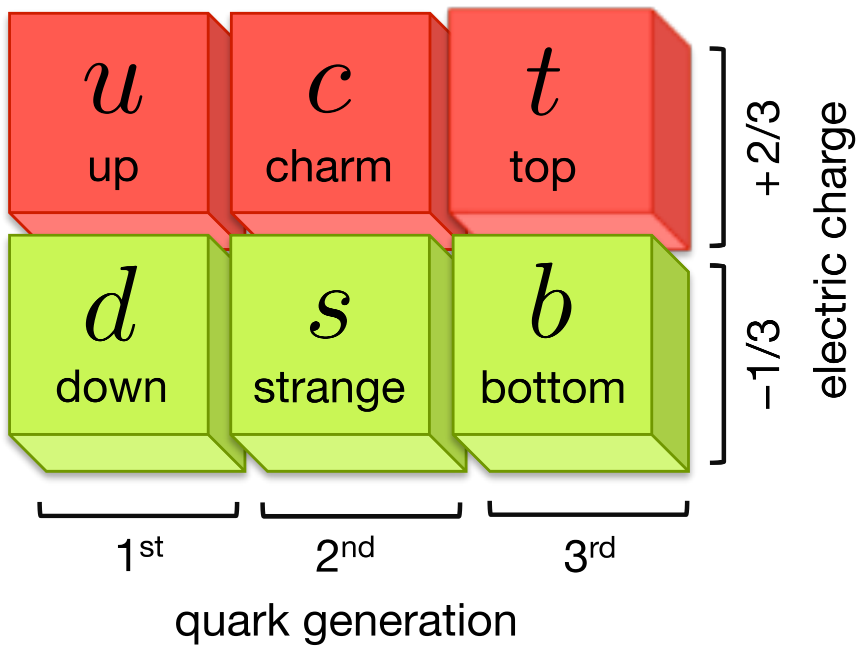
Quark mass spectrum

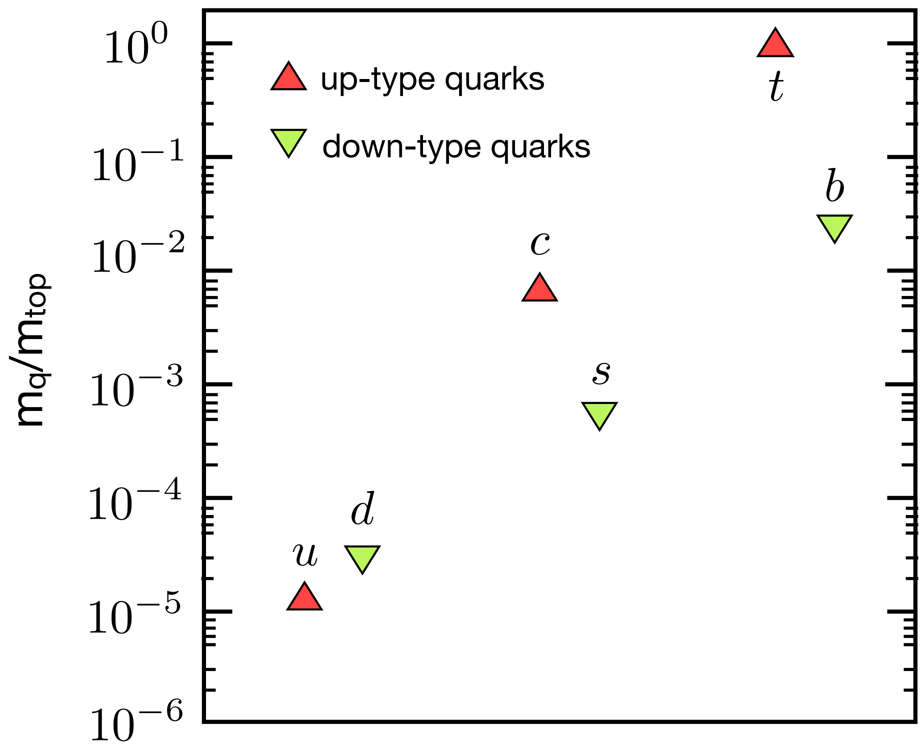




charge 2/3 mass=	up few MeV	charm ~1.6 GeV	top ~172 GeV
charge -1/3 mass =	down few MeV	strange ~100 MeV	bottom ~5 GeV

