Corfu lectures on supersymmetry and supergravity

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Two documents

- *On supergravity theories after ∼ 40 years*

  Proc. of DISCRETE 2014, London, December 2014
  J. Phys. Conf. Ser. **631** (2015) 1, 012009.

  arXiv:1509.01195 [hep-th]

  DOI: 10.1088/1742-6596/631/1/012009

- *Lecture notes on globally supersymmetric theories in four-dimensions and two-dimensions*

  Preprint ETH-TH/90-21, July 1990.

  Proceedings of the 3rd Hellenic School on Elementary Particle Physics, Corfu, September 1989, World Scientific Singapore, 1990
  pages 111–243.

  On the page “Document, publications, lecture notes" of my web site:
  http://www.derendinger.itp.unibe.ch/
  Documents,_publications,_lecture_notes_files/SUSY_nd.pdf

Contents

Space-time algebras: Poincaré

Relativistic quantum field theory has global Poincaré symmetry:
Lorentz and translations

- *On coordinates:*

$$x^\mu \quad \longrightarrow \quad x^{\mu\prime} = \Lambda^\mu{}_\nu \, x^\nu + a^\mu \qquad \eta_{\mu\nu} \, \Lambda^\mu{}_\rho \, \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$$

Variation: $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \eta^{\mu\rho}\omega_{\rho\nu} \qquad \omega_{\rho\nu} = -\omega_{\nu\rho}$

$$\delta x^\mu = \omega^{\mu\nu} \, x_\nu + a^\mu = \left[\frac{i}{2} \, \omega^{\rho\sigma} M_{\rho\sigma} + i \, a^\nu P_\nu \right] x^\mu$$

Poincaré generators on coordinates: linear differential operators

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \qquad P_\mu = -i\partial_\mu$$

Poincaré Lie algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i \, (\eta^{\mu\rho} M^{\nu\sigma} + \eta^{\nu\sigma} M^{\mu\rho} - \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\rho} M^{\mu\sigma})$$

$$[P^\mu, M^{\nu\rho}] = i \, (\eta^{\mu\nu} P^\rho - \eta^{\mu\rho} P^\nu) \qquad [P^\mu, P^\nu] = 0$$

Space-time algebras: Poincaré

- *On fields:* $\Phi(x)$: a set of fields

Translations:

$$\Phi'(x+a) = \Phi(x) \qquad \Phi'(x+a) = \Phi'(x) + a^\mu \partial_\mu \Phi(x)$$

$$\delta\Phi(x) = \Phi'(x) - \Phi(x) = -i\, a^\mu P_\mu \Phi(x) \qquad P_\mu = -i\partial_\mu$$

Lorentz:

$$\Phi'(\Lambda^\mu{}_\nu x^\nu) = S(\Lambda)\Phi(x) \qquad S(\Lambda) = \mathbb{I} - \frac{i}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}$$

$$\delta\Phi(x) = -\frac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}\Phi(x) - \delta x^\mu \partial_\mu \Phi(x)$$

- The Casimir operator $P^\mu P_\mu = -\partial^\mu\partial_\mu = -\Box$ gives the field masses^2
- The matrix representation $\Sigma_{\mu\nu}$ contains the spins (for $P^2 > 0$) and/or helicities ($P^2 = 0$) of the fields.

Space-time algebras: Poincaré

There are ten conserved currents:

Translations:	$\tau_{\mu\nu}$	$\partial^\mu \tau_{\mu\nu} = 0$
Lorentz:	$j_{\mu,\nu\rho} = -j_{\mu,\rho\nu}$	$\partial^\mu j_{\mu,\nu\rho} = 0$

The energy-momentum tensor $\tau_{\mu\nu}$ can be improved: (Belinfante)

- Use the six Lorentz symmetries to obtain a new symmetric energy-momentum tensor $T_{\mu\nu} = T_{\nu\mu}$
- The corresponding Lorentz currents are $j_{\mu,\nu\rho} = x_\rho T_{\mu\nu} - x_\nu T_{\mu\rho}$

Summary: for fields $\Phi(x)$ the information of Poincaré symmetry is:

- in the eigenvalues of P^2 (masses^2, Klein-Gordon equation),
- in Lorentz representation $\Sigma_{\mu\nu}$ (spins/helicities) and
- in the symmetric energy-momentum tensor $T_{\mu\nu}$.

Space-time algebras

- Poincaré algebra: a contraction of either de Sitter (dS) or Anti de Sitter (AdS) algebras.

$$
\begin{aligned}
[M^{\mu\nu}, M^{\rho\sigma}] &= -i\,(\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma}) \\
[M^{\mu\nu}, P^{\rho}] &= -i\eta^{\mu\rho}P^{\nu} + i\eta^{\nu\rho}P^{\mu} \\
[P^{\mu}, P^{\nu}] &= -i\,v^2\Delta\,M^{\mu\nu}
\end{aligned}
$$

v is an energy-scale, an inverse radius
$\Delta = 1$: Anti-de Sitter algebra, $SO(2,3)$
$\Delta = -1$: de Sitter algebra, $SO(1,4)$

The infinite radius limit $v = 0$ for both Δ is Poincaré algebra, as Minkowski space-time is the infinite radius limit of dS or AdS space-time.

