Corfu lectures on supersymmetry and supergravity

September 8, 2015

Jean-Pierre Derendinger

Albert Einstein Center for Fundamental Physics, ITP, Bern University

Corfu Summer School on the Standard Model and Beyond
Two documents

- **On supergravity theories after ~ 40 years**
  
  
  arXiv:1509.01195 [hep-th]  
  
  DOI: 10.1088/1742-6596/631/1/012009

- **Lecture notes on globally supersymmetric theories in four-dimensions and two-dimensions**
  
  
  
  On the page “Document, publications, lecture notes" of my web site:  
  http://www.derendinger.itp.unibe.ch/  
  Documents,_publications,_lecture_notes_files/SUSY_nd.pdf
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   \textit{supersymmetry algebra and representations}

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5 Some simple theories, supergravity gaugings
Space-time algebras: Poincaré

Relativistic quantum field theory has global Poincaré symmetry: Lorentz and translations

- **On coordinates:**

\[ x^\mu \longrightarrow x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \]

\[ \eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = \eta_{\rho\sigma} \]

Variation:

\[ \Lambda^\mu_\nu = \delta^\mu_\nu + \eta^{\mu\rho} \omega_{\rho\nu} \quad \omega_{\rho\nu} = -\omega_{\nu\rho} \]

\[ \delta x^\mu = \omega^{\mu\nu} x^\nu + a^\mu = \left[ \frac{i}{2} \omega^{\rho\sigma} M_{\rho\sigma} + i a^\nu P_\nu \right] x^\mu \]

**Poincaré generators on coordinates:**

\[ M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \]

\[ P_\mu = -i \partial_\mu \]

**Poincaré Lie algebra**

\[ [M^{\mu\nu}, M^{\rho\sigma}] = -i \left( \eta^{\mu\rho} M^{\nu\sigma} + \eta^{\nu\sigma} M^{\mu\rho} - \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\rho} M^{\mu\sigma} \right) \]

\[ [P_\mu, M^{\nu\rho}] = i \left( \eta^{\mu\nu} P^\rho - \eta^{\mu\rho} P^\nu \right) \]

\[ [P_\mu, P_\nu] = 0 \]
Space-time algebras: Poincaré

- **On fields:** \( \Phi(x) \): a set of fields

**Translations:**

\[
\Phi'(x + a) = \Phi(x) \quad \Rightarrow \quad \Phi'(x + a) = \Phi'(x) + a^\mu \partial_\mu \Phi(x)
\]

\[
\delta \Phi(x) = \Phi'(x) - \Phi(x) = -i a^\mu P_\mu \Phi(x)
\]

**Lorentz:**

\[
\Phi'(\Lambda^\mu_\nu x^\nu) = S(\Lambda) \Phi(x) \quad \Rightarrow \quad S(\Lambda) = \mathbb{I} - \frac{i}{2} \omega^{\mu\nu} \Sigma_{\mu\nu}
\]

\[
\delta \Phi(x) = -\frac{i}{2} \omega^\rho_\sigma \Sigma^{\rho\sigma} \Phi(x) - \delta x^\mu \partial_\mu \Phi(x)
\]

- The Casimir operator \( P^\mu P_\mu = -\partial^\mu \partial_\mu = -\Box \) gives the field masses

- The matrix representation \( \Sigma_{\mu\nu} \) contains the spins (for \( P^2 > 0 \)) and/or helicities (\( P^2 = 0 \)) of the fields.
Space-time algebras: Poincaré

There are ten conserved currents:

- **Translations**:
  \[ \tau_{\mu\nu} \]
  \[ \partial^{\mu} \tau_{\mu\nu} = 0 \]

- **Lorentz**:
  \[ j_{\mu,\nu\rho} = -j_{\mu,\rho\nu} \]
  \[ \partial^{\mu} j_{\mu,\nu\rho} = 0 \]

The energy-momentum tensor \( \tau_{\mu\nu} \) can be improved: (Belinfante)

- Use the six Lorentz symmetries to obtain a new symmetric energy-momentum tensor
  \[ T_{\mu\nu} = T_{\nu\mu} \]

- The corresponding Lorentz currents are
  \[ j_{\mu,\nu\rho} = x_{\rho} T_{\mu\nu} - x_{\nu} T_{\mu\rho} \]

**Summary:** for fields \( \Phi(x) \) the information of Poincaré symmetry is:

- in the eigenvalues of \( P^2 \) (masses\(^2\), Klein-Gordon equation),
- in Lorentz representation \( \Sigma_{\mu\nu} \) (spins/helicities) and
- in the symmetric energy-momentum tensor \( T_{\mu\nu} \).
Space-time algebras

- Poincaré algebra: a contraction of either de Sitter (dS) or Anti de Sitter (AdS) algebras.

\[
[M^{\mu\nu}, M^{\rho\sigma}] = -i (\eta^{\mu\rho} M^{\nu\sigma} + \eta^{\nu\sigma} M^{\mu\rho} - \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\rho} M^{\mu\sigma})
\]

\[
[M^{\mu\nu}, P^\rho] = -i\eta^{\mu\rho} P^\nu + i\eta^{\nu\rho} P^\mu
\]

\[
[P^\mu, P^\nu] = -i v^2 \Delta M^{\mu\nu}
\]

\(v\) is an energy-scale, an inverse radius

\(\Delta = 1\): Anti-de Sitter algebra, \(SO(2, 3)\)

\(\Delta = -1\): de Sitter algebra, \(SO(1, 4)\)

The infinite radius limit \(v = 0\) for both \(\Delta\) is Poincaré algebra, as Minkowski space-time is the infinite radius limit of dS or AdS space-time.

Background geometry has cosmological constant \(\Lambda = -3\Delta v^2\)
Quantum field theory admits (in principle) the extension of Poincaré algebra to the conformal algebra $SO(2, 4) \sim SU(2, 2)$

\[
\begin{align*}
[M^\mu{}^\nu, M^\rho{}^\sigma] &= -i (\eta^{\mu\rho} M^{\nu\sigma} + \eta^{\nu\sigma} M^{\mu\rho} - \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\rho} M^{\mu\sigma}), \\
[M^\mu{}^\nu, P^\rho] &= -i (\eta^{\mu\rho} P^{\nu} - \eta^{\nu\rho} P^{\mu}), \\
[M^\mu{}^\nu, D] &= 0, \\
[M^\mu{}^\nu, K^\rho] &= -i (\eta^{\mu\rho} K^{\nu} - \eta^{\nu\rho} K^{\mu}), \\
[P^\mu, K^\nu] &= -2i (\eta^{\mu\nu} D + M^{\mu\nu}),
\end{align*}
\]

\[D\]: generator of scale transformations (dilatation generator)

\[K_\mu\]: generator of conformal boosts (or conformal transformations)

Either explicitly broken (Higgs mass for instance), or spontaneously (scalar expectation values) or generically by quantum effects (scale dependence of interactions, renormalisation-group effects, scale anomalies . . . )
Space-time superalgebras, supersymmetry

Space-time superalgebras are extensions of the space-time algebras with a fermionic sector. Schematically:

\[ \begin{align*}
  i : & \quad [B, B] \subset B \\
  ii : & \quad [B, F] \subset F \\
  iii : & \quad \{F, F\} \subset B
\end{align*} \]

\( i : \) \( B \) is a subalgebra \( \supset \) a space-time algebra
\( ii : \) \( F \) is a representation of \( B \), the fermionic sector: \( F \) are Lorentz spinors
\( iii : \) Fermionic operators: anticommutators

Two cases:

- **Poincaré or Anti-de Sitter supersymmetry**: superalgebra \( OSp(\mathcal{N}, 4) \).
  \[ B = Sp(4, \mathbb{R}) \times SO(\mathcal{N}) \] or its infinite radius (super-Poincaré) contraction. Anti-de Sitter since \( Sp(4, \mathbb{R}) \sim SO(2, 3) \).

- **Superconformal algebra**: superalgebra \( SU(2, 2|\mathcal{N}) \).
  \[ B = SU(2, 2) \times SU(\mathcal{N}) \times U(1) \] and \( SU(2, 2) \sim SO(2, 4) \).
  No \( U(1) \) if \( \mathcal{N} = 4 \)

**De Sitter supersymmetry**?
Supersymmetry algebra

**Construction:**

- Start with Lorentz algebra
  \[
  [M^{\mu\nu}, M^{\rho\sigma}] = -i \left( \eta^{\mu\rho} M^{\nu\sigma} + \eta^{\nu\sigma} M^{\mu\rho} - \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\rho} M^{\mu\sigma} \right)
  \]

- Add supercharges \( Q_{\alpha}^i \) \((i = 1, \ldots, \mathcal{N})\), Lorentz spinors:
  \[
  [M^{\mu\nu}, Q_{\alpha}^i] = -\frac{i}{4}(\sigma^\mu, \bar{\sigma}^\nu) Q^i_{\alpha}
  \]

- Obtain \([P_\mu, Q_{\alpha}^i]\) from Jacobi identities.

- Find the most general \(\{Q_{\alpha}^i, Q_{\beta}^j\}\) which solves all Jacobi identities.