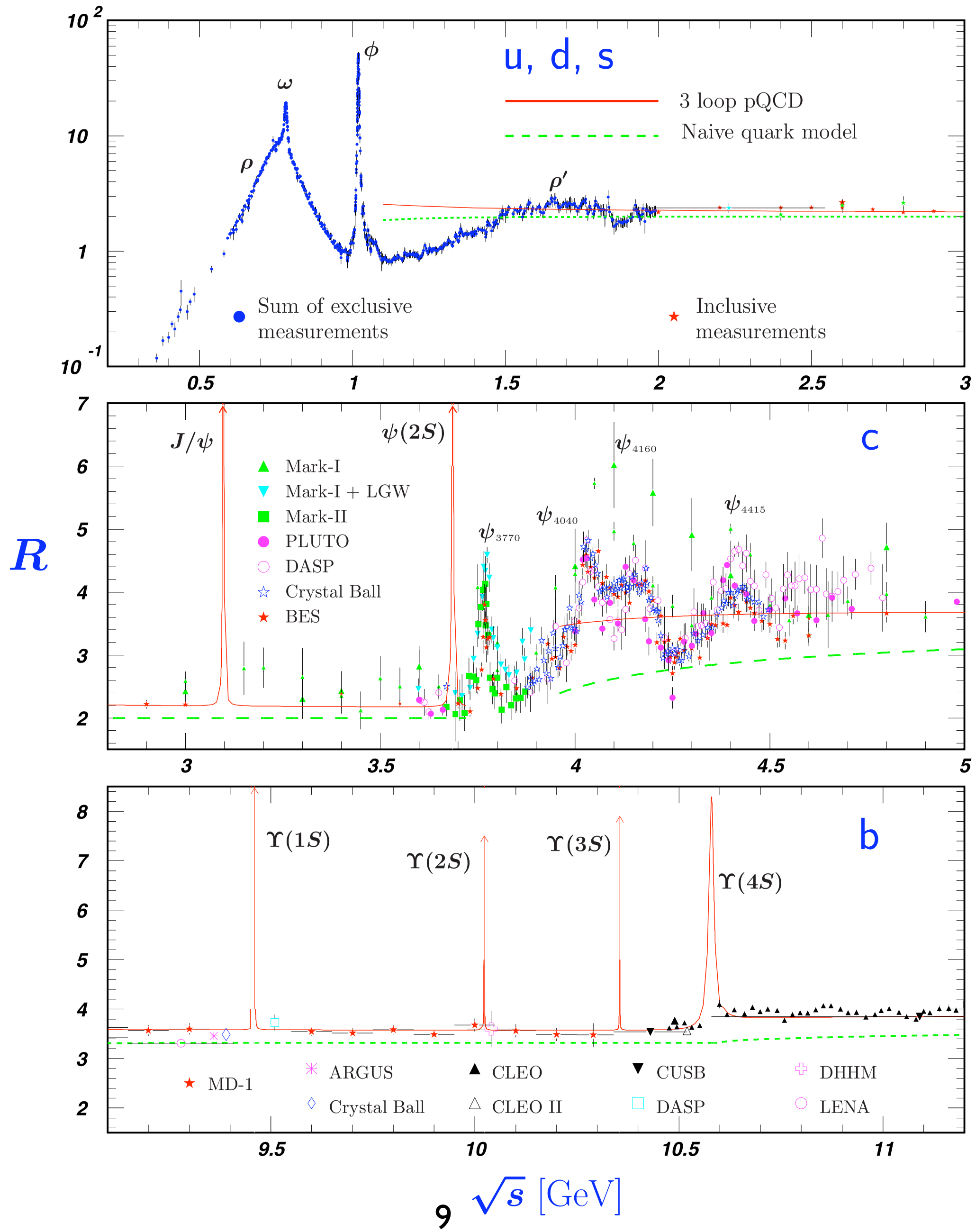
Quark mass spectrum





Six known quarks. Their masses of different quark flavors range from around 2 MeV for the up quark to around 173 GeV for the top. Why these masses are split by almost six orders of magnitude is one of the big mysteries of particle physics

The R-ratio: comparison to data



QED and QCD

- QED and QCD are very similar, yet very different theories
- quarks are a bit like leptons, but there are three of each
- gluons are a bit like photons, but there are eight of them
- gluons interact with themselves
- the QCD coupling is also small at collider energies, but larger then the QED one
- the similarities and differences are evident from the two Lagrangians

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So, let's start by looking at the QED Lagrangian

The QED Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{QED}} &= \mathcal{L}_{\mathrm{Dirac}} + \mathcal{L}_{\mathrm{Maxwell}} + \mathcal{L}_{\mathrm{int}} \\ &= \bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi - \frac{1}{2}(F_{\mu\nu})^2 - e\bar{\psi}\gamma^\mu\psi A_\mu \\ &= \bar{\psi}\,(iD\!\!\!\!/ - m)\,\psi - \frac{1}{2}(F_{\mu\nu})^2\end{aligned}$$

electromagnetic vector potential A_μ

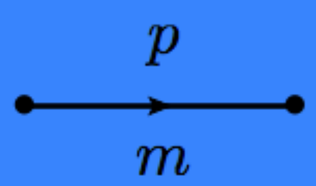
field strengh tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

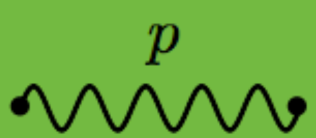
covariant derivative $D_\mu = \partial_\mu + ieA_\mu$

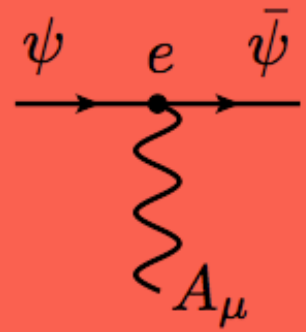
QED Feynman rules

$$\mathcal{L}_{\text{QED}} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{int}}$$

$$= \bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi - \frac{1}{2}(F_{\mu\nu})^2 - e\bar{\psi}\gamma^\mu\psi A_\mu$$

ψ  $\bar{\psi}$ $= \dfrac{i\,(p\!\!\!/ + m)}{p^2 - m^2}$

A_μ  $A_\nu = \dfrac{-ig^{\mu\nu}}{p^2}$

 $= ie\gamma^\mu$

QED gauge invariance

$$\mathcal{L}_{\mathrm{QED}} \;=\; \bar{\psi}\,(i\not{D} - m)\,\psi - \frac{1}{2}(F_{\mu\nu})^2$$

A crucial property of the QED Lagrangian is that it is invariant under

$$\boxed{\psi(x) \to e^{i\alpha(x)}\,\psi(x)\,, \qquad A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)}$$

which acts on the Dirac field as a local phase transformation

Exercise:

Check that the QED Lagrangian is invariant under the above transformations

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Exercise:
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Yang and Mills (1954) proposed that the local phase rotation in QED could be generalized to invariance under any continuos symmetry

[C. N. Yang and R. L. Mills, Phys. Rep. 96 (1954) 191]