Background geometry has cosmological constant $\Lambda = -3\Delta v^2$

Space-time algebras

- Quantum field theory admits (in principle) the extension of Poincaré algebra to the conformal algebra $SO(2,4) \sim SU(2,2)$

$$
\begin{aligned}
[M^{\mu\nu}, M^{\rho\sigma}] &= -i\left(\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma}\right), \\
[M^{\mu\nu}, P^{\rho}] &= -i\left(\eta^{\mu\rho}P^{\nu} - \eta^{\nu\rho}P^{\mu}\right), & [P^{\mu}, P^{\nu}] &= 0, \\
[M^{\mu\nu}, D] &= 0, & [D, P^{\mu}] &= iP^{\mu}, \\
[M^{\mu\nu}, K^{\rho}] &= -i\left(\eta^{\mu\rho}K^{\nu} - \eta^{\nu\rho}K^{\mu}\right), & [K^{\mu}, K^{\nu}] &= 0, \\
[P^{\mu}, K^{\nu}] &= -2i\left(\eta^{\mu\nu}D + M^{\mu\nu}\right), & [D, K^{\mu}] &= -iK^{\mu}.
\end{aligned}
$$

D: generator of scale transformations (dilatation generator)
K_{μ}: generator of conformal boosts (or conformal transformations)

Either explicitly broken (Higgs mass for instance), or spontaneously (scalar expectation values) or generically by quantum effects (scale dependence of interactions, renormalisation-group effects, scale anomalies . . .)

Space-time superalgebras, supersymmetry

Space-time superalgebras are extensions of the space-time algebras with a fermionic sector. Schematically:

$$i: \; [B,B] \subset B \qquad ii: \; [B,F] \subset F \qquad iii: \; \{F,F\} \subset B$$

$i:$ B is a subalgebra $\supset$ a space-time algebra
$ii:$ F is a representation of B, the fermionic sector: F are Lorentz spinors
$iii:$ Fermionic operators: anticommutators

Two cases:

- Poincaré or Anti-de Sitter supersymmetry: superalgebra $OSp(\mathcal{N},4)$. $B = Sp(4,\mathbb{R}) \times SO(\mathcal{N})$ or its infinite radius (super-Poincaré) contraction. Anti-de Sitter since $Sp(4,\mathbb{R}) \sim SO(2,3)$.
- Superconformal algebra: superalgebra $SU(2,2|\mathcal{N})$. $B = SU(2,2) \times SU(\mathcal{N}) \times U(1)$ and $SU(2,2) \sim SO(2,4)$. No $U(1)$ if $\mathcal{N} = 4$

De Sitter supersymmetry ?

Supersymmetry algebra

Construction:

- Start with Lorentz algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\,(\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma})$$

- Add supercharges Q^i_α $(i = 1, \ldots, \mathcal{N})$, Lorentz spinors:

$$[M^{\mu\nu}, Q^i_\alpha] = -\frac{i}{4}([\sigma^\mu, \overline{\sigma}^\nu]Q^i)_\alpha$$

- Obtain $[P_\mu, Q^i_\alpha]$ from Jacobi identities.

- Find the most general $\{Q^i_\alpha, Q^j_\beta\}$ which solves all Jacobi identities.

For the $OSp(4, \mathcal{N})$ superalgebra, assume that the spinor charges Q^i_α are Majorana

Supersymmetry algebra: $[P_\mu, Q^i_\alpha]$

- Jacobi identity

$$0 = [M^{\mu\nu}, [P^\rho, Q^i_\alpha]] + [P^\rho, [Q^i_\alpha, M^{\mu\nu}]] + [Q^i_\alpha, [M^{\mu\nu}, P^\rho]]$$

is solved by $[P^\mu, Q^i_\alpha] = [(a + ib\gamma_5)\gamma^\mu]_{\alpha\beta} Q^i_\beta$

for arbitrary real (Majorana condition) numbers a and b

- Using then $[P^\mu, P^\nu] = -i\, v^2 \Delta\, M^{\mu\nu}$

Jacobi identity

$$0 = [P^\mu, [P^\nu, Q^i_\alpha]] + [P^\nu, [Q^i_\alpha, P^\mu]] + [Q^i_\alpha, [P^\mu, P^\nu]]$$

implies $\underline{a^2 + b^2 = \frac{1}{4} v^2 \Delta}$

Minkowski supersymmetry: $v = 0$
Anti-de Sitter supersymmetry: $v^2 > 0 \quad \Delta = 1$
De Sitter supersymmetry, $v^2 > 0 \quad \Delta = -1$ is not allowed

$\Longrightarrow$ With parity $[P^\mu, Q^i_\alpha] = \frac{1}{2}\, v\, (\gamma^\mu Q^i)_\alpha$

Supersymmetry algebra

From here on: Minkowski-space supersymmetry only

$\mathcal{N}$–extended supersymmetry algebra (with central charges)

$$
\begin{aligned}
[M^{\mu\nu}, M^{\rho\sigma}] &= -i\,(\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma}) \\
[M^{\mu\nu}, P^{\rho}] &= -i\,(\eta^{\mu\rho}P^{\nu} - \eta^{\nu\rho}P^{\mu}) \qquad [P^{\mu}, P^{\nu}] = 0, \\
[M^{\mu\nu}, Q^i_\alpha] &= -\tfrac{i}{4}([\gamma^\mu, \gamma^\nu]Q^i)_\alpha \qquad [P^{\mu}, Q^i_\alpha] = 0 \\
\{Q^i_\alpha, Q^j_\beta\} &= -2(\gamma^\mu C)_{\alpha\beta}P_\mu\delta^{ij} + iC_{\alpha\beta}V^{ij} + (\gamma_5 C)_{\alpha\beta}Z^{ij}
\end{aligned}
$$

P_μ: dimension $(\text{mass})^1$ $\qquad$ Q^i_α: $(\text{mass})^{1/2}$.

$V^{ij} = -V^{ji}$ and $Z^{ij} = -Z^{ji}$: central charges, commute with all operators.

Exist only for $\mathcal{N} \geq 2$ $\qquad$ Dimension $(\text{mass})^1$

Central charges introduce mass parameters in massive representations of the supersymmetry algebra.

