Noncommutative geometry

Spectral Action and the Higgs

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The aim of this talk is to show you that the standard model is, like general relativity, a geometric theory

The geometry which describes it is not that made of points, lines, curves, but some generalization which involves matrices, Hilbert spaces, γ matrices and the like. Something with roots in quantum mechanics, where the geometry of phase space is substituted by operators. Except that now I need something able to describe fields, not particles.

The aim (and we are not too far from this goal), is that in the end this view of the standard model may help unifying it with gravity, and help make testable predictions

The first step will be to find such a generalized geometry

This already exists. It is based on algebras (you may consider them continuous fields) acting on a Hilbert space (you may consider it to be the space of matter fields), with the geometry encoded by a self-adjoint operator, a generalization of the Dirac operator.

When the algebra is commutative this framework enables an algebraic description of ordinary geometry. When the algebra is non abelian, we have a Noncommutative Geometry.

Moreover, the machinery I am building should be able to describe also a quantum spacetime, which should be the arena of a quantum gravity theory

But for the moment I will "limit" myself to the standard model coupled to background gravity, and see if I can say something useful

Throughout the seminar I will be very sketchy, skipping all of the mathematical details, and many of the physical ones. A recent book by Walter Van Suijlekom (Noncommutative Geometry and Particle Physics, Springer 2014) has a full description from a mathematical physicist point of view.

Most of the seminar is based on work of Connes for the mathematical part, and Chamseddine, Connes and Marcolli for the physical part. Important contributions were made by several people including Barrett, Boyle, Martin, Farnsworth, Gracia-Bondia, Iochum, Kastler, Lott, Schucker, Stephan, Van Suijlekom, Varilly. And by my collaborators on this topic: Andrianov, Devastato, Kurkov, Martinetti, Mangano, Miele, Sakellariadou, Sparano, Valcarcel, Vassilievich, Watcharangkool.

Disclaimer: For this seminar noncommutative geometry is not the one given by noncommutative coordinates $\boxed{[x^\mu, x^\nu] = i\theta^{\mu\nu}}$, in fact for most of the talk I will have a recognizable spacetime with usual symmetries.

The starting point is that geometry and its (noncommutative) generalizations are described by the spectral data of three basic ingredients:

- An algebra $\boxed{\mathcal{A}}$ which describes the topology of spacetime. In the commutative case is the algebra of continuous functions

- A Hilbert space $\boxed{\mathcal{H}}$ on which the algebra act as operators, and which also describes the matter fields of the theory.

- A (generalized) Dirac Operator $\boxed{D_0}$ which gives the metric of the space, and other information about the fermions.

An important role is also played by two other operators: $\boxed{\Gamma}$ and $\boxed{J}$. For physicists they are chirality and charge conjugation

There is a profound mathematical result (Gelfand-Najmark) which says that all topological informations on a space can be reconstructed algebraically from the algebra of continuous functions. Points can be reconstructed from the algebraic data, as irreducible representations of the algebra

Algebraic concepts are more robust than those based on “pointwise” geometry and they survive when the algebra is noncommutative, enabling us to do noncommutative geometry

The geometric aspects are encoded in the Dirac operator. Connes and other mathematicians are building some sort of dictionary translating all concepts or ordinary geometry (differential forms, vector bundles, metric distances . . .) into algebraic structures so that they can be generalized to the noncommutative case

Let us build a noncommutative geometry corresponding to a gauge theory. As algebra we take matrix valued functions on spacetime

$$\boxed{\mathcal{A} = C(M) \otimes A_F}$$

This must be an algebra. more precisely a $\boxed{C^*}$-algebra, not just a group. In the finite dimensional case these are only matrices with real, complex or quaternions entries, or they combination.

Represent this algebra as operators on a Hilbert space of spinors with extra indices corresponding to the representations of the algebra

$$\boxed{\mathcal{H} = Sp(M) \otimes \mathcal{H}_F}$$

The operators $\boxed{\Gamma = \gamma_5 \otimes \gamma}$ and $\boxed{J}$ split $\boxed{\mathcal{H}}$ into left-right and particle-antiparticle subpaces respectively

Note that since we are dealing with algebras, and not groups, the allowed representations are either the trivial one, or the fundamental one.

The gauge group, i.e. the invariances of the action which I will introduce shortly, is the unimodular group of the algebra, the unitary elements of the algebra with unit determinate , and the representations for the group as well have to be the same. This is true for the standard model, and for $\boxed{\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{SU}(4)_\mathrm{PS}}$, but is not true for generic GUT's

Define a proper generalized Dirac operator. For the commutative case it is $D_0 = \partial\!\!\!/ + m$, and the fermionic action is $\langle\psi|D_0\psi\rangle$

The coupling with a background is done adding to D_0 a potential, i.e. a connection one-form, which in this framework is a well defined object, generically defined as $D = D_0 + A\!\!\!/ = D_0 + \sum_i a_i[D_0, b_i]$,with $a_i, b_i \in \mathcal{A}$

This is the commutative case, when the internal parts $\mathcal{A}_F, \mathcal{H}_F$ are absent.