The QCD Lagrangian

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu} + \sum_f \bar{\psi}_i^{(f)} \left(i \not{D}_{ij} - m_f \delta_{ij} \right) \psi_j^{(f)}$$

$$D^\mu_{ij} \equiv \partial^\mu \delta_{ij} + i g_s t^a_{ij} A^\mu_a \,, \qquad F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f_{abc} A^b_\mu A^c_\nu$$

⇒ covariant derivative ⇒ field strength

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- QCD flavour blind (differences only due to EW)

The generators of SU(N)

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☛ unitarity ⇒ N^2 conditions　　☛ unit determinant ⇒ 1 condition

So, the fundamental representation of SU(N) has N^2-1 generators t^a :
N×N traceless hermitian matrices ⇒ N^2-1 gluons

$$U = e^{i\theta_a(x)t^a} \qquad\qquad a = 1, \cdots N^2 - 1$$

The Gell-Mann matrices

One explicit representation: $t^A = \frac{1}{2}\lambda^A$

λ^A are the Gell-Mann matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \; \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \; \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Standard normalization: $\mathrm{Tr}(t^a t^b) = T_R\, \delta^{ab} \quad T_R = \frac{1}{2}$

Notice that the first three Gell-Mann matrices contain the three Pauli matrices in the upper-left corner

The generators of SU(N)

Infinitesimal transformations (close to the identity) give complete information about the group structure. The most important characteristic of a group is the commutator of two transformations:

$$\begin{aligned}[U(\delta_1), U(\delta_2)] &\equiv U(\delta_1)U(\delta_2) - U(\delta_2)U(\delta_1) \\ &= (i\delta_1^a)\,(i\delta_2^b)\,[t^a, t^b] + \mathcal{O}(\delta^3)\end{aligned}$$

The two matrices to not commute, therefore the transformations don't. Such a group is called non-abelian.

- Familiar abelian groups: translations, phase transformations U(1) ...
- Familiar non-abelian groups: 3D-rotations

The generators of SU(N)

Consider the commutator

$$Tr([t_a, t_b]) = 0 \quad \Rightarrow \quad [t_a, t_b] = i f_{abc} t^c$$

f_{abc} are the (real) structure constants of the SU(N_c) algebra, they generate a representation of the algebra called adjoint representation

Clearly, f_{abc} is anti-symmetric in (ab). It is easy to show that it is fully antisymmetric

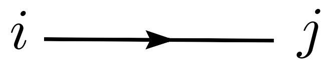
$$i f_{abc} = 2\,\mathrm{Tr}\left([t_a, t_b] t_c\right)$$

and that hence it is fully antisymmetric

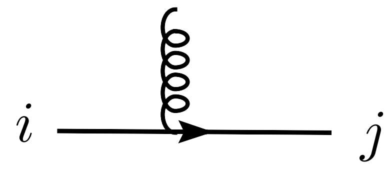
$$f_{abc} = -f_{bac} = -f_{acb}$$

Color algebra: fundamental identities

Fundamental representation 3:

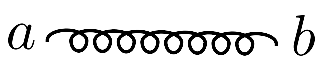


$= \delta_{ij}$



$= t^a_{ij}$

Adjoint representation 8:



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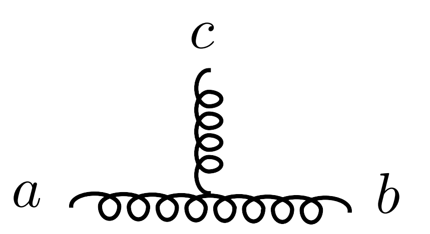
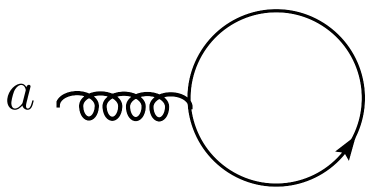


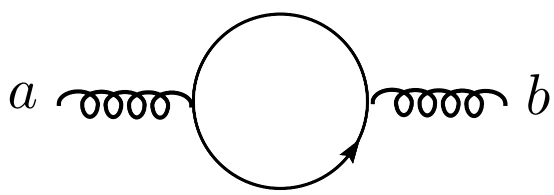
$= i f_{abc}$

Trace identities:



$= 0$

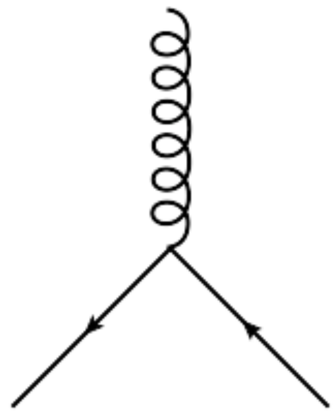
$\mathrm{Tr}(t^a) = 0$



$= T_R$

$\mathrm{Tr}(t^a\, t^b) = \mathrm{T_R}\delta^{ab}$

What do color identities mean physically



$$\bar{\psi}_i \, t^A_{ij} \, \psi_j$$

What does this really mean?