Representations of the supersymmetry algebra

Two general properties:

1. All states have same mass, since $[P^2, Q^i_\alpha] = 0$
2. In each representation, same number of bosons and fermions, $n_B = n_F$

$[P_\mu, P_\nu] = 0$: consider eigenstates of P_μ with momentum p_μ, $\quad p^2 = M^2$

- Massless supermultiplets: $M = 0 \qquad p^\mu = (E, 0, 0, E)$
  $\mathcal{N}$ fermionic creation–annihilation pairs, $SU(\mathcal{N})$ invariance:
  $$\{Q^i, Q^{j\dagger}\} = 4E\delta^{ij} \qquad \{Q^i, Q^j\} = \{Q^{i\dagger}, Q^{j\dagger}\} = 0$$
  $2^{\mathcal{N}}$ states with helicities $\quad \widehat{\lambda}, \widehat{\lambda} - 1/2, \widehat{\lambda} - 1, \ldots, \widehat{\lambda} - \mathcal{N}/2$
- Massive representations: $M \neq 0 \qquad p^\mu = (M, 0, 0, 0)$
  $2\mathcal{N}$ fermionic creation–annihilation pairs, $Sp(2\mathcal{N})$ invariance,
  A multiple of $2^{2\mathcal{N}}$ states in representation

+ Doubling of the representation usually required by CPT: $\widehat{\lambda} \Longleftrightarrow \mathcal{N}/2 - \widehat{\lambda}$

Massless supermultiplets

Helicity states of supersymmetric gauge theory multiplets ($2^{\mathcal{N}+1}$ states)

States of given helicity λ are labelled by their $SU(\mathcal{N})$ representation.

λ	$\mathcal{N}=1$, $\widehat{\lambda}=1/2$	$\mathcal{N}=1$, $\widehat{\lambda}=1$	$\mathcal{N}=2$, $\widehat{\lambda}=1/2$	$\mathcal{N}=2$, $\widehat{\lambda}=1$	$\mathcal{N}=3$, $\widehat{\lambda}=1$	$\mathcal{N}=4$, $\widehat{\lambda}=1$
1		1		1	1	1
1/2	1	1	$\mathbf{1}+(\mathbf{1})$	2	$\mathbf{3}+\mathbf{1}$	4
0	$\mathbf{1}+\mathbf{1}$		$\mathbf{2}+(\mathbf{2})$	$\mathbf{1}+\mathbf{1}$	$\overline{\mathbf{3}}+\mathbf{3}$	6
-1/2	1	1	$\mathbf{1}+(\mathbf{1})$	2	$\mathbf{1}+\overline{\mathbf{3}}$	$\overline{4}$
-1		1		1	1	1
	i	ii	iii	iv	v	vi

i, iii: Matter multiplets with maximal helicity $1/2$ (chiral and hyper multiplets)

ii, iv, vi: Vector or gauge multiplets (maximal helicity 1) of $\mathcal{N}=1,2,4$

v: Same as vi, a lagrangian with $\mathcal{N}=3$ has actually $\mathcal{N}=4$

The chiral multiplet i only admits chiral representations ($R\neq\overline{R}$), as in SM

Supergravity supermultiplets

All massless multiplets with one state at maximal helicity 2.

States are labelled by $SU(\mathcal{N})$ representations (antisymmetric tensors).

λ	$\mathcal{N}=1$	$\mathcal{N}=2$	$\mathcal{N}=3$	$\mathcal{N}=4$	$\mathcal{N}=5$	$\mathcal{N}=6$	$\mathcal{N}=8$
2	1	1	1	1	1	1	1
$\frac{3}{2}$	1	2	3	4	5	6	8
1		1	$\overline{\mathbf{3}}$	6	10	$\mathbf{15}+\mathbf{1}$	28
1/2			1	$\overline{\mathbf{4}}$	$\overline{\mathbf{10}}+\mathbf{1}$	$\mathbf{20}+\mathbf{6}$	56
0				$\mathbf{1}+\mathbf{1}$	$\overline{\mathbf{5}}+\mathbf{5}$	$\overline{\mathbf{15}}+\mathbf{15}$	70
-1/2			1	4	$\mathbf{1}+\mathbf{10}$	$\overline{\mathbf{6}}+\mathbf{20}$	$\overline{\mathbf{56}}$
-1		1	3	6	$\overline{\mathbf{10}}$	$\mathbf{1}+\overline{\mathbf{15}}$	$\overline{\mathbf{28}}$
-3/2	1	2	$\overline{\mathbf{3}}$	$\overline{\mathbf{4}}$	$\overline{\mathbf{5}}$	$\overline{\mathbf{6}}$	$\overline{\mathbf{8}}$
-2	1	1	1	1	1	1	1
	4	8	16	32	64	128	256

- $\mathcal{N}=7$ leads to the same multiplet and theory as $\mathcal{N}=8$.
- In contrast to gauge theories, $\mathcal{N}=3$ and $\mathcal{N}=4$ are different theories.
- Multiplets have $2^{\mathcal{N}}{}_B + 2^{\mathcal{N}}{}_F$ states. But for $\mathcal{N}=8$: $2^{7}{}_B + 2^{7}{}_F$.

Some massive supermultiplets

Massive multiplets with maximal spin 2 for $\mathcal{N} = 1, 2, 4$ theories

Zero central charges.

States are labelled by $Sp(2\mathcal{N})$ representations (antisymmetric traceless tensors).

Spin	$\mathcal{N}=1$				$\mathcal{N}=2$			$\mathcal{N}=4$
2				1			1	1
3/2			1	2		1	4	8
1		1	2	1	1	4	$5+1$	27
1/2	1	2	1		4	$5+1$	4	48
0	2	1			5	4	1	42
total =	4	8	12	16	16	32	48	256

These are “long multiplets". Massive multiplets with central charges have less states (“short multiplets") in general.