In the noncommutative setting as Dirac operator we take

$$D_0 = (\partial\!\!\!/ + \omega\!\!\!/) \otimes \mathbb{I} + \gamma_5 \otimes D_F$$

Where ω_μ is the spin connection (in the case of curved background)

D_F is a finite matrix containing all masses and mixings of the fermions. Calculating D gives the potentials corresponding to the unitary group of the algebra, which becomes automatically the gauge group

The procedure gives automatically an extra field, the Higgs, as a part of the connection one-form, obtained by D_F rater than $\partial\!\!\!/$

In this way the Higgs is the "vector" boson corresponding to the internal degrees of freedom

For the bosonic part of the action, we take, the spirit of noncommutative geometry a regularized trace of the Dirac operator, the spectral action:

$$S_B = \operatorname{Tr} \chi \left(\frac{D_A^2}{\Lambda^2} \right)$$

where Λ an energy cutoff scale, and χ is a cutoff function, for example the characteristic function of the unit interval. In this case the spectral action becomes the number of eigenvalued of D_A smaller than the cutoff Λ

The spectral action can be obtained from the fermionic action using finite mode regularization, and from zeta function regularization. Unfortunately there is no time to discuss this.

The spectral action is a function of a Laplacian, it can be expressed using known heat kernel techniques. If one uses as Dirac operator the usual covariant derivative then the first nontrivial order is the usual bosonic action.

Technically the bosonic spectral action is a sum of residues and can be expanded in a power series in terms of Λ^{-1} as

$$S_B = \sum_n f_n \, a_n(D^2/\Lambda^2)$$

where the f_n are the momenta of χ

$$\begin{aligned} f_0 &= \int_0^\infty \mathrm{d}x \, x\chi(x) \\ f_2 &= \int_0^\infty \mathrm{d}x \, \chi(x) \\ f_{2n+4} &= (-1)^n \partial_x^n \chi(x)\Big|_{x=0} \quad n \geq 0 \end{aligned}$$

the a_n are the Seeley-de Witt coefficients which vanish for n odd. For D^2 of the form

$$D^2 = -(g^{\mu\nu}\partial_\mu\partial_\nu \mathbb{1} + \alpha^\mu \partial_\mu + \beta)$$

Defining (in term of a generalized spin connection containing also the gauge fields)

$$
\begin{aligned}
\omega_\mu &= \frac{1}{2} g_{\mu\nu}\left(\alpha^\nu + g^{\sigma\rho}\Gamma^\nu_{\sigma\rho}\mathbb{1}\right) \\
\Omega_{\mu\nu} &= \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu] \\
E &= \beta - g^{\mu\nu}\left(\partial_\mu\omega_\nu + \omega_\mu\omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho\right)
\end{aligned}
$$

then

$$
\begin{aligned}
a_0 &= \frac{\Lambda^4}{16\pi^2}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\,\mathbb{1}_F \\
a_2 &= \frac{\Lambda^2}{16\pi^2}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\left(-\frac{R}{6} + E\right) \\
a_4 &= \frac{1}{16\pi^2}\frac{1}{360}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\,(-12\nabla^\mu\nabla_\mu R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} \\
&\quad +2R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - 60RE + 180E^2 + 60\nabla^\mu\nabla_\mu E + 30\Omega_{\mu\nu}\Omega^{\mu\nu}
\end{aligned}
$$

tr is the trace over the inner indices of the finite algebra $\mathcal{A}_F$ and in Ω and E are contained the gauge degrees of freedom including the gauge stress energy tensors and the Higgs, which is given by the inner fluctuations of D

The game is therefore to find a set of spectral data: Algebra, Hilbert Space and Dirac Operator and then 'crank a machine'

The first application was to consider the algebra of functions valued in diagonal 2×2 matrices (a two sheeted spacetime). As Hilbert space that of left and right Dirac spinor and

$$D_0 = \begin{pmatrix} \partial\!\!\!/ & m \\ m & \partial\!\!\!/ \end{pmatrix}$$

Then the covariant Dirac operator will be:

$$D_A = \begin{pmatrix} \partial\!\!\!/ + A\!\!\!/_L & \phi \\ \phi & \partial\!\!\!/ + A\!\!\!/_R \end{pmatrix}$$

This will give the Lagrangian of a $U(1)_L \times U(1)_R$ breaking to $U(1)$ theory. With the Higgs being the "vector" boson corresponding to the internal degree of freedom.

This space is "almost" noncommutative, in the sense that there still is an underlying spacetime, and and internal noncommutative but finite dimensional algebra. In the following we will consider algebras is of the kind $\boxed{\mathcal{A} = C(\mathbb{R}^4) \otimes \mathcal{A}_F}$, where $\boxed{\mathcal{A}_F}$ is a finite dimensional matrix algebra.

The algebra to describe the standard model is

$$\boxed{\mathcal{A}_F = \mathrm{Mat}(\mathbb{C})_3 \oplus \mathbb{H} \oplus \mathbb{C}}$$

$\boxed{\mathbb{H}}$ are the quaternions, which we represent as $\boxed{2 \times 2}$ matrices

The unimodular elements of the algebra which correspond to the symmetries of the standard model: $\boxed{SU(3) \times SU(2) \times U(1)}$

This algebra must be represented as operators on a Hilbert space, which also has a continuos infinite dimensional part (spinors on spacetime) times a finite dimensional one: $\boxed{\mathcal{H} = \mathrm{sp}(\mathbb{R}) \otimes \mathcal{H}_F}$. The grading given by $\boxed{\Gamma}$ splits it into a left and right subspace: $\boxed{\mathcal{H}_L \oplus \mathcal{H}_R}$

The $\boxed{J}$ operator basically exchange the two chiralities and conjugates, thus effectively making the algebra act form the right.

For $\boxed{\mathcal{H}_{\mathcal{F}}}$ we take the fermions: 32 degrees of freedom per generation.

Note the the full Hilbert space is the tensor product of this finite dimensional space times the usual spinorial degrees of freedom. So the states are overcounted. This is called fermion doubling.