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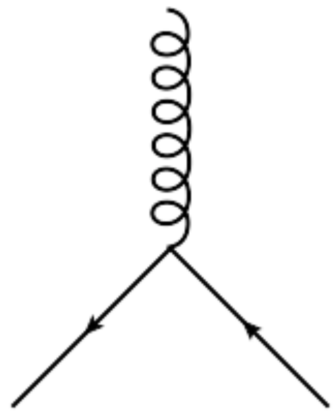

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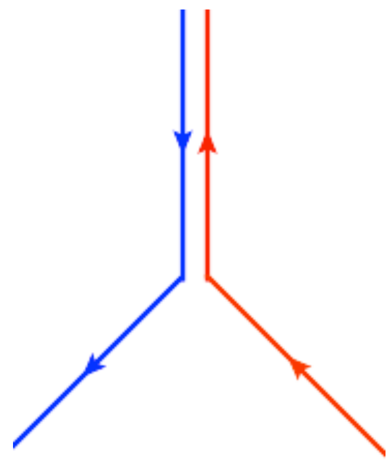


$$(1, 0, 0) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

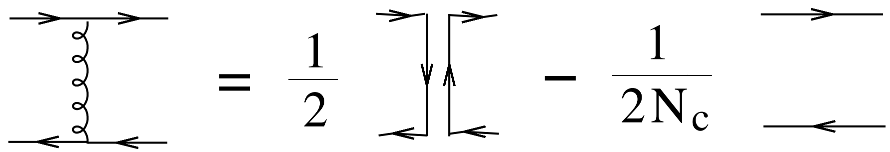
What does this really mean? $\qquad \bar{\psi}_i \qquad t^1_{ij} \qquad \psi_j$

Gluons carry color and anti-color. They repaint quarks and other gluons.

Color algebra: Casimirs & Fierz identity

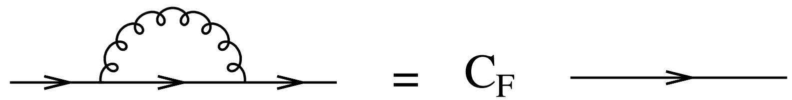
Fierz identity:

$$(t^a)^i_k (t^a)^l_j = \frac{1}{2}\delta^i_j \delta^l_k - \frac{1}{2N_c}\delta^i_k \delta^l_j$$



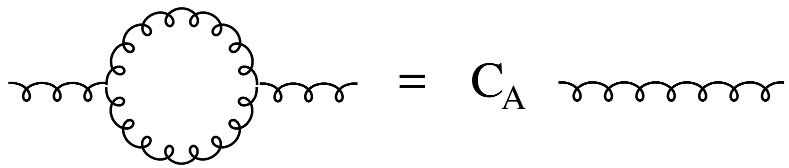
Fundamental representation 3:

$$\sum_a (t^a_{ij})(t^a_{kj}) = C_F \delta_{ij} \qquad C_F = \frac{N_c^2 - 1}{2N_c}$$



Adjoint representation 8:

$$\sum_{cd} f^{acd} f^{bdc} = C_A \delta^{ab} \qquad C_A = N_c$$



Exercises:

1) derive Fierz identity

2) use the Fierz identity to derive the value of C_F and C_A

Gauge invariance

The QCD Lagrangian is invariant under local gauge transformations, i.e. one can redefine the quark and gluon fields independently at every point in space and time without changing the physical content of the theory

- Gauge transformation for the quark field

$$\boxed{\psi \to \psi' = U(x)\psi}$$

- The covariant derivative $(D_\mu)_{ij} = \partial_\mu \delta_{ij} + i g_s t^a_{ij} A^\mu_a$ must transform as (covariant = transforms "with" the field)

$$\boxed{D_\mu \psi \to D'_\mu \psi' = U(x) D_\mu \psi}$$

- From which one derives the transformation property of the gluon field

$$\boxed{t^a A_a \to t^a A'_a = U(x) t^a A_a U^{-1}(x) + \frac{i}{g_s} \left(\partial U(x)\right) U^{-1}(x)}$$

Gauge invariance

The QCD Lagrangian is invariant under local gauge transformations, i.e. one can redefine the quark and gluon fields independently at every point in space and time without changing the physical content of the theory

- It follows that

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi} U^\dagger(x)$$

$$t^a F^a_{\mu\nu} \to t^a F^{a'}_{\mu\nu} = U(x) t^a F^a_{\mu\nu} U^{-1}(x)$$

e.g. because $i\, g_s\, t^a F^a_{\mu\nu} = [D_\mu, D_\nu]$

- Therefore the QCD Lagrangian is indeed gauge invariant

$$-\frac{1}{4} F'^{\mu\nu}_a F'^a_{\mu\nu} = -\frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu}$$

$$\sum_f \bar{\psi}'^{(f)}_i \left(i \not{D}'_{ij} - m_f \delta_{ij} \right) \psi'^{(f)}_j = \sum_f \bar{\psi}^{(f)}_i \left(i \not{D}_{ij} - m_f \delta_{ij} \right) \psi^{(f)}_j$$

Gauge invariance

The QCD Lagrangian is invariant under local gauge transformations, i.e. one can redefine the quark and gluon fields independently at every point in space and time without changing the physical content of the theory

Remarks:

- the field strength alone is not gauge invariant in QCD (unlike in QED) because of self interacting gluons (carries of the force carry colour, unlike the photon)

- a gluon mass term violate gauge invariance and is therefore forbidden (as for the photon). On the other hand quark mass terms are gauge invariant.

$$\cancel{m^2 A_\mu A^\mu}$$

Isospin symmetry

Isospin SU(2) symmetry: invariance under $u \leftrightarrow d$

Particles in the same isospin multiplet have very similar masses (proton and neutron, neutral and charged pions)

The QCD Lagrangian has isospin symmetry if $m_u = m_d$ or $m_u, m_d \to 0$

The fermionic Lagrangian becomes

$$\mathcal{L}_F = \sum_f \left(\bar{\psi}_L^{(f)} \not{D} \psi_L^{(f)} + \bar{\psi}_R^{(f)} \not{D} \psi_R^{(f)} \right) - \sum_f m_f \left(\bar{\psi}_R^{(f)} \psi_L^{(f)} + \bar{\psi}_L^{(f)} \psi_R^{(f)} \right)$$

$$\psi_L = P_L \psi \,, \quad \psi_R = P_R \psi \,, \quad P_{L/R} = \frac{1}{2}(1 \mp \gamma_5)$$