For the \(\text{OSp}(4, \mathcal{N})\) superalgebra, assume that the spinor charges \(Q_{\alpha}^i\) are Majorana
Supersymmetry algebra: \([P_\mu, Q^i_\alpha]\)

- Jacobi identity

\[
0 = [M^{\mu\nu}, [P^\rho, Q^i_\alpha]] + [P^\rho, [Q^i_\alpha, M^{\mu\nu}]] + [Q^i_\alpha, [M^{\mu\nu}, P^\rho]]
\]

is solved by

\[
[P_\mu, Q^i_\alpha] = [(a + ib\gamma_5)\gamma^\mu]_{\alpha\beta} Q^i_\beta
\]

for arbitrary real (Majorana condition) numbers \(a\) and \(b\)

- Using then

\[
[P_\mu, P^\nu] = -i v^2 \Delta M^{\mu\nu}
\]

Jacobi identity

\[
0 = [P_\mu, [P^\nu, Q^i_\alpha]] + [P^\nu, [Q^i_\alpha, P_\mu]] + [Q^i_\alpha, [P_\mu, P^\nu]]
\]

implies

\[
a^2 + b^2 = \frac{1}{4} v^2 \Delta
\]

Minkowski supersymmetry: \(v = 0\)

Anti-de Sitter supersymmetry: \(v^2 > 0\) \(\Delta = 1\)

De Sitter supersymmetry, \(v^2 > 0\) \(\Delta = -1\) is not allowed

\[\implies\] With parity \([P_\mu, Q^i_\alpha] = \frac{1}{2} v (\gamma^\mu Q^i)_\alpha\]
Supersymmetry algebra

From here on: Minkowski-space supersymmetry only

\[ [M_{\mu\nu}, M_{\rho\sigma}] = -i \left( \eta^{\mu\rho} M_{\nu\sigma} + \eta^{\nu\sigma} M_{\mu\rho} - \eta^{\mu\sigma} M_{\nu\rho} - \eta^{\nu\rho} M_{\mu\sigma} \right) \]

\[ [M_{\mu\nu}, P^\rho] = -i \left( \eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu \right) \]

\[ [P^\mu, P^\nu] = 0, \]

\[ [M_{\mu\nu}, Q^i_\alpha] = -\frac{i}{4} ([\gamma^\mu, \gamma^\nu] Q^i_\alpha) \]

\[ [P^\mu, Q^i_\alpha] = 0 \]

\[ \{ Q^i_\alpha, Q^j_\beta \} = -2(\gamma^\mu C)_{\alpha\beta} P_\mu \delta^{ij} + iC_{\alpha\beta} V^{ij} + (\gamma_5 C)_{\alpha\beta} Z^{ij} \]

\( P_\mu \): dimension (mass)\(^1\)

\( Q^i_\alpha \): (mass)\(^{1/2}\).

\( V^{ij} = -V^{ji} \) and \( Z^{ij} = -Z^{ji} \): central charges, commute with all operators.

Exist only for \( \mathcal{N} \geq 2 \)

Dimension (mass)\(^1\)

Central charges introduce mass parameters in massive representations of the supersymmetry algebra.
Representations of the supersymmetry algebra

Two general properties:

1. All states have **same mass**, since \([P^2, Q^i_\alpha] = 0\)

2. In each representation, **same number of bosons and fermions**, \(n_B = n_F\)

\([P_\mu, P_\nu] = 0\): consider eigenstates of \(P_\mu\) with momentum \(p_\mu, \quad p^2 = M^2\)

- **Massless supermultiplets**: \(M = 0\) \(p^\mu = (E, 0, 0, E)\)
  \(N\) fermionic creation–annihilation pairs, \(SU(N)\) invariance:
  \[
  \{Q^i, Q^j_\dagger\} = 4E\delta^{ij} \quad \{Q^i, Q^j\} = \{Q^i_\dagger, Q^j_\dagger\} = 0
  \]
  \(2^N\) states with helicities \(\lambda, \lambda - 1/2, \lambda - 1, \ldots, \lambda - N/2\)

- **Massive representations**: \(M \neq 0\) \(p^\mu = (M, 0, 0, 0)\)
  \(2N\) fermionic creation–annihilation pairs, \(Sp(2N)\) invariance,
  A multiple of \(2^{2N}\) states in representation

\(\pm\) Doubling of the representation usually required by CPT: \(\lambda \leftrightarrow N/2 - \lambda\)
Massless supermultiplets

**Helicity states of supersymmetric gauge theory multiplets**

(\(2^{\mathcal{N}+1}\) states)

States of given helicity \(\lambda\) are labelled by their \(SU(\mathcal{N})\) representation.

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>(\mathcal{N} = 1)</th>
<th>(\mathcal{N} = 1)</th>
<th>(\mathcal{N} = 2)</th>
<th>(\mathcal{N} = 2)</th>
<th>(\mathcal{N} = 3)</th>
<th>(\mathcal{N} = 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\hat{\lambda} = 1/2)</td>
<td>(\hat{\lambda} = 1)</td>
<td>(\hat{\lambda} = 1/2)</td>
<td>(\hat{\lambda} = 1)</td>
<td>(\hat{\lambda} = 1)</td>
<td>(\hat{\lambda} = 1)</td>
</tr>
<tr>
<td>1/2</td>
<td>1</td>
<td>1</td>
<td>1 + (1)</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1 + 1</td>
<td>2 + (2)</td>
<td>1 + 1</td>
<td>3 + 1</td>
<td>4</td>
</tr>
<tr>
<td>-1/2</td>
<td>1</td>
<td>1</td>
<td>1 + (1)</td>
<td>2</td>
<td>1 + 3</td>
<td>6</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

i, iii: Matter multiplets with maximal helicity 1/2 (chiral and hyper multiplets)

ii, iv, vi: Vector or gauge multiplets (maximal helicity 1) of \(\mathcal{N} = 1, 2, 4\)

v: Same as vi, a lagrangian with \(\mathcal{N} = 3\) has actually \(\mathcal{N} = 4\)

The chiral multiplet i only admits chiral representations (\(R \neq \overline{R}\)), as in SM
Supergravity supermultiplets

All massless multiplets with one state at maximal helicity 2.

States are labelled by $SU(\mathcal{N})$ representations (antisymmetric tensors).

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\mathcal{N} = 1$</th>
<th>$\mathcal{N} = 2$</th>
<th>$\mathcal{N} = 3$</th>
<th>$\mathcal{N} = 4$</th>
<th>$\mathcal{N} = 5$</th>
<th>$\mathcal{N} = 6$</th>
<th>$\mathcal{N} = 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{3}{2}$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>15 + 1</td>
<td>28</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>4</td>
<td>$10 + 1$</td>
<td>20 + 6</td>
<td>56</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1 + 1</td>
<td>5 + 5</td>
<td>15 + 15</td>
<td>70</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{-1}{2}$</td>
<td>1</td>
<td>4</td>
<td>$1 + 10$</td>
<td>$6 + 20$</td>
<td>56</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>1 + 15</td>
<td>28</td>
<td></td>
</tr>
<tr>
<td>$\frac{-3}{2}$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>-2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

$\mathcal{N} = 7$ leads to the same multiplet and theory as $\mathcal{N} = 8$.

In contrast to gauge theories, $\mathcal{N} = 3$ and $\mathcal{N} = 4$ are different theories.

Multiplets have $2^\mathcal{N}_B + 2^\mathcal{N}_F$ states. But for $\mathcal{N} = 8$: $2^7_B + 2^7_F$. 

Jean-Pierre Derendinger (AEC, Bern)
Some massive supermultiplets

Massive multiplets with maximal spin 2 for $\mathcal{N} = 1, 2, 4$ theories

Zero central charges.

States are labelled by $Sp(2\mathcal{N})$ representations (antisymmetric traceless tensors).

<table>
<thead>
<tr>
<th>Spin</th>
<th>$\mathcal{N} = 1$</th>
<th></th>
<th>$\mathcal{N} = 2$</th>
<th></th>
<th>$\mathcal{N} = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td></td>
<td>1</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>3/2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>27</td>
</tr>
<tr>
<td>1/2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>48</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>1</td>
<td>5</td>
<td>4</td>
<td>42</td>
</tr>
<tr>
<td>total =</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>256</td>
</tr>
</tbody>
</table>

These are “long multiplets”. Massive multiplets with central charges have less states (“short multiplets”) in general.
$\mathcal{N} = 1$ supersymmetric field theories

A supersymmetric $\mathcal{N} = 1$ field theory describes:

- Vector multiplets (gauge fields $A^a_\mu +$ gauginos $\lambda^a$) in the adjoint representation of a gauge group $G$.
- Chiral multiplets (Weyl fermions $\psi_i +$ complex scalars $z_i$) in representation $r$ of the gauge group.

Condition: $r$ should be free of chiral anomalies.

Supersymmetry relates the interactions (and the masses) of superpartners in the lagrangian.