One has to represent the algebra on this Hilbert space (in a reducible way). There are several restrictions imposed by the theory, namely the Dirac operator, one forms, chirality $\boxed{\Gamma}$ and charge conjugation $\boxed{J}$ have to commute or anticommute according to given rules dictated by the fact that we are generalizing a geometry to the noncommutative setting.

This imposes several restrictions, mainly to the representation of the algebra. The scheme works for the standard model, but not for a generic gauge theory, even enlarging the Hilbert space or the algebra

In fact the scheme almost singles out the standard model as the smallest non trivial gauge group which works.

Then one cranks the machine and obtains the lagrangian of the standard model coupled with gravity

$$\begin{gathered}
\mathcal{L}_{SM} = -\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4}g_s^2 f^{abc} f^{ade} g^b_\mu g^c_\nu g^d_\mu g^e_\nu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \\
\tfrac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c_w^2} M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - igc_w(\partial_\nu Z^0_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu (W^+_\mu \partial_\nu W^-_\mu - \\
W^-_\mu \partial_\nu W^+_\mu) + Z^0_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - igs_w(\partial_\nu A_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu (W^+_\mu \partial_\nu W^-_\mu - \\
W^-_\mu \partial_\nu W^+_\mu) + A_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - \tfrac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \tfrac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + \\
g^2 c_w^2 (Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2 (A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w (A_\mu Z^0_\nu (W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \tfrac{1}{2}\partial_\mu H \partial_\mu H - 2M^2 \alpha_h H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - \tfrac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \\
\beta_h \left(\tfrac{2M^2}{g^2} + \tfrac{2M}{g} H + \tfrac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)\right) + \tfrac{2M^4}{g^2}\alpha_h - g\alpha_h M \left(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-\right) - \\
\tfrac{1}{8}g^2 \alpha_h \left(H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2 H^2\right) - gM W^+_\mu W^-_\mu H - \\
\tfrac{1}{2}g\tfrac{M}{c_w^2} Z^0_\mu Z^0_\mu H - \tfrac{1}{2}ig \left(W^+_\mu (\phi^0 \partial_\mu \phi^- - \phi^- \partial_\mu \phi^0) - W^-_\mu (\phi^0 \partial_\mu \phi^+ - \phi^+ \partial_\mu \phi^0)\right) + \\
\tfrac{1}{2}g \left(W^+_\mu (H \partial_\mu \phi^- - \phi^- \partial_\mu H) + W^-_\mu (H \partial_\mu \phi^+ - \phi^+ \partial_\mu H)\right) + \tfrac{1}{2}g\tfrac{1}{c_w}(Z^0_\mu (H \partial_\mu \phi^0 - \phi^0 \partial_\mu H) + \\
M\, (\tfrac{1}{c_w} Z^0_\mu \partial_\mu \phi^0 + W^+_\mu \partial_\mu \phi^- + W^-_\mu \partial_\mu \phi^+) - ig\tfrac{s_w^2}{c_w} M Z^0_\mu (W^+_\mu \phi^- - W^-_\mu \phi^+) + igs_w M A_\mu (W^+_\mu \phi^- - W^-_\mu \phi^+) - \\
ig\tfrac{1-2c_w^2}{2c_w} Z^0_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) + igs_w A_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) - \tfrac{1}{4}g^2 W^+_\mu W^-_\mu \left(H^2 + (\phi^0)^2 + 2\phi^+\phi^-\right) - \\
\tfrac{1}{8}g^2 \tfrac{1}{c_w^2} Z^0_\mu Z^0_\mu \left(H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2 \phi^+\phi^-\right) - \tfrac{1}{2}g^2 \tfrac{s_w^2}{c_w} Z^0_\mu \phi^0 (W^+_\mu \phi^- + W^-_\mu \phi^+) - \\
\tfrac{1}{2}ig^2 \tfrac{s_w^2}{c_w} Z^0_\mu H (W^+_\mu \phi^- - W^-_\mu \phi^+) + \tfrac{1}{2}g^2 s_w A_\mu \phi^0 (W^+_\mu \phi^- + W^-_\mu \phi^+) + \tfrac{1}{2}ig^2 s_w A_\mu H (W^+_\mu \phi^- - W^-_\mu \phi^+) - \\
g^2 \tfrac{s_w}{c_w}(2c_w^2 - 1) Z^0_\mu A_\mu \phi^+\phi^- - g^2 s_w^2 A_\mu A_\mu \phi^+\phi^- + \tfrac{1}{2}ig_s\, \lambda^a_{ij} (\bar{q}^\sigma_i \gamma^\mu q^\sigma_j) g^a_\mu - \bar{e}^\lambda (\gamma\partial + m^\lambda_e) e^\lambda - \bar{\nu}^\lambda (\gamma\partial + \\
m^\lambda_\nu) \nu^\lambda - \bar{u}^\lambda_j (\gamma\partial + m^\lambda_u) u^\lambda_j - \bar{d}^\lambda_j (\gamma\partial + m^\lambda_d) d^\lambda_j + igs_w A_\mu \left(-(\bar{e}^\lambda \gamma^\mu e^\lambda) + \tfrac{2}{3}(\bar{u}^\lambda_j \gamma^\mu u^\lambda_j) - \tfrac{1}{3}(\bar{d}^\lambda_j \gamma^\mu d^\lambda_j)\right) + \\
\tfrac{ig}{4c_w} Z^0_\mu \{(\bar{\nu}^\lambda \gamma^\mu (1+\gamma^5) \nu^\lambda) + (\bar{e}^\lambda \gamma^\mu (4s_w^2 - 1 - \gamma^5) e^\lambda) + (\bar{d}^\lambda_j \gamma^\mu (\tfrac{4}{3}s_w^2 - 1 - \gamma^5) d^\lambda_j) + (\bar{u}^\lambda_j \gamma^\mu (1 - \tfrac{8}{3}s_w^2 + \\
\gamma^5) u^\lambda_j)\} + \tfrac{ig}{2\sqrt{2}} W^+_\mu \left((\bar{\nu}^\lambda \gamma^\mu (1+\gamma^5) {U^{lep}}_{\lambda\kappa} e^\kappa) + (\bar{u}^\lambda_j \gamma^\mu (1+\gamma^5) C_{\lambda\kappa} d^\kappa_j)\right) + \\
\tfrac{ig}{2\sqrt{2}} W^-_\mu \left((\bar{e}^\kappa {U^{lep}}^\dagger_{\kappa\lambda} \gamma^\mu (1+\gamma^5) \nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\kappa\lambda} \gamma^\mu (1+\gamma^5) u^\lambda_j)\right) + \\
\tfrac{ig}{2M\sqrt{2}} \phi^+ \left(-m^\kappa_e (\bar{\nu}^\lambda {U^{lep}}_{\lambda\kappa} (1-\gamma^5) e^\kappa) + m^\lambda_\nu (\bar{\nu}^\lambda {U^{lep}}_{\lambda\kappa} (1+\gamma^5) e^\kappa)\right) + \\
\tfrac{ig}{2M\sqrt{2}} \phi^- \left(m^\lambda_e (\bar{e}^\lambda {U^{lep}}^\dagger_{\lambda\kappa} (1+\gamma^5) \nu^\kappa) - m^\kappa_\nu (\bar{e}^\lambda {U^{lep}}^\dagger_{\lambda\kappa} (1-\gamma^5) \nu^\kappa)\right) - \tfrac{g}{2}\tfrac{m^\lambda_\nu}{M} H (\bar{\nu}^\lambda \nu^\lambda) - \tfrac{g}{2}\tfrac{m^\lambda_e}{M} H (\bar{e}^\lambda e^\lambda) + \\
\tfrac{ig}{2}\tfrac{m^\lambda_\nu}{M} \phi^0 (\bar{\nu}^\lambda \gamma^5 \nu^\lambda) - \tfrac{ig}{2}\tfrac{m^\lambda_e}{M} \phi^0 (\bar{e}^\lambda \gamma^5 e^\lambda) - \tfrac{1}{4}\, \bar{\nu}_\lambda\, M^R_{\lambda\kappa}\, (1-\gamma_5) \hat{\nu}_\kappa - \tfrac{1}{4}\, \overline{\bar{\nu}_\lambda\, M^R_{\lambda\kappa}\, (1-\gamma_5) \hat{\nu}_\kappa} + \\
\tfrac{ig}{2M\sqrt{2}} \phi^+ \left(-m^\kappa_d (\bar{u}^\lambda_j C_{\lambda\kappa} (1-\gamma^5) d^\kappa_j) + m^\lambda_u (\bar{u}^\lambda_j C_{\lambda\kappa} (1+\gamma^5) d^\kappa_j)\right) + \\
\tfrac{ig}{2M\sqrt{2}} \phi^- \left(m^\lambda_d (\bar{d}^\lambda_j C^\dagger_{\lambda\kappa} (1+\gamma^5) u^\kappa_j) - m^\kappa_u (\bar{d}^\lambda_j C^\dagger_{\lambda\kappa} (1-\gamma^5) u^\kappa_j)\right) - \tfrac{g}{2}\tfrac{m^\lambda_u}{M} H (\bar{u}^\lambda_j u^\lambda_j) - \tfrac{g}{2}\tfrac{m^\lambda_d}{M} H (\bar{d}^\lambda_j d^\lambda_j) + \\
\tfrac{ig}{2}\tfrac{m^\lambda_u}{M} \phi^0 (\bar{u}^\lambda_j \gamma^5 u^\lambda_j) - \tfrac{ig}{2}\tfrac{m^\lambda_d}{M} \phi^0 (\bar{d}^\lambda_j \gamma^5 d^\lambda_j)
\end{gathered}$$