So neglecting fermion masses the Lagrangian has the larger symmetry

$$SU_L(N_f) \times SU_R(N_f) \times U_L(1) \times U_R(1)$$

Feynman rules: propagators

Obtain quark/gluon propagators from free piece of the Lagrangian

<u>Quark propagator</u>: replace i∂ → k and take the i × inverse

$$\mathcal{L}_{\mathrm{q,free}} = \sum_f \bar{\psi}_i^{(f)} \left(i\partial\!\!\!/ - m_f \right) \delta_{ij} \psi_j^{(f)}$$

$$\alpha, i \xrightarrow[k,m]{\qquad\qquad} \beta, j \quad = \quad \left(\frac{i}{k\!\!\!/ - m} \right)_{\alpha\beta} \delta_{ij}$$

<u>Gluon propagator</u>: replace i∂ → k and take the i × inverse ?

$$\mathcal{L}_{\mathrm{g,free}} = \frac{1}{2} A^\mu \left(\Box g_{\mu\nu} - \partial_\mu \partial_\nu \right) A^\nu$$

➡ inverse does not exist, since $\left(\Box g_{\mu\nu} - \partial_\mu \partial_\nu \right) \partial_\mu = \Box \partial_\nu - \Box \partial_\nu = 0$

How can one to define the propagator ?

Gauge fixing

Solution:
add to the Lagrangian a gauge fixing term which depends on an arbitrary parameter ξ

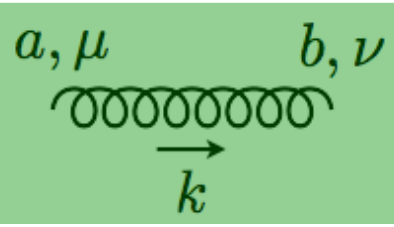
In covariant gauges:

$$\mathcal{L}_{\text{gauge fixing}} = -\frac{1}{\xi}\left(\partial^\mu A_\mu^A\right)^2$$

ξ=1 Feynman gauge
ξ=0 Landau gauge

Gluon propagator:

$$\frac{-i}{k^2}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\delta^{ab} =$$

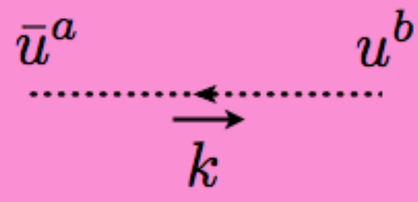


Gauge fixing explicitly breaks gauge invariance. However, in the end physical results are independent of the gauge choice. Powerful check of higher order calculations: verify that the ξ dependence fully cancels in the final result

Ghosts

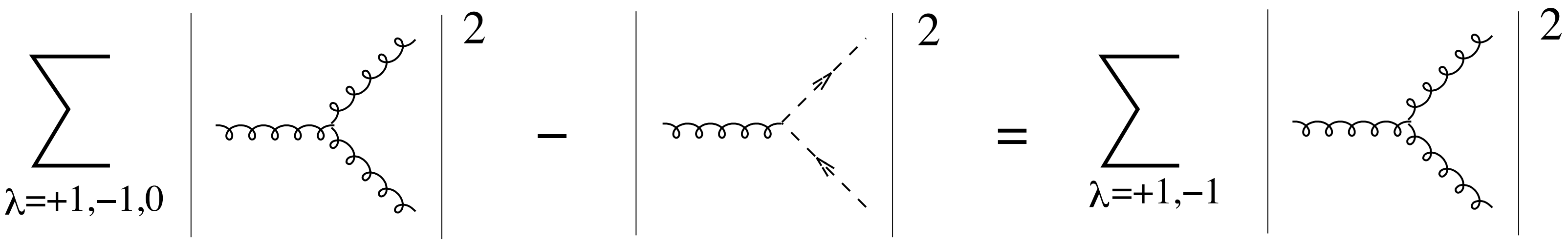
In covariant gauges gauge fixing term must be supplemented with ghost term to cancel unphysical longitudinal degrees of freedom which should not propagate

$$\mathcal{L}_{\text{ghost}} = \partial_\mu \eta^{a\dagger} D^\mu_{ab} \eta^b$$



$$= \frac{i}{k^2}\delta^{ab}$$

η: complex scalar field which obeys Fermi statistics



Axial gauges

Alternative: choose an axial gauge (introduce an arbitrary direction n)

$$\mathcal{L}_{\text{axial gauge}} = -\frac{1}{\xi}\left(n^\mu A_\mu^A\right)^2$$

The gluon propagator becomes

$$d_{\mu\nu} = \frac{i}{k^2}\left(-g_{\mu\nu} + \frac{n_\mu k_\nu + n_\nu k_\mu}{n\cdot k} + \frac{(n^2+\xi k^2)k_\mu k_\nu}{(n\cdot k)^2}\right)\delta_{ab}$$

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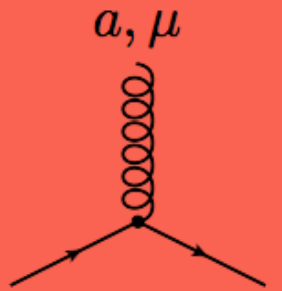
Light cone gauge: $n^2 = 0$ and $\xi = 0$

Axial gauges for $k^2 \to 0$

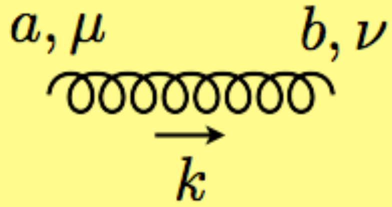
$$d_{\mu\nu} k^\mu = d_{\mu\nu} n^\mu = 0$$

i.e. only two physical polarizations propagate, that's why often the term physical gauge is used