$\mathcal{N} = 1$ supersymmetric field theories

A supersymmetric $\mathcal{N} = 1$ field theory describes:

- Vector multiplets (gauge fields A_μ^a + gauginos λ^a) in the adjoint representation of a gauge group G
- Chiral multiplets (Weyl fermions ψ_i + complex scalars z_i) in representation r of the gauge group.

Condition: r should be free of chiral anomalies.

Supersymmetry relates the interactions (and the masses) of superpartners in the lagrangian.

Exists in two forms:

- *Renormalizable:* defined by G, r and a gauge-invariant cubic polynomial (the superpotential) [MSSM, NMSSM, . . .]
- *SUSY sigma-model:* defined by G, r and three gauge-invariant functions

The simplest, chiral supermultiplet

Describes $2_B + 2_F$ on-shell states, helicities $0, 0, \pm 1/2$

Off-shell fields	z	ψ	f
$M = 0$, helicity:	$0, 0$	$\pm 1/2$	$0, 0$
$M \neq 0$, spin:	$0, 0$	$1/2$	$0, 0$

f is auxiliary, see later

1st step: Free, massless lagrangian

$$\mathcal{L}_0 = (\partial_\mu \overline{z})(\partial^\mu z) + \frac{i}{2}\psi\sigma^\mu\partial_\mu\overline{\psi} - \frac{i}{2}\partial_\mu\psi\sigma^\mu\overline{\psi}$$

Invariant, up to a derivative, under [ϵ: susy parameter, Majorana spinor]

$$\delta z = \sqrt{2}\,\epsilon\psi \qquad \delta\psi_\alpha = -\sqrt{2}i\,\partial_\mu z(\sigma^\mu\overline{\epsilon})_\alpha,$$

Susy algebra: $[\delta_1, \delta_2]z = -2i(\epsilon_2\sigma^\mu\overline{\epsilon}_1 - \epsilon_1\sigma^\mu\overline{\epsilon}_2)\partial_\mu z$

Result is a translation $\delta z = i\,\Delta^\mu\,P_\mu z$ $\qquad P_\mu = -i\partial_\mu$

$$\Delta^\mu = 2(\epsilon_2\sigma^\mu\overline{\epsilon}_1 - \epsilon_1\sigma^\mu\overline{\epsilon}_2)$$

The simplest, chiral supermultiplet

For the spinor, algebra holds on-shell only: Dirac: $\partial_\mu \psi \sigma^\mu = 0$

$$[\delta_1, \delta_2]\psi_\alpha = -2i(\epsilon_2 \sigma^\mu \overline{\epsilon}_1 - \epsilon_1 \sigma^\mu \overline{\epsilon}_2)\partial_\mu \psi_\alpha + 2i(\partial_\mu \psi \sigma^\mu \overline{\epsilon}_2)\epsilon_{1\alpha} - 2i(\partial_\mu \psi \sigma^\mu \overline{\epsilon}_1)\epsilon_{2\alpha}$$

2nd step: Modify with the auxiliary field f:

$$\delta z = \sqrt{2}\,\epsilon\psi \qquad \delta\psi_\alpha = -\sqrt{2} f \epsilon_\alpha - \sqrt{2} i \partial_\mu z (\sigma^\mu \overline{\epsilon})_\alpha$$

$$\delta f = -\sqrt{2} i\, \partial_\mu (\psi \sigma^\mu \overline{\epsilon})$$ $\Leftarrow$ a derivative

and then:

$$[\delta_1, \delta_2] z = -i\Delta^\mu \partial_\mu z \quad [\delta_1, \delta_2]\psi_\alpha = -i\Delta^\mu \partial_\mu \psi_\alpha \quad [\delta_1, \delta_2] f = -i\Delta^\mu \partial_\mu f$$

as expected: *a linear, off-shell representation of susy*

The simplest, chiral supermultiplet

3rd step: The modified δf imposes to modify the lagrangian:

$$\mathcal{L} = (\partial_\mu \overline{z})(\partial^\mu z) + \frac{i}{2}\psi\sigma^\mu\partial_\mu\overline{\psi} - \frac{i}{2}\partial_\mu\psi\sigma^\mu\overline{\psi} + \overline{f}f$$

$\Longrightarrow$ $\mathcal{L}$ invariant (up to a derivative), f *auxiliary* with field equation $f = 0$

4th step: Introduce masses:

$$-m[fz + \frac{1}{2}\psi\psi] - m[\overline{f}\overline{z} + \frac{1}{2}\overline{\psi\psi}]$$

is invariant (up to a derivative) under the same susy variations

Eliminate the auxiliary f with field equation $\overline{f} = mz$ leads to the free lagrangian of z and ψ with mass m.

The simplest, chiral supermultiplet

5th step: Introduce interactions: *Superpotential* $W(z) = \frac{m}{2}z^2 + \frac{\lambda}{3}z^3$.

$$-f\frac{dW}{dz} - \frac{1}{2}\frac{d^2W}{dz^2}\psi\psi \qquad + \text{h.c.}$$

is invariant (up to a derivative) under the same susy variations

Eliminating f with field equation $\overline{f} = mz + \lambda z^2$ (nonlinear now) leads to

$$\begin{aligned}\mathcal{L}_{m,\lambda} &= (\partial_\mu \overline{z})(\partial^\mu z) - V(z,\overline{z}) \\ &+\tfrac{i}{2}\psi\sigma^\mu\partial_\mu\overline{\psi} - \tfrac{i}{2}\partial_\mu\psi\sigma^\mu\overline{\psi} - \tfrac{m}{2}[\psi\psi + \overline{\psi\psi}] - \lambda z\psi\psi - \overline{\lambda}\overline{z}\overline{\psi\psi}\end{aligned}$$