Here the notation is as in [46], as follows.

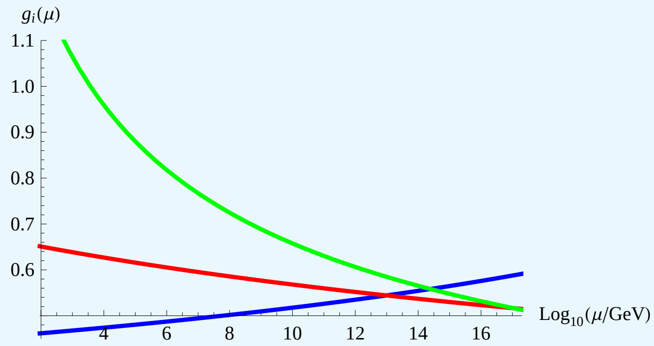
In the Dirac operator there are, as input all data relative to the fermions, but no information on the Higgs mass (vev and quartic coupling coefficient). In this scheme can be calculated from the fermion mass parameters (Yukawa couplings), which are dominated by the top quark coupling. Therefore we have a “prediction” for these quantities

The action we have written is classical and it depend on the cutoff $\boxed{\Lambda}$, and there is no symmetry to ensure that the renormalization flow conserves the relation among Yukawa couplings and Higgs parameters

In this scheme necessarily the coupling contents of the three gauge interactions come out to be equal (apart form the 3/5 normalization of the U(1) coupling)

Putting the previous comments together we will consider that the classical lagrangian coming from the spectral action is written at the scale Λ which is the scale of unification of the three constants.