Exists in two forms:

- **Renormalizable**: defined by $G$, $r$ and a gauge-invariant cubic polynomial (the superpotential) [MSSM, NMSSM, ...]
- **SUSY sigma-model**: defined by $G$, $r$ and three gauge-invariant functions
The simplest, chiral supermultiplet

Describes $2_B + 2_F$ on-shell states, helicities $0, 0, \pm 1/2$

<table>
<thead>
<tr>
<th>Off-shell fields</th>
<th>$z$</th>
<th>$\psi$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M = 0$, helicity:</td>
<td>0, 0</td>
<td>$\pm 1/2$</td>
<td>0, 0</td>
</tr>
<tr>
<td>$M \neq 0$, spin:</td>
<td>0, 0</td>
<td>1/2</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

$f$ is auxiliary, see later

1st step: Free, massless lagrangian

$$\mathcal{L}_0 = (\partial_\mu \bar{z})(\partial^\mu z) + \frac{i}{2} \psi \sigma^\mu \partial_\mu \bar{\psi} - \frac{i}{2} \partial_\mu \psi \sigma^\mu \bar{\psi}$$

Invariant, up to a derivative, under $[\epsilon$: susy parameter, Majorana spinor]$]

$$\delta z = \sqrt{2} \epsilon \psi$$

$$\delta \psi_\alpha = -\sqrt{2} i \partial_\mu z (\sigma^\mu \bar{\epsilon})_\alpha,$$

Susy algebra:

$$[\delta_1, \delta_2] z = -2i (\epsilon_2 \sigma^\mu \bar{\epsilon}_1 - \epsilon_1 \sigma^\mu \bar{\epsilon}_2) \partial_\mu z$$

Result is a translation $\delta z = i \Delta^\mu P_\mu z$

$$\Delta^\mu = 2(\epsilon_2 \sigma^\mu \bar{\epsilon}_1 - \epsilon_1 \sigma^\mu \bar{\epsilon}_2)$$
The simplest, chiral supermultiplet

For the spinor, algebra holds on-shell only:

\[ [\delta_1, \delta_2] \psi_\alpha = -2i(\epsilon_2 \sigma^\mu \bar{\epsilon}_1 - \epsilon_1 \sigma^\mu \bar{\epsilon}_2) \partial_\mu \psi_\alpha \]
\[ + 2i(\partial_\mu \psi \sigma^\mu \bar{\epsilon}_2) \epsilon_1 \alpha - 2i(\partial_\mu \psi \sigma^\mu \bar{\epsilon}_1) \epsilon_2 \alpha \]

2nd step: Modify with the auxiliary field \( f \):

\[ \delta z = \sqrt{2} \epsilon \psi \]
\[ \delta \psi_\alpha = -\sqrt{2} f \epsilon_\alpha - \sqrt{2} i \partial_\mu z (\sigma^\mu \bar{\epsilon})_\alpha \]
\[ \delta f = -\sqrt{2} i \partial_\mu (\psi \sigma^\mu \bar{\epsilon}) \]

\( \Leftarrow \) a derivative

and then:

\[ [\delta_1, \delta_2] z = -i \Delta^\mu \partial_\mu z \]
\[ [\delta_1, \delta_2] \psi_\alpha = -i \Delta^\mu \partial_\mu \psi_\alpha \]
\[ [\delta_1, \delta_2] f = -i \Delta^\mu \partial_\mu f \]

as expected: \( \text{a linear, off-shell representation of susy} \)
The simplest, chiral supermultiplet

**3rd step:** The modified $\delta f$ imposes to modify the lagrangian:

$$\mathcal{L} = (\partial_\mu \bar{z})(\partial^\mu z) + \frac{i}{2} \psi \sigma^\mu \partial_\mu \bar{\psi} - \frac{i}{2} \partial_\mu \psi \sigma^\mu \bar{\psi} + \bar{f} f$$

$\implies$ $\mathcal{L}$ invariant (up to a derivative), $f$ auxiliary with field equation $f = 0$

**4th step:** Introduce masses:

$$-m[fz + \frac{1}{2} \psi \psi] - m[\bar{f} \bar{z} + \frac{1}{2} \bar{\psi} \psi]$$

is invariant (up to a derivative) under the same susy variations

Eliminate the auxiliary $f$ with field equation $\bar{f} = mz$ leads to the free lagrangian of $z$ and $\psi$ with mass $m$. 
The simplest, chiral supermultiplet

5th step: Introduce interactions:  

\[ W(z) = \frac{m}{2} z^2 + \frac{\lambda}{3} z^3. \]

\[ -f \frac{dW}{dz} - \frac{1}{2} \frac{d^2 W}{dz^2} \psi \psi + \text{h.c.} \]

is invariant (up to a derivative) under the same susy variations

Eliminating \( f \) with field equation \( \bar{f} = mz + \lambda z^2 \) (nonlinear now) leads to

\[
\mathcal{L}_{m,\lambda} = (\partial_\mu \bar{z})(\partial^\mu z) - V(z, \bar{z})
\]

\[
+ \frac{i}{2} \psi \sigma^\mu \partial_\mu \bar{\psi} - \frac{i}{2} \partial_\mu \psi \sigma^\mu \bar{\psi} - \frac{m}{2} [\psi \psi + \bar{\psi} \bar{\psi}] - \lambda z \psi \psi - \bar{\lambda} \bar{z} \bar{\psi} \psi
\]

Scalar potential:

\[
V(z, \bar{z}) = |f|^2 = |mz + \lambda z^2|^2 = \left| \frac{d}{dz} W(z) \right|^2
\]
**The gauge, vector supermultiplet**

Describes $2_B + 2_F$ on-shell states, helicities $\pm 1, \pm 1/2$

<table>
<thead>
<tr>
<th></th>
<th>$A_\mu$</th>
<th>$\lambda$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M = 0$, helicity:</td>
<td>$\pm 1, 0$</td>
<td>$\pm 1/2$</td>
<td>0</td>
</tr>
<tr>
<td>$M \neq 0$, spin:</td>
<td>0, 0</td>
<td>1/2</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

$\lambda$: gaugino spinor  
$D$: auxiliary

Under supersymmetry variations

\[
\begin{align*}
\delta A_\mu &= i\epsilon \sigma_\mu \lambda - i\lambda \sigma_\mu \bar{\epsilon} \\
\delta \lambda &= iD\epsilon + \frac{1}{2} F_{\mu\nu} \sigma^{\mu} \bar{\sigma}^{\nu} \epsilon \\
\delta D &= \partial_\mu (\epsilon \sigma^{\mu} \lambda + \lambda \sigma^{\mu} \bar{\epsilon})
\end{align*}
\]

\[\Leftarrow \text{a derivative again}\]

the super-Yang-Mills lagrangian

\[
\mathcal{L}_{SYM} = \text{Tr} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \lambda \sigma^{\mu} \partial_\mu \lambda - \frac{i}{2} \partial_\mu \lambda \sigma^{\mu} \lambda + \frac{1}{2} DD \right]
\]

is invariant up to a derivative.
\( \mathcal{N} = 1 \) superspace, superfields

At this point:

- The component with highest mass dimension in an off-shell supermultiplet is an auxiliary scalar which transforms with a derivative.
- Precisely what is needed for a supersymmetric lagrangian.
- **Wanted**: a systematic method to combine supermultiplets into supermultiplets. Then, auxiliary fields provide contributions to lagrangian field theories.

Two options:
- Tensor calculus
- **Superspace and superfield techniques**
$\mathcal{N} = 1$ superspace, superfields

- **Space-time translations:** $\phi(x) \rightarrow \phi(x + a)$  
  $\delta \phi(x) = i a^\mu P_\mu \phi(x)$
  
  Generators are derivatives $P_\mu = -i \partial_\mu$  
  \[ [P_\mu : \text{energy}, \ a_\mu : \text{energy}^{-1}] \]

- **Leibniz rule:** Combine fields into fields:
  \[
  \delta \phi(x) = a^\mu \partial_\mu \phi \implies \delta F(\phi(x)) = \frac{dF}{d\phi} \delta \phi = a^\mu \frac{dF}{d\phi} \partial_\mu \phi = a^\mu \partial_\mu F
  \]

- **Supersymmetry ~ “square root of translation”:**
  \[
  \{Q_\alpha, Q_\dot{\alpha}\} = -2i (\sigma^\mu)_{\alpha \dot{\alpha}} \partial_\mu \quad \delta \Phi = (i\epsilon Q + \overline{\epsilon Q}) \Phi
  \]
  $\Phi$: a linear (off-shell) representation (supermultiplet)

- **Superspace:** extend formally space-time to superspace with coordinates
  \[
  ( x^\mu, \theta_\alpha, \overline{\theta}_{\dot{\alpha}} ) \quad \{\theta_\alpha, \theta_\beta\} = \{\overline{\theta}_{\dot{\alpha}}, \overline{\theta}_{\dot{\beta}}\} = \{\theta_\alpha, \overline{\theta}_{\dot{\beta}}\} = 0
  \]

$\theta_\alpha$ ($\overline{\theta}_{\dot{\alpha}}$) left-handed (right-handed) Weyl spinor:

Grassmann (anticommuting) coordinates.
### $\mathcal{N} = 1$ superspace, superfields

**Lorentz algebra,** $SO(1, 3) \sim Sl(2, \mathbb{C})$:

A superfield is a function in superspace, with coordinates $(x, \theta, \bar{\theta})$ (or with any other set of coordinates):

$$\Phi(x, \theta, \bar{\theta})$$

It has a **polynomial expansion** in $\theta, \bar{\theta}$ which stops at $\theta\theta\bar{\theta}\bar{\theta}$.