Scalar potential:

$$V(z,\overline{z}) = |f|^2 = |mz + \lambda z^2|^2 = \left|\frac{d}{dz}W(z)\right|^2$$

The gauge, vector supermultiplet

Describes $\mathbf{2}_B + \mathbf{2}_F$ on-shell states, helicities $\pm 1, \pm 1/2$

	A_μ	λ	D
$M = 0$, helicity:	$\pm 1, 0$	$\pm 1/2$	0
$M \neq 0$, spin:	$0, 0$	$1/2$	$0, 0$

λ: gaugino spinor
D: auxiliary

Under supersymmetry variations

$$\delta A_\mu = i\epsilon\sigma_\mu\overline{\lambda} - i\lambda\sigma_\mu\overline{\epsilon} \qquad \delta\lambda = iD\epsilon + \frac{1}{2} F_{\mu\nu}\,\sigma^\mu\overline{\sigma}^\nu\,\epsilon$$

$$\delta D = \partial_\mu(\epsilon\sigma^\mu\overline{\lambda} + \lambda\sigma^\mu\overline{\epsilon})$$ $\Leftarrow$ a derivative again

the super-Yang-Mills lagrangian

$$\mathcal{L}_{SYM} = \mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\lambda\sigma^\mu\partial_\mu\overline{\lambda} - \frac{i}{2}\partial_\mu\lambda\sigma^\mu\overline{\lambda} + \frac{1}{2}DD\right]$$

is invariant up to a derivative.

$\mathcal{N} = 1$ superspace, superfields

At this point:

- The component with highest mass dimension in an off-shell supermultiplet is an auxiliary scalar which transforms with a derivative
- Precisely what is needed for a supersymmetric lagrangian
- *Wanted:* a systematic method to combine supermultiplets into supermultiplets.
  Then, auxiliary fields provide contributions to lagrangian field theories
- Two options:
  - Tensor calculus
  - Superspace and superfield techniques

$\mathcal{N} = 1$ superspace, superfields

- Space-time translations: $\phi(x) \longrightarrow \phi(x+a)$ $\quad\delta\phi(x) = ia^\mu P_\mu \phi(x)$

  Generators are <u>derivatives</u> $\quad P_\mu = -i\partial_\mu$ $\quad$ [P_μ: energy, a_μ: energy^{-1}]

- *Leibniz rule:* Combine fields into fields:

$$\delta\phi(x) = a^\mu \partial_\mu \phi \qquad \Longrightarrow \qquad \delta\, F(\phi(x)) = \frac{dF}{d\phi}\delta\phi = a^\mu \frac{dF}{d\phi}\partial_\mu\phi = a^\mu \partial_\mu F$$

- <u>Supersymmetry $\sim$ “square root of translation"</u>:

$$\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = -2i\,(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu \qquad\qquad \delta\Phi = (i\epsilon Q + \overline{\epsilon Q})\Phi$$

  Φ: a linear (off-shell) representation (supermultiplet)

- Superspace: extend formally space-time to superspace with coordinates

$$(\ x^\mu\ ,\ \theta_\alpha\ ,\ \overline{\theta}_{\dot\alpha}\ ) \qquad\qquad \{\theta_\alpha, \theta_\beta\} = \{\overline{\theta}_{\dot\alpha}, \overline{\theta}_{\dot\beta}\} = \{\theta_\alpha, \overline{\theta}_{\dot\beta}\} = 0$$

θ_α ($\overline{\theta}_{\dot\alpha}$) left-handed (right-handed) Weyl spinor:

Grassmann (anticommuting) coordinates.

$\mathcal{N} = 1$ superspace, superfields

Lorentz algebra, $SO(1,3) \sim Sl(2,\mathbb{C})$: θ_α: $(2,1)$ $\overline{\theta}_{\dot\alpha}$: $(1,2)$

$$\theta_\alpha\theta_\beta = -\theta_\beta\theta_\alpha = \tfrac{1}{2}\epsilon_{\alpha\beta}\theta\theta: \quad [(2,1)\times(2,1)]_A \to (1,1), \quad \theta\theta = \epsilon^{\beta\alpha}\theta_\alpha\theta_\beta$$

$$\overline{\theta}_{\dot\alpha}\overline{\theta}_{\dot\beta} = -\overline{\theta}_{\dot\beta}\overline{\theta}_{\dot\alpha} = \tfrac{1}{2}\epsilon_{\alpha\beta}\overline{\theta\theta}: \quad [(1,2)\times(1,2)]_A \to (1,1), \quad \overline{\theta\theta} = \epsilon^{\dot\alpha\dot\beta}\overline{\theta}_{\dot\alpha}\overline{\theta}_{\dot\beta}$$

$\theta_\alpha \times \overline{\theta}_{\dot\beta}$: $(2,1)\times(1,2) = (2,2)$, a vector: $\theta\sigma^\mu\overline{\theta}\ \sigma_{\mu\,\alpha\dot\beta}$

$$\theta_\alpha\theta_\beta\theta_\gamma = 0 \qquad\qquad \overline{\theta}_{\dot\alpha}\overline{\theta}_{\dot\beta}\overline{\theta}_{\dot\rho} = 0$$

- A superfield is a function in superspace, with coordinates $(x, \theta, \overline{\theta})$ (or with any other set of coordinates): $\Phi(x,\theta,\overline{\theta})$
- It has a polynomial expansion in θ, $\overline{\theta}$ which stops at $\theta\theta\overline{\theta\theta}$.
- The expansion includes 16 fields (functions of x)
- 8 fields are bosons, 8 fields are fermions