This is almost true. In the absence of new physics the constants show a pattern in which the three meeting point are roughy at the same value, but at scales ranging form $10^{13} - 10^{17}$ GeV



If one uses the values of the quartic coupling λ calculated form the top Yukawa at a scale in the proper range, and run back to low energy. This to a prediction for the Higgs mass

$$M_H = 175.1 + 5.8 - 7.2 \text{ GeV}$$

Ironically, a few months after the publications of the Chamseddine-Connes paper, Fermilab excluded this value

Now it depends how you consider this theory. if you take it as a mature fully formed theory then the result is wrong. If you take it (as I do) as a tool to investigate the standard model starting from first principles, then I think it is remarkable that a theory based on pure mathematical result gets reasonable numbers

There are several areas in which one could improve, or in other words, if you were pessimist, there were several shortcoming: the theory is Euclidean, it needs a compact spacetime (to define a discrete spectrum), the heat kernel is trustable only to first order, the three couplings do not meet exactly, the Hilbert space needs fermion doubling . . .

Now LHC gave $\boxed{M_H}$, and we know in which direction to move

In some sense the answer was already present, together with some positive features

I mentioned earlier that noncommutative geometry allows the translation of geometric concepts into algebraic language. In the commutative case it is possible to characterize a manifold with properties of the elements of the triple (all five of them)

There is a list of conditions and a theorem (Connes) which proves this.

Since the conditions are all purely algebraic there remain valid in the noncommutative case, defining a noncommutative manifold. And we apply these to the noncommutative matrix part of the triple

In case you want to see the conditions:

1. Dimension There is a nonnegative integer n such that the eigenvalues of D_0 grow os $O(\frac{1}{n})$.

2. Regularity For any $a \in \mathcal{A}$ both a and $[D_0, a]$ belong to the domain of δ^k for any integer k, where δ is the derivation given by $\delta(T) = [|D|, T]$.

3. Finiteness The space $\bigcap_k \mathrm{Dom}(D^k)$ is a finitely generated projective left $\mathcal{A}$ module.

4. Reality There exist J with the commutation relation fixed by the number of dimensions with the property

   (a) Commutant $[a, Jb^* J^{-1}] = 0, \forall a, b$

   (b) First order $[[D, a], b^o = Jb^* J^{-1}] = 0 \ , \forall a, b$

5. Orientation There exists a Hochschild cycle c of degree n which gives the grading γ , This condition gives an abstract volume form.

6. Poincaré duality A Certain intersection form determined by D_0 and by the K-theory of $\mathcal{A}$ and its opposite is nondegenrate.

The conditions are purely algebraic. They can be applied to finite dimensional (matrix) algebras. The result singles out one class of algebras:

$$\boxed{\mathcal{A}_{\mathcal{F}} = \mathbb{M}_a(\mathbb{H}) \oplus \mathbb{M}_{2a}(\mathbb{C}) \qquad a \in \mathbb{N}.}$$

Matrices of quaternions plus matrices of complex numbers. Since quaternions can be represented as $\boxed{2 \times 2}$ matrices, the two algebras are matrices of equal rank

This algebra acts on a finite Hilbert space of dimension $\boxed{2(2a)^2}$. For a non trivial grading it must be $\boxed{a \geq 2}$

The simplest case is:

$$\boxed{\mathcal{A}_F = \mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{C})}$$

Hence an Hilbert space of dimension $\boxed{2(2 \cdot 2)^2 = 32}$, the dimension of $\boxed{\mathcal{H}_F}$ for one generation.

The grading condition $\boxed{[a, \Gamma] = 0}$ reduces the algebra to the left-right algebra

$$\boxed{\mathcal{A}_{LR} = \mathbb{H}_L \oplus \mathbb{H}_R \oplus \mathbb{M}_4(\mathbb{C})}$$

The order one condition reduces further the standard model algebra to $\mathcal{A}_{sm}$, i.e. the algebra whose gauge group is U(1)×SU(2)×U(3)

You may have recognized before a Pati-Salam kind of symmetry. Its presence suggests the presence of a field which causes the breaking to the standard model

This field, which we call $\boxed{\sigma}$ should appear in the Dirac operator in the position corresponding to the neutrino Majorana mass. But unfortunately if I put a nonzero entry in that position the cracking of the machine does not produce a field. Hence one has to include it by hand. Which is unpleasant.

Doing again the running of the physical quantitates with this field does change the Higgs mass, making it compatible with the experimental value

Physics is therefore telling us that into his framework right handed neutrinos, and Majorana masses are crucial

Can we do better? Avoid adding this field by hand? There are three solutions on the market

- Enlarge the Hilbert space introducing new fermions and new interactions. Stephan

- Consider a Grand Symmetry based on $\boxed{\mathbb{M}(\mathbb{H})_4 \oplus \mathbb{M}(\mathbb{C})_8}$ Devastato Lizzi Martinetti

- Violate one of the conditions (the order one conditions) Chamseddine, Connes Van Suijlekom

The latter solutions allow the introduction of a new field $\boxed{\sigma}$ which not only fixes the mass of the Higges making it compatible with 126 GeV, but also solves the possible instability of the theory.