**The expansion includes 16 fields** (functions of $x$)

- 8 fields are bosons, 8 fields are fermions

\[\begin{align*}
\theta_\alpha \theta_\beta &= -\theta_\beta \theta_\alpha = \frac{1}{2} \epsilon_{\alpha\beta} \theta \theta : \quad [(2, 1) \times (2, 1)]_A \rightarrow (1, 1), \\
\bar{\theta}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} &= -\bar{\theta}_{\dot{\beta}} \bar{\theta}_{\dot{\alpha}} = \frac{1}{2} \epsilon_{\alpha\beta} \bar{\theta} \bar{\theta} : \quad [(1, 2) \times (1, 2)]_A \rightarrow (1, 1), \\
\theta_\alpha \times \bar{\theta}_{\dot{\beta}} : \quad (2, 1) \times (1, 2) = (2, 2), \text{ a vector: } \quad \theta_\sigma^\mu \bar{\theta} \sigma^\mu_{\alpha\dot{\beta}} \\
\theta_\alpha \theta_\beta \theta_\gamma &= 0 \\
\bar{\theta}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} \bar{\theta}_{\dot{\rho}} &= 0
\end{align*}\]
An example: the real, or vector (since it includes a vector field) superfield:

\[
V(x, \theta, \bar{\theta}) = C(x) + i\theta \chi(x) - i\bar{\theta} \bar{\chi}(x) + \theta \sigma^\mu \bar{\theta} \nu_\mu(x)
\]

\[
+ \frac{i}{2} \theta \theta [M(x) + iN(x)] - \frac{i}{2} \bar{\theta} \bar{\theta} [M(x) - iN(x)]
\]

\[
+ i\theta \theta \bar{\theta} [\bar{\lambda}(x) + \frac{i}{2} \partial_\mu \chi(x) \sigma^\mu] - i\bar{\theta} \bar{\theta} \theta [\lambda(x) - \frac{i}{2} \sigma^\mu \partial_\mu \chi(x)]
\]

\[
+ \frac{1}{2} \theta \theta \bar{\theta} [D(x) - \frac{1}{2} \Box C(x)]
\]

- \(V(x, \theta, \bar{\theta})\) is Lorentz invariant (scalar, zero spin)

  (by assumption, it could be spinor, vector, . . .)

- Since \(V\) is scalar, bosons are red, fermions are green:

  \(C, M, N, D\): four real scalars (4\(_B\)), \(\nu_\mu\): vector field (4\(_B\)),
  \(\chi, \lambda\): two Weyl spinors (8\(_F\)).
$\mathcal{N} = 1$ superspace, superfields

On superfields, supersymmetry variations are represented by derivatives in superspace:

\[
iQ_\alpha = \frac{\partial}{\partial \theta^\alpha} + i(\sigma^\mu \overline{\theta})_\alpha \partial_\mu \\
i\overline{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial \theta^{\dot{\alpha}}} - i(\theta \sigma^\mu)_{\dot{\alpha}} \partial_\mu \\
P_\mu = -i \partial_\mu \\
\{ Q_\alpha, \overline{Q}_{\dot{\alpha}} \} = -2i (\sigma^\mu)_{\alpha \dot{\alpha}} \partial_\mu
\]

\[
\delta \Phi(x, \theta, \overline{\theta}) = i[a^\mu P_\mu + \epsilon^\alpha Q_\alpha + \overline{\epsilon}_{\dot{\alpha}} \overline{Q}_{\dot{\alpha}}] \Phi
\]

A supersymmetry variation induces a translation is superspace:

\[
\begin{align*}
x^\mu & \quad \rightarrow \quad x^\mu + a^\mu - i\theta \sigma^\mu \overline{\epsilon} + i\epsilon \sigma^\mu \overline{\theta} \\
\theta_\alpha & \quad \rightarrow \quad \theta_\alpha + \epsilon_\alpha \\
\overline{\theta}_{\dot{\alpha}} & \quad \rightarrow \quad \overline{\theta}_{\dot{\alpha}} + \overline{\epsilon}_{\dot{\alpha}}
\end{align*}
\]

And a function of superfields is a superfield.
$\mathcal{N} = 1$ superspace, superfields

Chiral superfields describe chiral supermultiplets (helicities $\pm 1/2, 0, 0$)

Susy covariant derivatives

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} - i(\sigma^\mu \overline{\theta})_\alpha \partial_\mu$$
$$\overline{D}_{\dot{\alpha}} = \frac{\partial}{\partial \theta^\dot{\alpha}} - i(\theta \sigma^\mu)_{\dot{\alpha}} \partial_\mu$$

Since $\{D_\alpha, Q_\beta\} = \{D_\alpha, \overline{Q}_\dot{\beta}\} = \{\overline{D}_{\dot{\alpha}}, Q_\beta\} = \{\overline{D}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\} = 0$, constraints

$$\overline{D}_{\dot{\alpha}} \Phi = 0 \quad \text{(chiral superfield)}$$
$$D_\alpha \overline{\Phi} = 0 \quad \text{(antichiral superfield)}$$

are compatible with supersymmetry variations.

Expansion is:

$$[ y^\mu = x^\mu - i\theta \sigma^\mu \overline{\theta} ]$$

$$\Phi(y, \theta) = z(y) + \sqrt{2} \theta \psi(y) - \theta \theta f(y)$$
$$= z(x) - i \theta \sigma^\mu \overline{\theta} \partial_\mu z(x) - \frac{1}{4} \theta \theta \overline{\theta} \overline{\theta} \Box z(x)$$
$$+ \sqrt{2} \theta \psi(x) + \frac{i}{\sqrt{2}} \theta \theta \partial_\mu \psi(x) \sigma^\mu \overline{\theta} - \theta \theta f(x)$$
Supersymmetric gauge theories

Field content:

- **Chiral superfields** in representation $r$ of the gauge group (“matter”)

  $$\Phi \rightarrow e^\Lambda \Phi, \quad \bar{D}_{\dot{\alpha}} \Lambda = 0, \quad \Lambda = \Lambda^a T^a_r$$

- Gauge fields in **real superfield** $A = A^a T^a_r$, with gauge transformation

  $$e^A \rightarrow e^{-\bar{\Lambda}} e^A e^{-\Lambda}$$

The abelian (Maxwell) case:

$$A \rightarrow A - \Lambda - \bar{\Lambda}$$

Then:

- $\bar{\Phi} e^A \Phi$ is gauge invariant. Its highest component is the gauge-invariant (renormalizable) kinetic lagrangian of the chiral multiplet:

$$L_{kin.} = [\bar{\Phi} e^A \Phi]_{\dot{\theta} \theta \dot{\bar{\theta}} \bar{\theta}} \quad \text{or} \quad L_{kin.} = \int d^2 \theta d^2 \bar{\theta} \bar{\Phi} e^A \Phi$$

- The same holds for the real superfield $\mathcal{K}(\bar{\Phi} e^A, \Phi)$. 

Jean-Pierre Derendinger (AEC, Bern)
A parenthesis on integrals over $\theta, \bar{\theta}$

- Under $\int d^4x$, all derivatives $\partial_\mu(\ldots)$ are irrelevant.
- In a lagrangian, a derivative $\partial_\mu(\ldots)$ is irrelevant.
- For a chiral superfield $\Phi$:

$$\int d^4x \left[ \Phi \right]_{\theta\theta} = -\frac{1}{4} \int d^4x \, DD \, \Phi \equiv \int d^4x \int d^2\theta \, \Phi$$

- For a real superfield $A$:

$$\int d^4x \left[ A \right]_{\theta\theta\bar{\theta}\bar{\theta}} = \frac{1}{16} \int d^4x \, DDDD \, A \equiv \int d^4x \int d^2\theta d^2\bar{\theta} \, A$$

These equalities can be used as definitions of the integration over Grassmann variables $\theta, \bar{\theta}$ (Berezin integral).
Next, we need gauge kinetic terms and the super-Yang-Mills lagrangian

- Gauge field strengths (or curvatures) $F_{\mu\nu}^{\alpha}$ are in the chiral superfields:

$$W_{\alpha} = -\frac{1}{4}DDe^{-A}D_{\alpha}e^{A}$$

$$\bar{W}_{\dot{\alpha}} = \frac{1}{4}DDe^{A}\bar{D}_{\dot{\alpha}}e^{-A}$$

with $\bar{D}_{\dot{\alpha}}W_{\alpha} = D_{\alpha}\bar{W}_{\dot{\alpha}} = 0$ (and susy Bianchi identity)

- Then:

$$L_{SYM} = \frac{1}{4} \int d^2\theta \ Tr \ W_{\alpha}W_{\alpha} + \frac{1}{4} \int d^2\bar{\theta} \ Tr \ \bar{W}_{\dot{\alpha}}\bar{W}_{\dot{\alpha}}$$

Or:

$$L_{SYM} = \text{Tr} \left[ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\lambda\sigma^{\mu}\partial_{\mu}\bar{\lambda} - \frac{i}{2}\partial_{\mu}\lambda\sigma^{\mu}\bar{\lambda} + \frac{1}{2}DD \right]$$
Supersymmetric gauge theories

The (almost) most general two-derivative lagrangian with $\mathcal{N} = 1$ supersymmetry and gauge invariance:

$$
\mathcal{L} = \int d^2 \theta d^2 \theta \left[ \mathcal{K}(\Phi^e A, \Phi) + \xi^a A^a \right] \\
+ \int d^2 \theta \left[ \mathcal{W}(\Phi) + \frac{1}{4} f(\Phi) \text{Tr} \mathcal{W} \mathcal{W} \right] \\
+ \int d^2 \theta \left[ \overline{\mathcal{W}}(\Phi) + \frac{1}{4} \overline{f}(\Phi) \text{Tr} \mathcal{W} \mathcal{W} \right]
$$

$\mathcal{K}$: Kähler potential
$\mathcal{W}$: superpotential
$\overline{f}$: gauge kinetic function

Defined in terms of three gauge-invariant functions.

$\xi^a$: Fayet-Iliopoulos terms for abelian gauge fields only.

A moderately interesting generalization is

$$
f(\Phi) \text{Tr} \mathcal{W} \mathcal{W} \quad \Rightarrow \quad \tilde{f}(\Phi, \text{Tr} \mathcal{W} \mathcal{W})$$
Supersymmetric gauge theories

- **Scalar kinetic terms:**
  \[ \mathcal{K}_{zz} \left( \partial_\mu \overline{z} \right) \partial^\mu z \]
  \[ \mathcal{K}_{zz} = \frac{\partial^2 \mathcal{K}}{\partial z \partial \overline{z}} \]

A scalar field theory on a Kähler manifold.