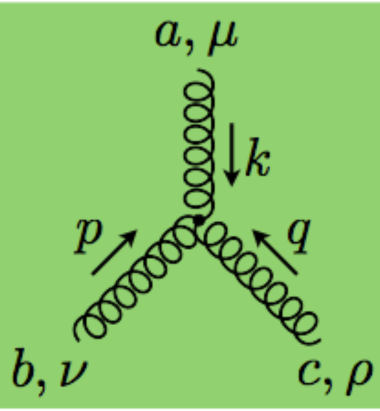
QCD Feynman rules: the vertices



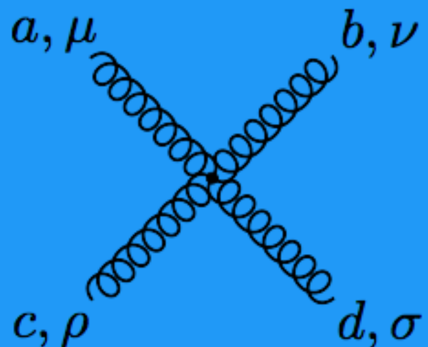
$$= \quad i g_s \gamma^\mu t^a$$



$$= \left(\frac{-i g_{\mu\nu}}{k^2} \right) \delta^{ab}$$



$$= \quad g_s f^{abc} \Big[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu \Big]$$

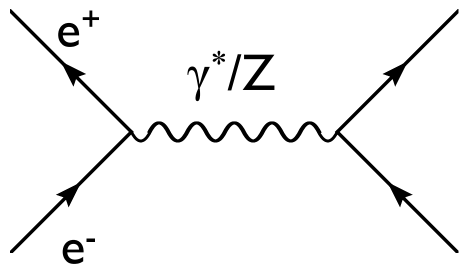


$$= \quad -i g_s^2 \Big[f^{abe} f^{cde} \left(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho} \right) + f^{ace} f^{bde} \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho} \right) + f^{ade} f^{bce} \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} \right) \Big]$$

Perturbative expansion of the R-ratio

The R-ratio is defined as

$$R \equiv \frac{\sigma(e^+e^- \to \mathrm{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$



At lowest order in perturbation theory

$$\sigma(e^+e^- \to \mathrm{hadrons}) = \sigma_0(e^+e^- \to q\bar{q})$$

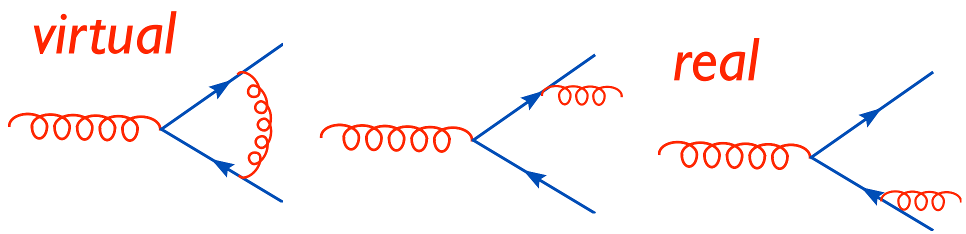
The PT treatment works since the scattering happens at large momentum transfer (short time), while hadronization happens at low momentum transfer, i.e. too late to change the original probability distribution

Since common factors cancel in numerator/denominator, to lowest order one finds

$$R_0 = \frac{\sigma_0(\gamma^* \to \mathrm{hadrons})}{\sigma_0(\gamma^* \to \mu^+\mu^-)} = N_c \sum_f q_f^2$$

The R-ratio: perturbative expansion

First order correction



Real and virtual do not interfere since they have a different # of particles.
The amplitude squared becomes

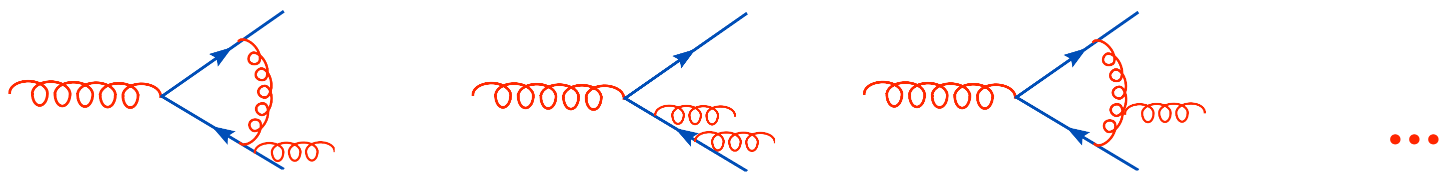
$$|A_1|^2 = |A_0|^2 + \alpha_s \left(|A_{1,r}|^2 + 2\mathrm{Re}\{A_0 A_{1,v}^*\}\right) + \mathcal{O}(\alpha_s^2) \qquad \alpha_s = \frac{g_s^2}{4\pi}$$

Integrating over phase space, the first order result reads

$$R_1 = R_0 \left(1 + \frac{\alpha_s}{\pi}\right)$$

R-ratio and UV divergences

To compute the second order correction one has to compute diagrams like these and many more



One gets

$$R_2 = R_0 \left(1 + \frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi}\right)^2 \left(c + \pi b_0 \ln \frac{M_{\rm UV}^2}{Q^2} \right) \right) \qquad b_0 = \frac{11 N_c - 4 n_f T_R}{12\pi}$$

Ultra-violet divergences do not cancel. Result depends on UV cut-off.