$\mathcal{N} = 1$ superspace, superfields

An example: the real, or vector (since it includes a vector field) superfield:

$$\begin{aligned} V(x,\theta,\overline{\theta}) &= C(x) + i\theta\chi(x) - i\overline{\theta}\overline{\chi}(x) + \theta\sigma^\mu\overline{\theta}\, v_\mu(x) \\ &\quad + \frac{i}{2}\theta\theta[M(x) + iN(x)] - \frac{i}{2}\overline{\theta\theta}[M(x) - iN(x)] \\ &\quad + i\theta\theta\overline{\theta}[\overline{\lambda}(x) + \frac{i}{2}\partial_\mu\chi(x)\sigma^\mu] - i\overline{\theta\theta}\theta[\lambda(x) - \frac{i}{2}\sigma^\mu\partial_\mu\overline{\chi}(x)] \\ &\quad + \frac{1}{2}\theta\theta\overline{\theta\theta}[D(x) - \frac{1}{2}\Box C(x)] \end{aligned}$$

- $V(x,\theta,\overline{\theta})$ is Lorentz invariant (scalar, zero spin)
  (by assumption, it could be spinor, vector, ...)
- Since V is scalar, bosons are red, fermions are green:
  C, M, N, D: four real scalars (4_B), v_μ: vector field (4_B),
  χ, λ: two Weyl spinors (8_F).

$\mathcal{N} = 1$ superspace, superfields

On superfields, supersymmetry variations are represented by derivatives in superspace:

$$iQ_\alpha = \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\overline{\theta})_\alpha\partial_\mu \qquad i\overline{Q}_{\dot\alpha} = -\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} - i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu$$

$$P_\mu = -i\partial_\mu \qquad \{Q_\alpha, \overline{Q}_{\dot\alpha}\} = -2i\,(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu$$

$$\delta\Phi(x,\theta,\overline{\theta}) = i[a^\mu P_\mu + \epsilon^\alpha Q_\alpha + \overline{\epsilon}_{\dot\alpha}\overline{Q}^{\dot\alpha}]\,\Phi$$

A supersymmetry variation induces a translation is superspace:

$$\begin{cases} x^\mu & \longrightarrow & x^\mu + a^\mu - i\theta\sigma^\mu\overline{\epsilon} + i\epsilon\sigma^\mu\overline{\theta} \\ \theta_\alpha & \longrightarrow & \theta_\alpha + \epsilon_\alpha \\ \overline{\theta}_{\dot\alpha} & \longrightarrow & \overline{\theta}_{\dot\alpha} + \overline{\epsilon}_{\dot\alpha} \end{cases}$$

And a function of superfields is a superfield.

$\mathcal{N} = 1$ superspace, superfields

Chiral superfields describe chiral supermultiplets (helicities $\pm 1/2, 0, 0$)

Susy covariant derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu\overline{\theta})_\alpha\partial_\mu \qquad \overline{D}_{\dot\alpha} = \frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} - i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu$$

Since $\{D_\alpha, Q_\beta\} = \{D_\alpha, \overline{Q}_{\dot\beta}\} = \{\overline{D}_{\dot\alpha}, Q_\beta\} = \{\overline{D}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} = 0$, constraints

$\overline{D}_{\dot\alpha}\Phi = 0$ (chiral superfield) $\qquad D_\alpha\overline{\Phi} = 0$ (antichiral superfield)

are compatible with supersymmetry variations.

Expansion is: $\qquad [\ y^\mu = x^\mu - i\theta\sigma^\mu\overline{\theta}\ ]$

$$\begin{aligned}\Phi(y,\theta) &= z(y) + \sqrt{2}\,\theta\psi(y) - \theta\theta\, f(y) \\ &= z(x) - i\theta\sigma^\mu\overline{\theta}\,\partial_\mu z(x) - \tfrac{1}{4}\theta\theta\overline{\theta\theta}\,\Box z(x) \\ &\quad + \sqrt{2}\,\theta\psi(x) + \tfrac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\overline{\theta} - \theta\theta\, f(x)\end{aligned}$$

Supersymmetric gauge theories

Field content:

- Chiral superfields in representation r of the gauge group ("matter")

$$\Phi \quad \longrightarrow \quad e^{\Lambda}\Phi, \qquad \overline{D}_{\dot{\alpha}}\Lambda = 0, \qquad \Lambda = \Lambda^a T_r^a$$

- Gauge fields in real superfield $\mathcal{A} = \mathcal{A}^a T_r^a$, with gauge transformation

$$e^{\mathcal{A}} \quad \longrightarrow \quad e^{-\overline{\Lambda}} e^{\mathcal{A}} e^{-\Lambda}$$

The abelian (Maxwell) case: $\mathcal{A} \quad \longrightarrow \quad \mathcal{A} - \Lambda - \overline{\Lambda}$

Then:

- $\overline{\Phi} e^{\mathcal{A}} \Phi$ is gauge invariant. Its highest component is the gauge-invariant (renormalizable) kinetic lagrangian of the chiral multiplet:

$$\mathcal{L}_{kin.} = [\overline{\Phi} e^{\mathcal{A}} \Phi]_{\theta\theta\overline{\theta}\overline{\theta}} \qquad \text{or} \qquad \mathcal{L}_{kin.} = \int d^2\theta d^2\overline{\theta}\, \overline{\Phi} e^{\mathcal{A}} \Phi$$

- The same holds for the real superfield $\mathcal{K}(\overline{\Phi} e^{\mathcal{A}}, \Phi)$.