- **Scalar potential:**
  \[ V(z, \overline{z}) = \mathcal{K}_{zz} \overline{f} f + \frac{1}{2} f(z) D^a D^a \geq 0 \]
  (of course, the value “0” in the bound is not meaningful)

Auxiliary fields (field equations):

\[ f = (\mathcal{K}_{zz})^{-1} \frac{\partial W}{\partial z} \]
\[ D^a = \text{Re} f(z)^{-1} \left[ \frac{\partial \mathcal{K}}{\partial z} T^a_r z + \xi^a \right] \]

- If \( \langle f \rangle = \langle D^a \rangle = 0 \): the true ground state, supersymmetric.
- If no such solution, \( \langle f \rangle \) or \( \langle D^a \rangle \neq 0 \) do not vanish, susy spontaneously broken

\[ \delta \psi = -\sqrt{2} f \epsilon + \ldots \]
\[ \delta \lambda^a = i D^a \epsilon + \ldots \]

and there is a massless Goldstone spinor (the *Goldstino*)
The renormalizable theory

Renormalizability is obtained if $\mathcal{K} = \Phi e^{A} \Phi$ and $f(\Phi) = 1$ (to get canonical kinetic terms) and with a cubic, gauge-invariant, polynomial for the superpotential:

$$W(\Phi^i) = \alpha_i \Phi^i + \frac{1}{2} m_{ij} \Phi^i \Phi^j + \frac{1}{3} \lambda_{ijk} \Phi^i \Phi^j \Phi^k$$

Linear terms only exist for gauge-singlet chiral superfields.

This theory has exceptional renormalization properties:

- **Non-renormalization theorems**: only wave function renormalization for gauge and chiral multiplets needed: the parameters of the superpotential are not renormalized. Holds to all orders of perturbation theory.

- **Soft breaking terms**: terms breaking susy which only affect logarithmic divergences are gaugino masses, scalar masses ($\bar{z}z$ and $\mu^2 z^2 + \text{h.c.}$), analytic trilinear couplings $\beta z^3 + \text{h.c.}$.

- Generated by susy breaking in supergravity, as required for realistic models.

- **Non-perturbative results, subtleties with massless chiral superfields, ...
Supergravity

Local supersymmetry: variation parameter $\epsilon_\alpha$ local $\implies \epsilon_\alpha(x)$

The space-time translation induced by $[\delta_1, \delta_2]$ is local

$$\Delta^\mu = 2(\epsilon_2 \sigma^\mu \bar{\epsilon}_1 - \epsilon_1 \sigma^\mu \bar{\epsilon}_2) = \Delta^\mu(x)$$

Follows from the superalgebra

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -2i (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu = 2(\sigma^\mu)_{\alpha\dot{\alpha}} P_\mu$$

Local supersymmetry $\iff$ local translations or general coordinate transformations (GCT)

and the gauge theory of supersymmetry is a theory of GRAVITATION.

- Non-renormalizable . . . an effective theory of “something”
- Natural cut-off scale where gravitation is expected to feel quantum physics, the Planck scale $M_P \sim 10^{19}$ GeV
- The maximal $\mathcal{N} = 8$ theory has exceptional finiteness properties, under difficult investigations (but not the right physics)
There are independent motivations to consider supergravity theories

- **Bottom-up**: It provides a source and suggests structures for the supersymmetry breaking needed in (realistic) supersymmetric quantum field theories (like MSSM, NMSSM, . . .)

- **Top-down**: It can be used as an effective, low-energy ($E \ll M_P$) description of a more fundamental microscopic quantum theory with gravitation (like superstring theories)

- **Curiosity**: model for microscopic gravitation with gauge and matter fields (scattering amplitudes, . . .) . . .

A vast and complicated subject.

At first sight, for “realistic models”, the existence of fermions in chiral representations excludes $\mathcal{N} > 1$.

But, for instance, string models suggest more supersymmetries with quite elaborate breaking patterns, still under study.
Supersymmetric theories (and Nature) have **fermions** and **spinor fields**.

- **GCT:** space-time with (Riemann) metric $g_{\mu\nu}(x)$ in coordinates $x^\mu$ and GCT-invariant line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.

- Spinors $\psi$ live in the (Minkowski) tangent-space at each point $x$.

- Tangent-space coord. $\zeta^a(x)$, line element $ds^2 = \eta_{ab} d\zeta^a(x) d\zeta^b(x)$, $ds^2$ has local Lorentz invariance.

- **Vierbein:**
  
  \[ g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x) \quad e^a_\mu = \partial_\mu \zeta^a(x) \]

  **Inverse:**
  
  \[ e^a_\mu e^b_\nu = \delta^b_a \quad e^a_\mu e^a_\nu = \delta^\nu_\mu \]

- **Local Lorentz:**
  
  \[ \delta \psi(x) = -\frac{i}{2} \omega_{ab}(x) \sigma^{ab} \psi(x) \quad \sigma^{ab} = \frac{i}{4} [\gamma^a, \gamma^b] \]

**Dirac lagrangian:** requires the gauge field of local Lorentz: the **spin connection**

\[ \mathcal{L} = ie \bar{\psi} \gamma^\mu D_\mu \psi \quad D_\mu \psi = \partial_\mu \psi + \frac{1}{2} \omega_\mu{}^{ab} \sigma^{ab} \psi \quad e = \det e^a_\mu \]
Simple $\mathcal{N} = 1$ supergravity

Simply the sum of the \textit{covariantized} Einstein and Rarita-Schwinger lagrangians

- **Symmetries** are:
  - Local coordinate transformations (GCT) \implies General relativity
  - Local Lorentz (tangent space)
  - Local supersymmetry

- **Fields** are the gauge fields of:
  - Translations, GCT: the vierbein $e^a_{\mu}$
  - Local Lorentz: the spin connection $\omega^{ab}_{\mu} = -\omega^{ba}_{\mu}$
  - Supersymmetry: the gravitino $\psi_{\alpha\mu}$

[ This is the first-order formalism ]

Explicitly:

$$\gamma^{\mu\nu\rho} = \gamma^{[\mu} \gamma^{\nu} \gamma^{\rho]}$$

$$\mathcal{L} = e e^a_{\alpha} e^b_{\beta} R_{\mu\nu}^{ab} (\omega) + e \bar{\psi}_{\mu} \gamma^{\mu\nu\rho} \tilde{D}_{\nu} \psi_{\rho}$$

$$e = \text{det} e^a_{\mu}$$
Gravitino, Rarita-Schwinger lagrangian

\[ \psi_{\mu \alpha} : \ \text{gravitino, gauge field of supersymmetry, vector–spinor} : \]

\[
\begin{align*}
\text{spinor} & \otimes \text{vector} = \text{gravitino} \oplus \text{spinor} \\
[(2, 1) \oplus (1, 2)] & \otimes (2, 2) = [(3, 2) \oplus (2, 3)] \oplus [(1, 2) \oplus (2, 1)]
\end{align*}
\]

To isolate the spinor, projection condition:

\[
(\gamma^a \psi_a)_\alpha = (\gamma^\mu \psi_\mu)_\alpha = 0 \quad \Rightarrow \quad \tilde{\psi}_\alpha a = \psi_\alpha a - \frac{1}{4} (\gamma_a \gamma^b \psi_b)_\alpha
\]

This condition follows from the lagrangian (in Minkowski space)

\[
\mathcal{L}_{RS} = \frac{1}{2\kappa^2} \overline{\psi}_a \gamma^{abc} \partial_b \psi_c \quad \text{Rarita-Schwinger} \quad [\kappa^{-1}: \text{mass scale}]
\]

- **Majorana condition on** \(\psi_a\)
- **Gauge invariance** \(\delta \psi_a = \partial_a \lambda\)
- **Field equation:** \(\gamma^{abc} \partial_b \psi_c = 0\)
- **Propagates two massless states with** helicities \(\pm 3/2\)
Gravitino, Rarita-Schwinger lagrangian

Counting states: starting with $4 \times 4 = 16_F$ hermitian fields in $\psi_{\alpha a}$

Rarita-Schwinger:

$$\mathcal{L}_{RS} = \frac{1}{2\kappa^2} \bar{\psi}_a \gamma^{abc} \partial_b \psi_c$$

$$\delta \psi_a = \partial_a \lambda$$

$$\gamma^{abc} \partial_b \psi_c = 0$$

- Use gauge invariance with $\gamma^a \partial_a \lambda = -\gamma^a \psi_a$ to impose $\gamma^a \psi_a = 0$
- Defines $\lambda$ up to a solution of $\gamma^a \partial_a \tilde{\lambda} = 0$ (massless Dirac)
- In gauge $\gamma^a \psi_a = 0$: field equation $\gamma^a \partial_b \psi^b = \gamma^b \partial_b \psi^a$
- And then multiply by $\gamma_a$ to obtain:

$$\gamma^a \psi_a = 0$$ (gauge choice)  
$$\partial_b \psi^b = 0$$  
$$\gamma^b \partial_b \psi_a = 0$$ (Dirac)

$$\delta \psi_a = \partial_a \tilde{\lambda}$$  
$$\gamma^a \partial_a \tilde{\lambda} = 0$$ (residual gauge symmetry)

- Counting on-shell states: $16_F - 4_F - 4_F - 4_F - 2_F = 2_F$
  and these two states have helicities $\pm 3/2$ (use plane waves to check)
The spin connection, Einstein-Hilbert lagrangian

Einstein gravitation formulated in terms of the vierbein $e^{a}_{\mu}$ and the spin connection $\omega^{ab}_{\mu}$.