A. Devastato, P. Martinetti and I proposed as solution: a grand symmetry.

In NCG the usual grand unified groups, such as $\boxed{SU(5)}$ or $\boxed{SO(10)}$ do not work. There are very few representations of algebra as opposed to groups. Finite dimensional algebras only have one nontrivial IRR

Fortunately in the standard model there are only weak doublets and colour triplets, so it works

Recall that a finite "manifold" is an algebra: $\boxed{M_a(\mathbb{H}) \oplus M_{2a}(\mathbb{C})}$ acting on a $\boxed{2(2a)^2}$ dimensional Hilbert space. So far we had $\boxed{a = 2,\ 2(2a)^2 = 32 \times 3 = 96}$

The numerology comes out correct

For $M_4(\mathbb{H}) \oplus M_8(\mathbb{C})$ one requires a $2(2 \cdot 4)^2 = 128$ dimensional space. (384 taking generations into account)

This is exactly the dimension of the Hilbert space if we take the fermion doubling into account

It is necessary to look at Hilbert space with different eyes

$$\mathcal{H} = sp(L^2(\mathbb{R}^4)) \otimes \mathcal{H}_F = L^2(\mathbb{R}^4) \otimes \mathsf{H}_F$$

where now the dimensions of H_F is 384

It is still possible to represent the grand algebra $\boxed{M_4(\mathbb{H}) \oplus M_8(\mathbb{C})}$ satisfying all of the manifold conditions. This is highly nontrivial if one keeps the same Hilbert space.

But this time the algebra does not act diagonally on the spinor indices. it mixes them.

If there is time I can show explicitly the details of the two representation (on particle and anti particles). The key point is that in the process spacetime indices, related to the Euclidean symmetries, mix with internal, gauge indices.

The suggestion is to consider this algebra to be some high energy description, so that the standard model is some sort of effective low energy theory, coming after the breaking due to the Dirac operator

In this scheme the $\boxed{\sigma}$ field comes related to the Majorana masses of right handed neutrinos, which are another "hot" (i.e. not well understood) topic in high energy physics

Unfortunately we had to pay the price of the introduction of a new parameter. And this limited our predictive power. So that we cannot say what the mass of the Higgs is, but only that the theory is compatible with experiments

There are however several directions in which the physics might evolve, and some input may again come form mathematics.

For example, what is the meaning of the violation of the order one condition proposed by CC&vS? Can the spectral triple we are proposing be a twisted one?

There is also more work being done on the spectral action itself, the expansion, its relations with anomalies. . .

Conclusions

- What remains to be seen is if from the particle physics point of view the noncommutative geometry is Kepler's law, the theory of gravitation cum differential calculus, the law of diminishing proportions of Hooke, some further epicycloid, Kant's theory of heavens, or phlogiston.

- I find that it is in any case a good example of how a view deeply rooted in mathematics can try to say something about the physical world.

- And I actually hope that sooner or later we can also say something useful for phenomenology

Details

There are several finite dimensional algebras in this game, and I want to look at their representations

Ultimately we want to ge to the the standard model algebra

$$\mathcal{A}_{sm} = \mathbb{C} \oplus \mathbb{H} \oplus \mathbb{M}_3(\mathbb{C}),$$

$\mathbb{H}$ are the quaternions, which we represent as 2×2 matrices

It is possible to have this emerge from the most general algebra which satisfies the condition of being a noncommutative manifold

$$\mathcal{A}_{\mathcal{F}} = \mathbb{M}_a(\mathbb{H}) \oplus \mathbb{M}_{2a}(\mathbb{C}) \qquad a \in \mathbb{N}.$$

This algebra acts on a finite Hilbert space of dimension $\boxed{2(2a)^2}$.
For a non trivial grading it must be $\boxed{a \leq 2}$

$$\boxed{\mathcal{A}_F = \mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{C})}$$

Hence an Hilbert space of dimension $\boxed{2(2 \cdot 2)^2 = 32}$, the dimension of $\boxed{\mathcal{H}_F}$ for one generation.

The grading condition $\boxed{[a, \Gamma] = 0}$ reduces the algebra to the left-right algebra

$$\boxed{\mathcal{A}_{LR} = \mathbb{H}_L \oplus \mathbb{H}_R \oplus \mathbb{M}_4(\mathbb{C})}$$

The order one condition reduces further the algebra to $\mathcal{A}_{sm}$, i.e. the algebra whose unimodular group is U(1)×SU(2)×U(3)

Let us look in detail to a vector in the Hilbert space:

$$\Psi^{\mathsf{C}\mathrm{I}m}_{s\dot{s}\alpha}(x) \in \mathcal{H} = L^2(\mathcal{M}) \otimes \mathsf{H}_F = sp(L^2(\mathcal{M})) \otimes \mathcal{H}_F$$

Note the difference between $\mathcal{H}_F$, which is 96 dimensional, and H_F which is 384 dimensional. The meaning of the indices is as follows:

$$\Psi^{\mathsf{C}\mathrm{I}m}_{s\dot{s}\alpha}(x)$$

$$\begin{aligned} s &= r, l \\ \dot{s} &= \dot{0}, \dot{1} \end{aligned}$$

are the spinor indices. They are not internal indices in the sense that the algebra $\mathcal{A}_F$ acts diagonally on it. They take two values each, and together they make the four indices on an ordinary Dirac spinor.

$$\boxed{\Psi^{\mathsf{c}\mathrm{I}m}_{s\dot{s}\alpha}(x)}$$

$\boxed{\mathrm{I} = 0, \ldots 3}$ indicates a "lepto-colour" index. The zeroth "colour" actually identifies leptons while $\boxed{\mathrm{I} = 1, 2, 3}$ are the usual three colours of QCD.

$$\Psi^{\mathrm{cI}m}_{s\dot{s}\alpha}(x)$$

$\alpha = 1 \ldots 4$ is the flavour index. It runs over the set u_R, d_R, u_L, d_L when $\mathrm{I} = 1, 2, 3$, and ν_R, e_R, ν_L, e_L when $\mathrm{I} = 0$. It repeats in the obvious way for the other generations.

$$\Psi^{\mathsf{C} \mathrm{I} m}_{s \dot{s} \alpha}(x)$$

$\mathsf{C} = 0, 1$ indicates whether we are considering "particles" ($\mathsf{C} = 0$) or "antiparticles" ($\mathsf{C} = 1$).

$$\boxed{\Psi_{s\dot{s}\alpha}^{\mathrm{c}\mathrm{I}m}(x)}$$

$\boxed{m = 1, 2, 3}$ is the generation index. The representation of the algebra of the standard model is diagonal in these indices, the Dirac operator is not, due to Cabibbo-Kobayashi-Maskawa mixing parameters. For this seminar plays no role, and will ignored.