A parenthesis on integrals over θ, $\overline{\theta}$

- Under $\int d^4x$, all derivatives $\partial_\mu(\ldots)$ are irrelevant.
- In a lagrangian, a derivative $\partial_\mu(\ldots)$ is irrelevant.
- For a chiral superfield Φ:

$$\int d^4x\, [\Phi]_{\theta\theta} = -\frac{1}{4}\int d^4x\, DD\,\Phi \;\equiv \int d^4x \int d^2\theta\,\Phi$$

- For a real superfield $\mathcal{A}$:

$$\int d^4x\, [\mathcal{A}]_{\theta\theta\overline{\theta\theta}} = \frac{1}{16}\int d^4x\, DD\overline{DD}\,\mathcal{A} \;\equiv \int d^4x \int d^2\theta d^2\overline{\theta}\,\mathcal{A}$$

These equalities can be used as definitions of the integration over Grassmann variables θ, $\overline{\theta}$ (Berezin integral).

Supersymmetric gauge theories

Next, we need gauge kinetic terms and the super-Yang-Mills lagrangian

- Gauge field strengths (or curvatures) $F^a_{\mu\nu}$ are in the chiral superfields:

$$\mathcal{W}_\alpha = -\frac{1}{4}\overline{DD}e^{-\mathcal{A}}D_\alpha e^{\mathcal{A}} \qquad \overline{\mathcal{W}}_{\dot\alpha} = \frac{1}{4}DDe^{\mathcal{A}}\overline{D}_{\dot\alpha}e^{-\mathcal{A}}$$

with $\overline{D}_{\dot\alpha}\mathcal{W}_\alpha = D_\alpha\overline{\mathcal{W}}_{\dot\alpha} = 0$ (and susy Bianchi identity)

- Then:

$$\mathcal{L}_{SYM} = \frac{1}{4}\int d^2\theta \ \mathrm{Tr}\,\mathcal{W}^\alpha\mathcal{W}_\alpha + \frac{1}{4}\int d^2\overline{\theta} \ \mathrm{Tr}\,\overline{\mathcal{W}}_{\dot\alpha}\overline{\mathcal{W}}^{\dot\alpha}$$

Or:

$$\mathcal{L}_{SYM} = \mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\lambda\sigma^\mu\partial_\mu\overline{\lambda} - \frac{i}{2}\partial_\mu\lambda\sigma^\mu\overline{\lambda} + \frac{1}{2}DD\right]$$

Supersymmetric gauge theories

The (almost) most general two-derivative lagrangian with $\mathcal{N} = 1$ supersymmetry and gauge invariance:

$$\begin{aligned}
\mathcal{L} &= \int d^2\theta d^2\overline{\theta} \, \left[\mathcal{K}(\overline{\Phi} e^{\mathcal{A}}, \Phi) + \xi^a \mathcal{A}^a\right] \\
&\quad + \int d^2\theta \, \left[W(\Phi) + \frac{1}{4} f(\Phi) \, \mathrm{Tr} \, \mathcal{W}\mathcal{W}\right] \\
&\quad + \int d^2\overline{\theta} \, \left[\overline{W}(\overline{\Phi}) + \frac{1}{4} \overline{f}(\overline{\Phi}) \, \mathrm{Tr} \, \overline{\mathcal{W}\mathcal{W}}\right]
\end{aligned}$$

$\mathcal{K}$: Kähler potential

W: superpotential

f: gauge kinetic function

Defined in terms of three gauge-invariant functions.

ξ^a: Fayet-Iliopoulos terms for abelian gauge fields only.

A moderately interesting generalization is

$$f(\Phi) \, \mathrm{Tr} \, \mathcal{W}\mathcal{W} \quad \Longrightarrow \quad \widetilde{f}(\Phi, \mathrm{Tr} \, \mathcal{W}\mathcal{W})$$

Supersymmetric gauge theories

- *Scalar kinetic terms:* $\mathcal{K}_{z\overline{z}}(\partial_\mu \overline{z})\partial^\mu z)$ $\qquad \mathcal{K}_{z\overline{z}} = \dfrac{\partial^2 \mathcal{K}}{\partial z \partial \overline{z}}$

A scalar field theory on a Kähler manifold.

- *Scalar potential:* $V(z, \overline{z}) = \mathcal{K}_{z\overline{z}}\, \overline{f} f + \dfrac{1}{2} f(z)\, D^a D^a \geq 0$

(of course, the value “0“ in the bound is not meaningful)

Auxiliary fields (field equations):

$$f = (\mathcal{K}_{z\overline{z}})^{-1} \frac{\partial W}{\partial z} \qquad\qquad D^a = \mathrm{Re}\, f(z)^{-1} \left[\frac{\partial \mathcal{K}}{\partial z} T_r^a z + \xi^a \right]$$

- If $\langle f \rangle = \langle D^a \rangle = 0$: the true ground state, supersymmetric.
- If no such solution, $\langle f \rangle$ or $\langle D^a \rangle \neq 0$ do not vanish, susy spontaneously broken

$$\delta\psi = -\sqrt{2}\, f\, \epsilon + \ldots \qquad\qquad \delta\lambda^a = i\, D^a\, \epsilon + \ldots$$

and there is a massless Goldstone spinor (the *Goldstino*)

The renormalizable theory

Renormalizability is obtained if $\mathcal{K} = \overline{\Phi} e^{\mathcal{A}} \Phi$ and $f(\Phi) = 1$ (to get canonical kinetic terms) and with a cubic, gauge-invariant, polynomial for the superpotential:

$$W(\Phi^i) = \alpha_i \Phi^i + \frac{1}{2} m_{ij} \Phi^i \Phi^j + \frac{1}{3} \lambda_{ijk} \Phi^i \Phi^j \Phi^k$$

Linear terms only exist for gauge-singlet chiral superfields.