- Spin connection curvature (Lorentz gauge field)
  \[
  R^{ab}_{\mu\nu} = \partial_{\mu} \omega^{ab}_{\nu} - \partial_{\nu} \omega^{ab}_{\mu} + \omega^{ac}_{\mu} \omega^{b}_{\nu c} - \omega^{ac}_{\nu} \omega^{b}_{\mu c}
  \]

- Lagrangian
  \[
  L_{\text{grav.}} = \frac{1}{2\kappa^2} e R
  \]
  \[
  R = e^{\mu}_{a} e^{\nu}_{b} R^{ab}_{\mu\nu}
  \]

- The spin connection has an algebraic field equation (does not propagate):
  \[
  \omega^{cd}_{\mu} = -\frac{1}{2} (\partial_{\mu} e^{\nu}_{c} - \partial_{\nu} e^{\mu}_{c}) e^{\nu}_{d} + \frac{1}{2} (\partial_{\mu} e^{\nu}_{d} - \partial_{\nu} e^{\mu}_{d}) e^{\nu}_{c}
  - \frac{1}{2} e^{\rho}_{c} e^{\nu}_{d} (\partial_{\rho} e^{\nu}_{a} - \partial_{\nu} e^{\rho}_{a}) e^{a}_{\mu}
  \equiv \omega^{cd}_{\mu}(e)
  \]

- Rewrite then $R$ as a function of the vierbein and its derivative.
  and $L_{\text{grav.}}$ propagates two states with helicities $\pm 2$ (graviton)

- If the spin connection appears in other lagrangian terms, its algebraic field equation leads to contorsion:
  \[
  \omega^{ab}_{\mu} = \omega^{ab}_{\mu}(e) + \kappa^{ab}_{\mu}
  \]
Pure $\mathcal{N} = 1$ supergravity, construction

Pure $\mathcal{N} = 1$ supergravity is very simple:

Einstein-Hilbert + Rarita-Schwinger

$$S_{ERS}[e_\mu^a, \psi_\mu, \omega_\mu{}^{ab}] = \frac{1}{2\kappa^2_D} \int d^D x \, e \left( R + \overline{\psi}_\mu \gamma^{\mu\nu\rho} \tilde{D}_\nu \psi_\rho \right)$$

But: local symmetries imply covariant derivatives

$$\tilde{D}_\mu \psi_\nu = \partial_\mu \psi_\nu + \frac{1}{2} \omega_\mu{}^{ab} \sigma_{ab} \psi_\nu$$

(\text{spin connection})

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e) + \kappa_\mu{}^{ab}$$

(\text{contorsion tensor})

$$\tilde{D}_\mu \psi_\nu - \tilde{D}_\nu \psi_\mu = D_\mu \psi_\nu - D_\nu \psi_\mu + 2 S_\mu{}^{\lambda\nu} \psi_\lambda$$

(torsion tensor)

with gravitino torsion

(for a $D = 4$ Majorana gravitino)

$$S_\mu{}^{\lambda\nu} = -\frac{1}{4} \overline{\psi}_\mu \gamma^\lambda \psi_\nu$$

$$\kappa_\mu{}^{ab} = -\frac{1}{4} \left[ \overline{\psi}_\mu \gamma_a \psi_b - \overline{\psi}_\mu \gamma_b \psi_a + \overline{\psi}_a \gamma_\mu \psi_b \right]$$
Pure $\mathcal{N} = 1$ supergravity, construction

Covariantization $\implies$ four-gravitino interaction, and then:

Four-dimensional pure $\mathcal{N} = 1$ supergravity is not so simple:

$$
\mathcal{L} = \frac{1}{2\kappa^2} e R(\omega(e)) + \frac{1}{2\kappa^2} e \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu(\omega(e)) \psi_\rho \\
+ \frac{e}{32\kappa^2} \left[ 4(\bar{\psi}_\mu \gamma_\mu \psi_\rho)(\bar{\psi}_\nu \gamma_\nu \psi_\rho) - (\bar{\psi}_\mu \gamma_\nu \psi_\rho)(\bar{\psi}_\mu \gamma_\nu \psi_\rho) \\
- 2(\bar{\psi}_\mu \gamma_\nu \psi_\rho)(\bar{\psi}_\mu \gamma^\rho \psi_\nu) \right]
$$

with now $\tilde{D}_\nu \psi_\rho = \partial_\nu \psi_\rho + \frac{1}{2} \omega_\nu{}^{ab}(e) \sigma^{ab} \psi_\rho$, the usual spin connection of pure gravitation theory.

- In four space-time dimensions:
  - Gravitino: $4 \times 4 - 4 = 12_F$ off-shell. $2_F$ with helicities $\pm 3/2$ on-shell.
  - Graviton: $10 - 4 = 6_B$ on-shell. $2_B$ with helicities $\pm 2$ on-shell.
Pure $\mathcal{N} = 1$ supergravity, variations, auxiliary fields

More complications: auxiliary fields of $\mathcal{N} = 1$ supergravity

The $\mathcal{N} = 1$ supergravity action is invariant under local susy variations

\[
\begin{align*}
\delta e^a_\mu &= -\frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu \\
\delta e^a_\mu &= \frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu \\
\delta \psi_\mu &= D_\mu \epsilon \\
\delta \bar{\psi}_\mu &= D_\mu \bar{\epsilon}
\end{align*}
\]

With standard covariant derivative

\[
D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{2} \omega_\mu \sigma^{ab} \epsilon
\]

But it is not an off-shell representation of the supersymmetry algebra:

$[\delta_1, \delta_2]$ is a diffeomorphism only for fields solving the field equations

Another sign is the number of off-shell field components: $12_F \neq 6_B$.

More (auxiliary) fields needed for a linear off-shell representation, with

$\mathcal{n}^a_{aux} - \mathcal{n}^a_{aux} = 6$

By field equations, vanish for pure supergravity, but produce interactions when coupled to matter or gauge multiplets.
Pure $\mathcal{N} = 1$ supergravity, auxiliary fields

Several choices of auxiliary fields:

- **Minimal schemes with** $12_B + 12_F$ 
  ($n_B^{aux} = 6$, $n_F^{aux} = 0$)

  - **Old minimal**: $A^\mu$, not a gauge field ($4_B$), couples to a non-conserved (in general) current and $f_0$, complex scalar ($2_B$).

  - **New minimal**: $A^\mu$ with gauge symmetry ($3_B$), couples to a conserved current, antisymmetric tensor $B_{\mu\nu}$ with gauge symmetry ($3_B$).

- **Non minimal schemes** have $16_B + 16_F$ (somewhat, loosely relevant to superstring theories), $20_B + 20_F$, ... 

- Each scheme generates a particular class of interactions. For instance: $R$–symmetric with new minimal.

- The most general is the simplest, **old minimal** (describes all classes).

- Each scheme can be constructed from **superconformal theories** with various **compensating fields** to gauge-fix dilatation, special conformal and $R$ symmetries.
Supergravities, summary

Four-dimensional supersymmetry (linear) representations

| SUSY | Supergravity | |Hel.|≤ 1 | |Hel.|≤ 1/2 | Chirality |
|---|---|---|---|---|---|---|---|
| $\mathcal{N} = 1$ | $2_B + 2_F$ | ✓ | ✓ | ✓ | ✓ |
| $\mathcal{N} = 2$ | $4_B + 4_F$ | ✓ * | ✓ * | ✓ * | - |
| $\mathcal{N} = 3$ | $8_B + 8_F$ | ✓ * | - | - | $D = 6$ |
| $\mathcal{N} = 4$ | $16_B + 16_F$ * | ✓ * | - | - | $D = 10$ |
| $\mathcal{N} = 5$ | $32_B + 32_F$ * | - | - | - |
| $\mathcal{N} = 6$ | $64_B + 64_F$ * | - | - | - |
| $\mathcal{N} = 8$ | $128_B + 128_F$ * | - | - | - | $D = 11$ |

*: scalar fields in supermultiplet
- Number of supercharges is $4\mathcal{N}$
- 16 supercharges ($\mathcal{N} = 4$): type I, heterotic strings
- 32 supercharges ($\mathcal{N} = 8$): type IIA, IIB strings, M–“theory"
- $\mathcal{N} = 7$ does not exist (it is the $\mathcal{N} = 8$ theory)
- $\mathcal{N} = 0, 1$ only for realistic models, or nonlinear, (or truncated...)

Jean-Pierre Derendinger (AEC, Bern)
\( \mathcal{N} = 1 \) supergravity and matter couplings

\( \mathcal{N} = 1 \) supergravity couples to:

- all gauge groups (gauge superfield \( A_\mu, \lambda \), helicities \( \pm 1, \pm 1/2 \))
- all representations for chiral multiplets (\( \psi \) and \( z \), helicities \( \pm 1/2, 0, 0 \))

and allows chirality of fermion representations.

The idea is then:

- Couple the SSM to supergravity, add a "hidden" sector to break supersymmetry.
- Generate a susy breaking scale \( m_{3/2} \) and scalar vev’s in the hidden sector \( \left\langle \phi \right\rangle \)
- Decouple gravity: expand, take \( M_P \longrightarrow \infty \), keep \( m_{3/2} \) fixed, . . .
- The result is a global \( \mathcal{N} = 1 \) theory with soft breaking terms.
- However: \( \mathcal{N} = 1 \) Poincaré only if the cosmological constant at the breaking point is zero. In general, AdS global \( \mathcal{N} = 1 \) . . .
- A severe constraint on the hidden sector . . .
The complete Lagrangian has been obtained by Cremmer, Ferrara, Girardello and Van Proeyen (1982). It needs 1.5 pages in Nucl. Phys. B 212.