We can now give explicitly the algebra representations in term of these indices.

We start from $\mathcal{A}_F = \mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{C})$, a generic element will depend on 4×4 complex matrix m, and a 2×2 matrix of quaternions q , which we may also see as a 4×4 with some conditions

The representation in its fullness is

$$A_{s\dot{s}\mathsf{DJ}\alpha}^{t\dot{t}\,\mathsf{CI}\beta} = \delta_s^t \delta_{\dot{s}}^{\dot{t}} \left(\delta_0^{\mathsf{C}} \delta_{\mathsf{J}}^{\mathsf{I}} Q_\alpha^\beta + \delta_1^{\mathsf{C}} M_{\mathsf{J}}^{\mathsf{I}} \delta_\alpha^\beta \right)$$

Note the two δ's at the beginning which show that the algebra acts trivially on the spacetime indices, and the fact that the two matrices act on different indices. This ensures the order zero condition, namely exchanging particles with antiparticles, the job done by J, the two representations commute.

The representations of the other algebra are similar, in the case of the standard model there is a differentiation with the leptocolour indices.

The order one condition and a $\boxed{\nu}$ Majorana mass cause the reduction to $\boxed{C^\infty(\mathcal{M}) \otimes \mathcal{A}_{sm}}$, represented as

$$\boxed{a = \{m, q, c\} \text{ with } m \in C^\infty(\mathcal{M}) \otimes \mathbb{M}_3(\mathbb{C}),\ q \in C^\infty(\mathcal{M}) \otimes \mathbb{H},\ c \in C^\infty(\mathcal{M}) \otimes \mathbb{C}}$$

is

$$\boxed{a^{t\dot{t}\mathsf{CI}\beta}_{s\dot{s}\mathsf{DJ}\alpha} = \delta^t_s \delta^{\dot{t}}_{\dot{s}} \left(\delta^{\mathsf{C}}_0 \delta^{\mathsf{I}}_{\mathsf{J}} \left(\mathsf{q}^\beta_\alpha + \mathsf{c}^\beta_\alpha \right) + \delta^{\mathsf{C}}_1 \left(\mathsf{m}^{\mathsf{I}}_{\mathsf{J}} + \tilde{\mathsf{c}}^{\mathsf{I}}_{\mathsf{J}} \right) \delta^\beta_\alpha \right)}$$

where we use the following $\boxed{4 \times 4}$ complex matrices:

$$\mathsf{q} = \begin{pmatrix} 0_2 & \\ & q \end{pmatrix}_{\alpha\beta}, \qquad \mathsf{c} = \begin{pmatrix} c & & \\ & \bar{c} & \\ & & 0_2 \end{pmatrix}_{\alpha\beta}, \qquad \tilde{\mathsf{c}} = \begin{pmatrix} c & \\ & 0_3 \end{pmatrix}_{\mathsf{IJ}}, \qquad \mathsf{m} = \begin{pmatrix} 0 & \\ & m \end{pmatrix}_{\mathsf{IJ}}$$

The breaking from $\mathcal{A}_F$ to $\mathcal{A}_{sm}$ goes with the chirality and first order conditions

I can similarly write down the Dirac operator

$$D = \partial\!\!\!/ \otimes \mathbb{I}_{96} + \gamma^5 \otimes D_F$$

$$D_F = \begin{pmatrix} 0_{8N} & \mathcal{M} & \mathcal{M}_R & 0_{8N} \\ \mathcal{M}^\dagger & 0_{8N} & 0_{8N} & 0_{8N} \\ \mathcal{M}_R^\dagger & 0_{8N} & 0_{8N} & \bar{\mathcal{M}} \\ 0_{8N} & 0_{8N} & \mathcal{M}^T & 0_{8N} \end{pmatrix}.$$

$\mathcal{M}$ contains the Dirac-Yukawa couplings. It links left with right particles. $\mathcal{M}_R = \mathcal{M}_R^T$ contains Majorana masses and links righr particles with right antiparticles. $\mathcal{M} = \begin{pmatrix} M_u & 0_{4N} \\ 0_{4N} & M_d \end{pmatrix} \qquad \mathcal{M}_R = \begin{pmatrix} M_R & 0_{4N} \\ 0_{4N} & 0_{4N} \end{pmatrix}$ where M_u contains the masses of the up, charm and top quarks and the neutrinos (Dirac mass), M_R contains the Majorana neutrinos masses and M_d the remaining quarks and electrons, muon and tau masses, including mixings

I think by now you know the rules. With the algebra and D one builds the one-form, which are the fluctuations of the Dirac operator.The bosonic fields are coming from these one-form $\sum_i a_i[D, b_i]$

But here we run into a problem: the elements of $\mathcal{M}_R$ are the ones which should give rise to the field σ as intermediate boson, on a par with the Higgs, and relate to the breaking of the left-right symmetry.