Renormalization and running coupling

The divergence is dealt with by renormalization of the coupling constant

$$\alpha_s(\mu) = \alpha_s^{\rm bare} + b_0 \ln \frac{M_{UV}^2}{\mu^2} \left(\alpha_s^{\rm bare}\right)^2$$

R expressed in terms of the renormalized coupling is finite

$$R = R_0 \left(1 + \frac{\alpha_s(\mu)}{\pi} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left(c + \pi b_0 \ln \frac{\mu^2}{Q^2}\right) + \mathcal{O}(\alpha_s^3(\mu))\right)$$

Renormalizability of the theory guarantees that *the same redefinition of the coupling removes all UV divergences from all physical quantities (massless case)*

Will not cover renormalization in these lectures, but it suffices to know that renormalization of S-matrix elements is achieved by replacing bare masses and bare coupling with renormalized ones

- the coupling $\Rightarrow$ β function
- the masses $\Rightarrow$ anomalous dimensions γ_m

The beta-function

$$\boxed{\beta(\alpha_s^{\text{ren}}) \equiv \mu^2 \frac{d\alpha_s(\mu^2)}{d\mu^2}}$$

The renormalized coupling is

$$\alpha_s(\mu) = \alpha_s^{\text{bare}} + b_0 \ln \frac{M_{UV}^2}{\mu^2} \left(\alpha_s^{\text{bare}}\right)^2$$

So, one immediately gets

$$\beta = -b_0 \alpha_s^2(\mu) + \ldots$$

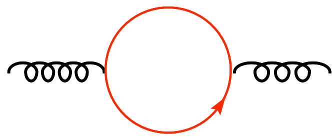
Integrating the differential equation one finds at lowest order

$$\frac{1}{\alpha_s(\mu)} = b_0 \ln \frac{\mu^2}{\mu_0^2} + \frac{1}{\alpha_s(\mu_0)} \qquad \Rightarrow \qquad \alpha_s(\mu) = \frac{1}{b_0 \ln \frac{\mu^2}{\Lambda^2}}$$

More on the beta-function

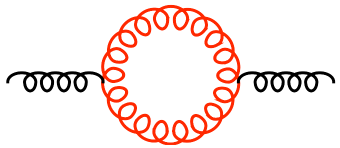
<u>Roughly speaking:</u>

(a) quark loop vacuum polarization diagram gives a negative contribution to $b_0 \sim n_f$



(a)

(b) gluon loop gives a positive contribution to $b_0 \sim N_c$



(b)

Since (b) > (a) $\Rightarrow$ $b_{0,QCD} > 0 \Rightarrow$ overall negative beta-function in QCD

While in QED (b) = 0 $\Rightarrow b_{0,QED} < 0$

$$\beta_{\mathrm{QED}} = \frac{1}{3\pi}\alpha^2 + \dots$$

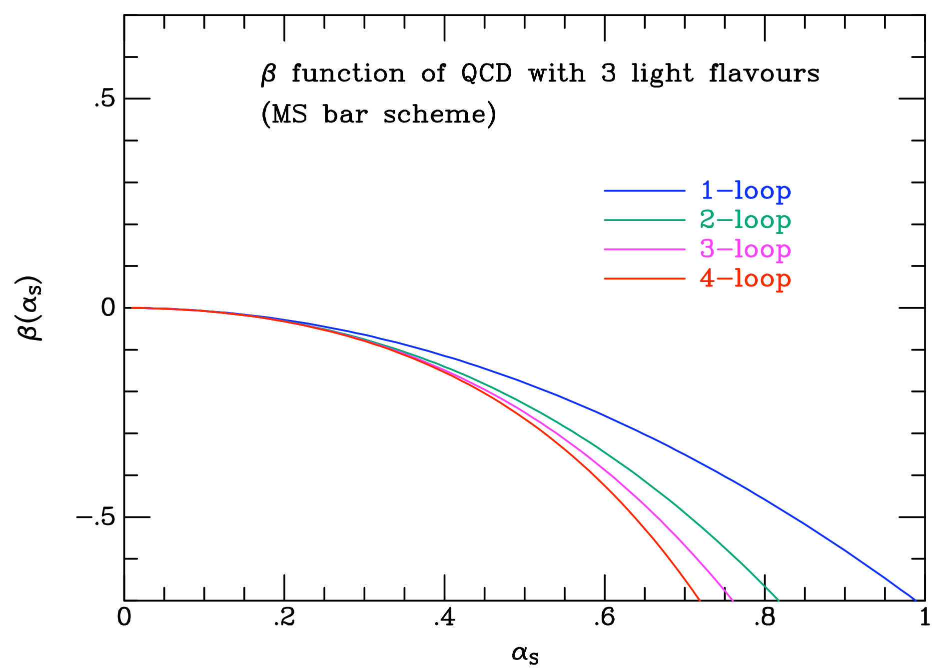
More on the beta-function

Perturbative expansion of the beta-function:

$$\beta = -\alpha_s^2(\mu) \sum_i b_i \alpha_s^i(\mu)$$

$$b_0 = \frac{11 N_c - 4 n_f T_R}{12\pi}$$

$$b_1 = \frac{17 N_c^2 - 5 N_c n_f - 3 C_F n_f}{24\pi^2}$$

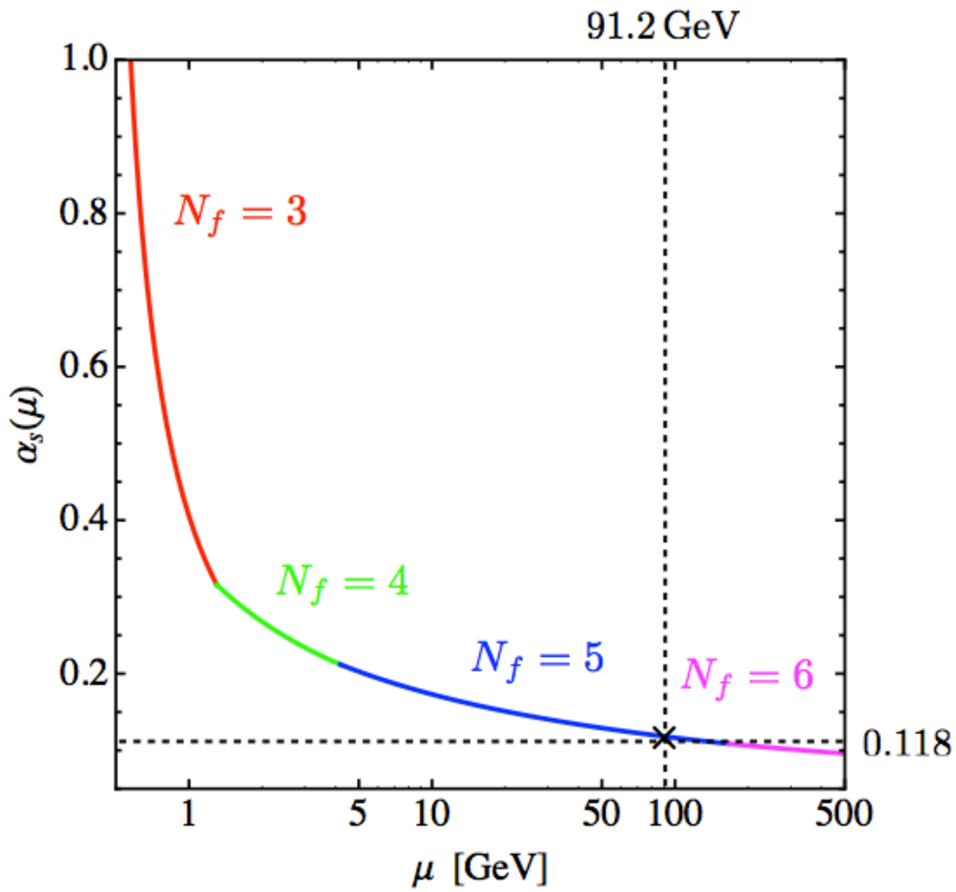


- n_f is the number of active flavours (depends on the scale)
- today, the beta-function known up to four loops, but only first two coefficients are independent of the renormalization scheme

<u>Exercise</u>: proof the above statement [hint: use the fact that at $O(\alpha_s)$ the coupling in two different schemes is related by a finite change]

Active flavours & running coupling

The active field content of a theory modifies the running of the couplings



Constrain New Physics by measuring the running at high scales?

Renormalization Group Equation

Consider a dimensionless quantity A, function of a single scale Q. The dimensionless quantity should be independent of Q. However in quantum field theory this is not true, as renormalization introduces a second scale μ

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So, for any observable A one can write a renormalization group equation

$$\boxed{\left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s}\right] A\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = 0}$$

$$\alpha_s = \alpha_s(\mu^2) \qquad \beta(\alpha_s) = \mu^2 \frac{\partial \alpha_s}{\partial \mu^2}$$

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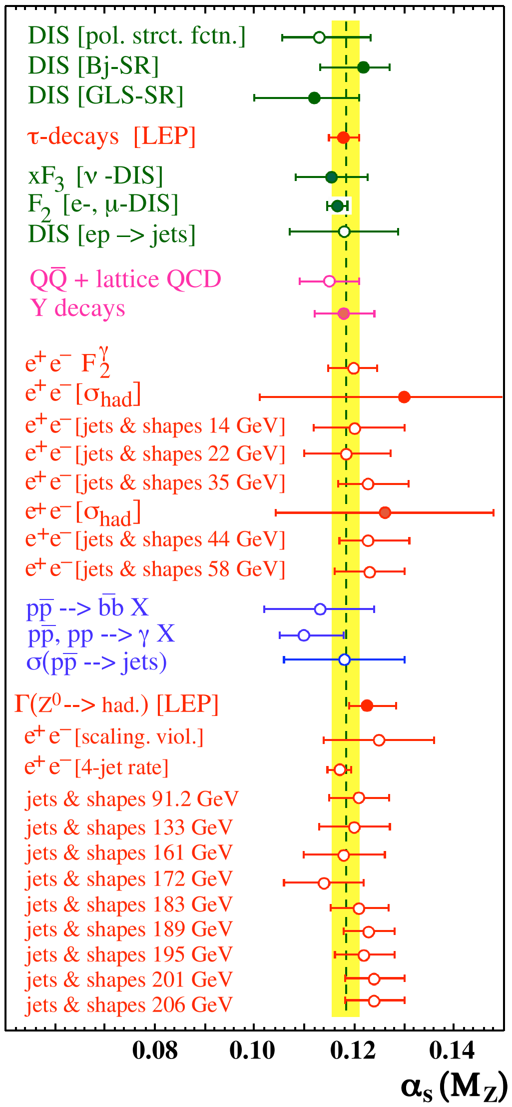
Scale dependence of A enters through the running of the coupling: knowledge of $A(1, \alpha_s(Q^2))$ allows one to compute the variation of A with Q given the beta-function

Measurements of the running coupling

<u>Summarizing:</u>

- overall consistent picture: α_s from very different observables compatible
- α_s is not so small at current scales
- α_s decreases slowly at higher energies (logarithmic only)
- higher order corrections are and will remain important



World average

$$\boxed{\alpha_s(M_{Z^0}) = 0.1184 \pm 0.007}$$