In the superconformal formulation, it is symbolically

\[ \mathcal{L} = -\frac{3}{2} \left[ S_0 \overline{S}_0 e^{-\kappa/3} \right]_D + \left[ S_0^3 W(\Phi) + \frac{1}{4} f(\Phi) \text{Tr} \overline{\mathcal{W}} \mathcal{W} \right]_F \]

where \( S_0 \) is the chiral compensating multiplet of the old minimal formalism.

- Similar to the global superspace Lagrangian

\[ \mathcal{L} = \int d^2 \theta d^2 \overline{\theta} \kappa(\overline{\Phi}^A, \Phi) + \int d^2 \theta \left[ W(\Phi) + \frac{1}{4} f(\Phi) \text{Tr} \overline{\mathcal{W}} \mathcal{W} \right] + \text{h.c.} \]

Superconformal and superspace calculus turn these symbolic expressions into Lagrangians . . . . . .

Curiously, most applications use only the scalar potential and few fermion mass terms.
• **Global supersymmetry:** three independent functions

\[ \mathcal{L} = \int d^2 \theta d^2 \bar{\theta} \mathcal{K}(\Phi^A, \Phi) + \int d^2 \theta [W(\Phi) + \frac{1}{4} f(\Phi) \text{Tr} \mathcal{W} \mathcal{W}] + \text{h.c.} \]

Invariant under Kähler transformations \( \mathcal{K} \rightarrow \mathcal{K} + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}) \).

• \( \mathcal{N} = 1 \) supergravity, in the superconformal formulation:

\[ \mathcal{L} = -\frac{3}{2} \left[ S_0 \bar{S}_0 e^{-\mathcal{K}/3} \right]_D + \left[ S_0^3 W + \frac{1}{4} f(\Phi) \text{Tr} \mathcal{W} \mathcal{W} \right] \]

Kähler transformation is a gauge invariance:

\[ \mathcal{K}' = \mathcal{K} + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}) \quad S_0' = S_0 e^{\Lambda(\Phi)/3} \quad W' = W e^{-\Lambda(\Phi)} \]

Supergravity depends then on two functions only:

\[ \mathcal{L} = -\frac{3}{2} \left[ S_0 \bar{S}_0 e^{-\mathcal{G}/3} \right]_D + \left[ S_0^3 + \frac{1}{4} \text{Tr} \mathcal{W} \mathcal{W} \right] \quad \mathcal{G} = \mathcal{K} + \ln |W|^2 \]

Also true for global susy in an Anti-de Sitter geometry, an effect of the large radius limit to Poincaré
The $\mathcal{N} = 1$ supergravity scalar potential

$$\begin{align*}
V &= \frac{1}{\kappa^4} \left[ e^K K^{-i} j (W_i + K_i W)(W^j + K^j W) - 3 e^K W W \right] \\
&\quad + \frac{1}{2} f(\Phi)^{-1} K_i (T^A z)^i K_j (T^A z)^j
\end{align*}$$

- **Blue** terms are positive or zero: they are generated by chiral ($f$) or gauge ($D$) auxiliary fields. Susy breaks if they are not zero.

- The **red** term is negative or zero, it is generated by a supergravity auxiliary field. **Unbroken susy:** Anti-de Sitter or Minkowski ($W = 0$).

- Supergravity and supersymmetry are actually extensions of **Anti-de Sitter symmetry:** $SO(2, 3) \sim Sp(4, \mathbb{R}) \rightarrow OSp(n|4)$
  
Poincaré is obtained in the large AdS radius limit only.

- **De Sitter** ground state: with broken supersymmetry only.
Supersymmetry breaking

In Nature, supersymmetry is at best a broken symmetry . . .

- Global $\mathcal{N} = 1$ supersymmetry

Spontaneous susy breaking induced by auxiliary field vev’s: $\langle f \rangle, \langle D^a \rangle$

Scalar potential:

$$V = f^* f + \frac{1}{2} D^a D^a \geq 0$$

An algebraic problem:

$$f = \frac{\partial W}{\partial z} = 0 \quad \text{and} \quad D^a = 0$$

should have no solution.

Disastrous mass relations:

$$\langle f \rangle \Rightarrow \mathrm{STr} \mathcal{M}^2 = 0 \quad (\mathrm{STr} \mathcal{M}^2 = \mathrm{Tr}_{\text{bosons}} \mathcal{M}^2 - \mathrm{Tr}_{\text{fermions}} \mathcal{M}^2)$$

$$\langle D^a \rangle: \quad \text{needs Fayet-Iliopoulos terms for gauged } U(1)$$

in conflict with data, phenomenology, anomalies, esthetics . . .

$$\mathrm{STr} \mathcal{M}^2 \sim -\langle D^a \rangle \mathrm{Tr} T^a$$

And anyway a massless Goldstino spinor.

Consider then spontaneous breaking of local susy in supergravity models
Spontaneous breaking of local supersymmetry

- Analogy: $U(1)$ gauge theory with a charged complex $\phi(x)$:

  \[
  \phi(x) = e^{i\sigma(x)}[v + h(x)]
  \]

  $v$: $U(1)$-breaking order parameter (gauge-invariant)

  Gauge symmetry:

  \[
  \delta \sigma = \Lambda, \quad \delta A_\mu = -\partial_\mu \Lambda
  \]

  \[
  (D_\mu \phi) \dagger (D^\mu \phi) \rightarrow v^2 (A_\mu + \partial_\mu \sigma)(A^{\mu} + \partial^{\mu} \sigma) + \ldots
  \]

  The gauge-invariant massive field is $\tilde{A}_\mu = A_\mu + \partial_\mu \sigma$, $\sigma$ is the Goldstone boson, and the helicity zero component of the spin one $\tilde{A}_\mu$

- Super-higgs mechanism: (neither Higgs nor "et al." found this)

  Induced by auxiliary field vev's $\langle f_i \rangle, \langle D^a \rangle$

  Massless Goldstino would be

  \[
  \eta_G = \langle \mathcal{K}_{\bar{j}}^i f_{\bar{j}} \rangle \psi^i - \frac{i}{2} \langle \text{Re} f(z) D^a \rangle \lambda^a
  \]

  Use local susy parameter $\epsilon$ to gauge-away $\eta_G$.

  Since $\delta \psi_\mu = \partial_\mu \epsilon + \ldots$ $\eta_G$ absorbed in $\psi_\mu$: a massive spin $3/2$ state.

  Susy-breaking order parameter.

  \[
  m_{3/2}^2 = \frac{1}{\kappa^2} \langle |W|^2 e^K \rangle
  \]
A “no-scale” example

Two chiral supermultiplets with scalar fields $S$ and $T$, and

$$\mathcal{K} = -n \ln(T + \bar{T}) + \hat{\mathcal{K}}(S, \bar{S})$$

Kähler potential,

$$W = W(S)$$

Superpotential.

Scalar potential (using $\kappa = 1$ in Einstein frame)

$$V = (T + \bar{T})^{-n} e^{\hat{\mathcal{K}}} \left[ \hat{\mathcal{K}}_{SS}^{-1} |W_S + \hat{\mathcal{K}} S W|^2 + (n - 3)W \bar{W} \right]$$

Positive if $n \geq 3$. Choose $n = 3$ and solve

$$\langle W_S + \hat{\mathcal{K}} S W \rangle = 0$$

$$\implies$$ A stable ground state with $\langle V \rangle = \Lambda = 0$

$$\langle W_S + \hat{\mathcal{K}} S W \rangle = 0:$$ fixes $\langle S \rangle$ (in general), $\langle f_S \rangle = 0$, leaves $\langle T + \bar{T} \rangle$ arbitrary, as well as

$$\langle f_T \rangle = \langle (T + \bar{T})^{-1/2} e^{\hat{\mathcal{K}}/2} W \rangle$$

$\langle W \rangle \neq 0$: susy broken by $T$ with arbitrary susy-breaking order parameter

$$m_{3/2} = \langle e^{\mathcal{K}/2} W \rangle = \langle (T + \bar{T})^{-3/2} e^{\hat{\mathcal{K}}/2} W \rangle$$

$m_{3/2}$ is the gravitino mass since $\Lambda = 0$. 
Supersymmetry breaking

**Supergravity scalar potentials again**

For all $\mathcal{N}$, the typical supergravity potential is the sum of three terms:

$$V = V_m + V_g + V_0$$

- **$V_m \geq 0$**: generated when matter fermions are present. Exists for $\mathcal{N} = 1, 2$ (chiral and hyper multiplets).
  $$\delta \psi = \mathcal{A} \epsilon + \ldots \quad \rightarrow \quad V_m \sim +|\mathcal{A}|^2$$

- **$V_g \geq 0$**: generated when gauginos are present. Exists for all $\mathcal{N} = 1, \ldots, 8$.
  $$\delta \lambda = \mathcal{B} \epsilon + \ldots \quad \rightarrow \quad V_g \sim +|\mathcal{B}|^2$$

- **$V_0 \leq 0$**: generated by the gravitinos (helicities $\pm 2, \pm 1/2$): Exists for all $\mathcal{N} = 1, \ldots, 8$.
  $$\delta \psi_\mu = \mathcal{C} \gamma_\mu \epsilon + \ldots \quad \rightarrow \quad V_0 \sim -|\mathcal{C}|^2$$

- Some or all supersymmetries break if $V_m$ and/or $V_g$ are positive on the vacuum.
Supergravity scalar potentials again

- The gauge potential $V_g$ plays a fundamental role in the vacuum structure of $\mathcal{N}$-extended supergravities.
- Produced by gauging a symmetry of the theory: abelian (for instance, $R$-symmetry in $\mathcal{N} = 1$) or non-abelian. Compact or non-compact.
- Flat directions in the potential, with no-scale behaviour and Minkowski space, algebraically characterized: a condition on the gauged algebra.