Except that this term either commutes with D or violates the first order condition!

One alternative would is to have a combination of algebra and Dirac operator violating the first order condition

Or we may look for a bigger algebra...

Consider the case of $\mathbb{M}_a(\mathbb{H}) \oplus \mathbb{M}_{2a}(\mathbb{C})$ for the case $a = 4$

In this case we need a $2 \cdot (2 \cdot 4)^2 = 128$ dimensional space, which for 3 generations gives a 384 dimensional Hilbert space.

I need a representation of the algebra $\mathbb{M}_4(\mathbb{H}) \oplus \mathbb{M}_8(\mathbb{C})$ acting on the spinors I gave earlier, and which can satisfy the stringent order zero conditions

Consider $Q \in \mathbb{M}_4(\mathbb{H})$ and $M \in \mathbb{M}_8(\mathbb{C})$ with indices

$$Q^{\dot{t}\beta}_{\dot{s}\alpha} = \begin{pmatrix} Q^{\dot{0}\beta}_{\dot{0}\alpha} & Q^{\dot{1}\beta}_{\dot{0}\alpha} \\ Q^{\dot{0}\beta}_{\dot{1}\alpha} & Q^{\dot{1}\beta}_{\dot{1}\alpha} \end{pmatrix}_{\dot{s}\dot{t}}, \qquad M^{t\mathsf{I}}_{s\mathsf{J}} = \begin{pmatrix} M^{r\mathsf{I}}_{r\mathsf{J}} & M^{l\mathsf{I}}_{r\mathsf{J}} \\ M^{r\mathsf{I}}_{l\mathsf{J}} & M^{l\mathsf{I}}_{l\mathsf{J}} \end{pmatrix}_{st}$$

where, for any $\dot{s}, \dot{t} \in \{\dot{0}, \dot{1}\}$ and $s, t \in \{l, r\}$, the matrices $Q^{\dot{t}\beta}_{\dot{s}\alpha} \in \mathbb{M}_2(\mathbb{H}), \quad M^{t\mathsf{I}}_{s\mathsf{J}} \in \mathbb{M}_4(\mathbb{C})$ have the index structure defined above

The representation of the element $\boxed{A = (Q, M) \in \mathcal{A}_G}$ is:

$$\boxed{\mathsf{A}^{t\dot{t}\mathsf{C}\;\mathrm{I}\beta}_{s\dot{s}\mathsf{D}\mathsf{J}\alpha} = \left(\delta^{\mathsf{C}}_{0}\delta^{t}_{s}\delta^{\mathrm{I}}_{\mathsf{J}}\,Q^{\dot{t}\beta}_{\dot{s}\alpha} + \delta^{\mathsf{C}}_{1}M^{t\mathrm{I}}_{s\mathsf{J}}\delta^{\dot{t}}_{\dot{s}}\delta^{\beta}_{\alpha}\right)}$$

compare with the previous case

$$\boxed{A^{t\dot{t}\,\mathsf{C}\mathrm{I}\beta}_{s\dot{s}\mathsf{D}\mathsf{J}\alpha} = \delta^{t}_{s}\delta^{\dot{t}}_{\dot{s}}\left(\delta^{\mathsf{C}}_{0}\delta^{\mathrm{I}}_{\mathsf{J}}Q^{\beta}_{\alpha} + \delta^{\mathsf{C}}_{1}M^{\mathrm{I}}_{\mathsf{J}}\delta^{\beta}_{\alpha}\right)}$$

The spinor indices and the internal gauge indices are mixed. We are in a phase in which the Euclidean structure of space time has not yet emerged.

The fermions are not yet fermions

We envisage this Grand Symmetry to belong to a pre geometric phase. At this stage all elements of D_F may be negligible, and the sponsorial part of the direct operator $\partial\!\!\!/$ will cause the "breaking" to a phase in which the symmetries of the phase space emerge

In particular, the order one condition for $\partial\!\!\!/$ causes the reduction of the algebra to $\mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{C})$

And there is an added bonus:

This grand algebra, and a corresponding $\boxed{D}$ operator, have "more room" to operate. Although the Hilbert space is the same, the fact that we abandoned the factorization of the internal indices, gives us more entries to accommodate the Majorana masses

Hence we can put a Majorana mass for the neutrino and at the same time satisfy the order one condition. Then the one form corresponding to this $\boxed{D_\nu}$ will give us the by now famous field $\boxed{\sigma}$, which can only appear before the transition to the geometric spacetime

The natural scale for this mass is to be above a transition which gives the geometric structure. Therefore it is natural that it may be at a high scale. How high we can discuss

The grand symmetry is no ordinary gauge symmetry, there is never a $\boxed{SU(8)}$ in the game for example

It represents a phase in which the internal noncommutative geometry contains also the spin structure, even the Lorentz (Euclidean) structure of space time in a mixed way

The differentiation between the spin structure of spacetime, and the internal gauge theory comes as a breaking of the symmetry, triggered by $\boxed{\sigma}$, which now appears naturally has having to do with the geometry of spacetime.

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The first version of the grand symmetry had a technical problem: the commutator of the D_0 with the elements of the algebra was not a bounded operator. A mathematical requirement for the consistency of the theory.

This has been solved in a more recent version (Devastato and Martinetti) defining a twisted commutator

$$[D_0, a]_\rho = D_0 a - \rho(a) D_0$$

where ρ is an automorphism of the algebra

Also in this case there is an added bonus, it is possible to see that the breaking from $\mathcal{A}_G \to \mathcal{A}_{sm}$ is dynamical

Conclusions

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