(...): number of abelian, ungauged, gauge fields in the supermultiplet:

| SUSY | Supergravity | $|\text{Hel.}| \leq 1$ | $|\text{Hel.}| \leq 1/2$ |
|------|--------------|-----------------|-----------------|
| $\mathcal{N} = 1$ | $2_B + 2_F$ (0) | ✓ $(n)$ | ✓ |
| $\mathcal{N} = 2$ | $4_B + 4_F$ (1) | ✓ $(n)$ | ✓ |
| $\mathcal{N} = 3$ | $8_B + 8_F$ (3) | ✓ $(n)$ | - |
| $\mathcal{N} = 4$ | $16_B + 16_F$ (6) | ✓ $(n)$ | - |
| $\mathcal{N} = 5$ | $32_B + 32_F$ (10) | - | - |
| $\mathcal{N} = 6$ | $64_B + 64_F$ (15) | - | - |
| $\mathcal{N} = 8$ | $128_B + 128_F$ (28) | - | - |
Supersymmetry breaking

Some dates and names, $\sim$ 40 years ago

- Field theories with linear supersymmetry, 1974 (Wess and Zumino).
- Soon found to have softer divergences than ordinary gauge theories (logarithmic renormalization only) and powerful all-order non-renormalization theorems (Iliopoulos, Zumino, Wess, Ferrara).
- Superspace techniques (Salam, Strathdee; Wess, Zumino, Ferrara).
- Spontaneous supersymmetry breaking (Fayet, Iliopoulos, 1974; O’Raifeartaigh, 1975).
- Currents and supercurrents, approaches to gravity coupling (Ferrara, Zumino, 1974).
- Supergravity was created in 1976 (Ferrara, Freedman and Van Nieuwenhuizen; Deser and Zumino).
- Matter and gauge couplings to $\mathcal{N} = 1$ supergravity, 1982, Cremmer, Ferrara, Girardello, Van Proeyen (also Arnowitt, Chamseddine, Nath; Bagger, Witten)
Dilaton supergravity, no-scale models

For a single chiral superfield $S$ and a constant superpotential $W$, 

$$V = \frac{1}{\kappa^4} e^\kappa \left[ K_{S\bar{S}}^{-1} \kappa S \kappa \bar{S} - 3 \right] \bar{W} W$$

is identically zero if 

$$\kappa = -3 \ln(S + \bar{S}) \quad \forall W$$

but the auxiliary field $f_S$ and the gravitino mass are 

$$f_S = \bar{W} (S + \bar{S})^{-1/2} \neq 0 \quad m_{3/2} = W (S + \bar{S})^{-3/2}$$

Hence, $W$ induces supersymmetry breaking in Minkowski space, to obtain: 

**Broken supersymmetry in Minkowski space with a free scale $\langle S + \bar{S} \rangle$**

The prototype of no-scale models:

Tree-level susy breaking scale arbitrary, radiative corrections may define it with some logarithmic factor and then with an induced scale hierarchy.

(Cremmer, Ferrara, Kounnas, Nanopoulos, 1983)
Consider now a string compactification:

In general, it produces a real dilaton scalar and an antisymmetric tensor $B_{\mu\nu}$ with gauge invariance $\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$ in the universal gravitation sector (in type II, in NS–NS sector).

The antisymmetric tensor is equivalent to a real scalar with shift symmetry:

$$\partial_{[\mu} B_{\nu\rho]} \leftrightarrow \partial_\mu \text{Im } s \quad C \leftrightarrow \text{Re } s$$

and there should be a description in terms of a chiral multiplet $S$, with however an auxiliary field $f_S$ which could be a source of supersymmetry breaking.

The relation is a Legendre transformation between supermultiplets.

The behaviour of the dilaton scalar in the effective supergravity Lagrangian is important: its value is the string coupling. Does it stabilize, does it slide to zero (run away), are further moduli fields needed?
Dilaton supergravity, no-scale models

Within supergravity, two descriptions and a duality generated by a Legendre transformation:

- **Description with** $B_{\mu\nu}$: (The superpotential is constant)

\[ \mathcal{L} = -\frac{3}{2} \left[ S_0 \bar{S}_0 \mathcal{H}(X) \right]_D + \left[ S_0^3 W \right]_F \]

\[ X = \frac{L}{S_0 \bar{S}_0} \]

- **Description with chiral multiplet** $S$:

\[ \tilde{\mathcal{L}} = -\frac{3}{2} \left[ S_0 \bar{S}_0 e^{-\frac{1}{3} \mathcal{K}(S+S)} \right]_D + \left[ S_0^3 W \right]_F \]

- **Legendre transformation**:

\[ e^{-\frac{1}{3} \mathcal{K}(S+S)} = \mathcal{H}(X) - X (S + \bar{S}) \]

- **Dilaton supergravities**:

  - Heterotic: $\mathcal{H} \sim X^{-1/2} \quad \mathcal{K} = -\ln(S + \bar{S})$
  
  - Type II: $\mathcal{H} \sim X^4 \quad \mathcal{K} = -4 \ln(S + \bar{S})$
The Legendre transformation implies:

\[ f_S = -C\mathcal{H}c_c z_0 f_0 \]

and \( f_S \) is not an independent auxiliary field. Generalization to many fields:

The auxiliary field \( f_S \) of a chiral multiplet dual to a linear superfield with an antisymmetric tensor is a linear combination of other auxiliary fields.

\[ f_S \sim \frac{\partial}{\partial z^i} \mathcal{H} c f^i \]

The dilaton is not stabilized. More fields and interactions required.

The single field no-scale model with \( \mathcal{K} = -3 \ln (S + \overline{S}) \) does not describe a \( B_{\mu\nu} \) + dilaton sector.

Hence, low-energy scenarios in which supersymmetry breaking is induced by the dilaton superfield \( S \) only are forbidden by supergravity arguments.
Gauged supergravities

- All \textit{ungauged} supergravities have been constructed long ago. They depend on the abelian field strengths $F_{\mu\nu}$ only and have then (in four dimensions) electric-magnetic duality.

- A symmetry of an ungauged theory can be \textit{gauged} using the abelian gauge fields of the theory. One selects an algebra and associates a (electric or magnetic) gauge field $A^M_\mu$ of the theory with each generator

$$[T_A, T_B] = f_{AB}^C \quad X_M = \Theta_M^A T_A \quad \Theta_M^A: \text{embedding tensor}$$

- The consistency conditions for the procedure have been established for a generic field theory in a fundamental paper by de Wit, Samtleben and Trigiante (hep-th/0507289).

- Large classes of gauged supergravities have been constructed, \textit{large classes are missing}.

- Particularly interesting for 16 ($\mathcal{N} = 4$) and 32 ($\mathcal{N} = 8$) supercharges related to superstrings and $M$ theories.
An example, maximal supergravity with $SO(8)$

- Can be obtained by $S_7$ sphere compactification of 11-dimensional supergravity.  

$(\text{de Wit, Nicolai})$

- $\mathcal{N} = 8$ supergravity has 28 abelian gauge fields $F_{\mu\nu}^I$ and then 28 duals $\tilde{F}_{\mu\nu}^I = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}^I$ and 70 scalars.

Obvious gauging: the 28 gauge fields in the adjoint of $SO(8)$: electric gauging.

- Is the gauging unique?

Starting point:

- The electric-magnetic duality group is $Sp(56, \mathbb{R})$  

$(\text{Gaillard, Zumino})$.

- The 70 scalar fields are in $E_{7,7}/SU(8)$ with $E_{7,7} \subset Sp(56, \mathbb{R})$.

- Fermions reduce the symmetry to $SU(8)$

- Gauge group $SO(8) \subset SU(8)$, $28 = 28$. 

Jean-Pierre Derendinger (AEC, Bern)
An example, maximal supergravity with $SO(8)$

Group theory:
First embedding chain, relevant to gauge fields:

$Sp(56, \mathbb{R}) \supset SU(28) \times U(1) \supset SU(8) \times U(1)$

$56 = 28_1 + 28_{-1} = 28_1 + 28_{-1}$

$1596 = 783_0 + 1_0 + 406_2 + 406_{-2}$

$= 63_0 + 1_0 + 720_0 + 336_2 + 336_{-2} + 70_2 + 70_{-2}$

Second embedding chain, relevant to scalar fields:

$Sp(56, \mathbb{R}) \supset E_{7,7} \supset SU(8)$

$56 = 56 = 28 + 28$

$1596 = 133 + \ldots = 63 + 70 + \ldots$

$E_{7,7}$ is not unique in $Sp(56, \mathbb{R})$: for a given $SU(8)$, the $70$ component is complex with a $U(1)$ charge: a phase choice to adapt the $E_{7,7}$ of the scalars inside the electric-magnetic duality group.
An example, maximal supergravity with $SO(8)$

- Leads to a one-parameter family of $SO(8), \mathcal{N} = 8$ gauged supergravity. (Dall’Agata, Inverso, Trigiante; Borghese, Guarino, Roest)

- Invisible at the $SO(8)$ level: there is only one $\mathcal{N} = 8$, $SO(8)$ theory, a different definition of electric/magnetic.

- But visible when a second parameter is introduced in the embedding tensor, reducing the gauged algebra.

A very simple (but surprising) example of the gauging procedure in extended supergravities, with the largest compact gauging $SO(8)$. 