This theory has exceptional renormalization properties:

- Non-renormalization theorems: only wave function renormalization for gauge and chiral multiplets needed: the parameters of the superpotential are not renormalized. Holds to all orders of perturbation theory.
- Soft breaking terms: terms breaking susy which only affect logarithmic divergences are gaugino masses, scalar masses ($\overline{z}z$ and $\mu^2 z^2 + \mathrm{h.c.}$), analytic trilinear couplings $\beta z^3 + \mathrm{h.c.}$
- Generated by susy breaking in supergravity, as required for realistic models
- Non-perturbative results, subtleties with massless chiral superfields, . . .

Supergravity

Local supersymmetry: variation parameter ϵ_α local $\Longrightarrow$ $\epsilon_\alpha(x)$

The space-time translation induced by $[\delta_1, \delta_2]$ is local

$$\Delta^\mu = 2(\epsilon_2\sigma^\mu\overline{\epsilon}_1 - \epsilon_1\sigma^\mu\overline{\epsilon}_2) = \Delta^\mu(x)$$

Follows from the superalgebra $\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = -2i\,(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu = 2(\sigma^\mu)_{\alpha\dot\alpha}P_\mu$

Local supersymmetry $\Longleftrightarrow$ local translations or general coordinate transformations (GCT)

and the gauge theory of supersymmetry is a theory of GRAVITATION.

- Non-renormalizable ... an effective theory of “something"
- Natural cut-off scale where gravitation is expected to feel quantum physics, the Planck scale $M_P \sim 10^{19}$ GeV
- The maximal $\mathcal{N} = 8$ theory has exceptional finiteness properties, under difficult investigations (but not the right physics)

Supergravity

There are independent motivations to consider supergravity theories

- *Bottom-up:* It provides a source and suggests structures for the supersymmetry breaking needed in (realistic) supersymmetric quantum field theories (like MSSM, NMSSM, . . .)
- Top-down: It can be used as an effective, low-energy ($E \ll M_P$) description of a more fundamental microscopic quantum theory with gravitation (like superstring theories)
- Curiosity, model for microscopic gravitation with gauge and matter fields (scattering amplitudes, . . .) . . .

A vast and complicated subject.
At first sight, for “realistic models", the existence of fermions in chiral representations excludes $\mathcal{N} > 1$.
But, for instance, string models suggest more supersymmetries with quite elaborate breaking patterns, still under study.

Preamble: spinors, vierbein

Supersymmetric theories (and Nature) have fermions and spinor fields.

- GCT: space-time with (Riemann) metric $g_{\mu\nu}(x)$ in coordinates x^μ and GCT-invariant line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$.
- Spinors ψ live in the (Minkowski) tangent-space at each point x.
- Tangent-space coord. $\zeta^a(x)$, line element $ds^2 = \eta_{ab}d\zeta^a(x)d\zeta^b(x)$, ds^2 has local Lorentz invariance.
- Vierbein:
  $$g_{\mu\nu}(x) = \eta_{ab}\, e^a_\mu(x) e^b_\nu(x) \qquad e^a_\mu = \partial_\mu \zeta^a(x)$$
  Inverse: $e^\mu_a e^b_\mu = \delta^b_a \qquad e^\mu_a e^a_\nu = \delta^\nu_\mu$
- Local Lorentz: $\delta\psi(x) = -\frac{i}{2}\omega_{ab}(x)\,\sigma^{ab}\psi(x) \qquad \sigma^{ab} = \frac{i}{4}[\gamma^a, \gamma^b]$

Dirac lagrangian: requires the gauge field of local Lorentz: the spin connection

$$\mathcal{L} = ie\,\overline{\psi}\gamma^\mu D_\mu\psi \qquad D_\mu\psi = \partial_\mu\psi + \frac{1}{2}\omega_{\mu\,ab}\,\sigma^{ab}\psi \qquad e = \det e^a_\mu$$

Simple $\mathcal{N} = 1$ supergravity

Simply the sum of the *covariantized* Einstein and Rarita-Schwinger lagrangians

- Symmetries are:
  - Local coordinate transformations (GCT) $\Longrightarrow$ General relativity
  - Local Lorentz (tangent space)
  - Local supersymmetry
- Fields are the gauge fields of:
  - Translations, GCT: the vierbein e_μ^a
  - Local Lorentz: the spin connection $\omega_\mu^{ab} = -\omega_\mu^{ba}$
  - Supersymmetry: the gravitino $\psi_{\alpha\mu}$

  [This is the first-order formalism]

Explicitly: $\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$

$$\mathcal{L} = e\, e_a^\mu e_b^\nu R_{\mu\nu}{}^{ab}(\omega) + e\,\overline{\psi}_\mu \gamma^{\mu\nu\rho} \widetilde{D}_\nu \psi_\rho \qquad e = \det e_\mu^a$$

Gravitino, Rarita-Schwinger lagrangian

$\psi_{\mu\alpha}$: gravitino, gauge field of supersymmetry, vector–spinor:

$$\begin{array}{ccccccccc} \textbf{spinor} & \otimes & \textbf{vector} & = & \textbf{gravitino} & \oplus & \textbf{spinor} \\ [(2,1)\oplus(1,2)] & \otimes & (2,2) & = & [(3,2)\oplus(2,3)] & \oplus & [(1,2)\oplus(2,1)] \end{array}$$

To isolate the spinor, projection condition:

$$(\gamma^a\psi_a)_\alpha = (\gamma^\mu\psi_\mu)_\alpha = 0 \qquad \Longrightarrow \qquad \widetilde{\psi}_{\alpha a} = \psi_{\alpha a} - \frac{1}{4}(\gamma_a\gamma^b\psi_b)_\alpha$$

This condition follows from the lagrangian (in Minkowski space)

$$\mathcal{L}_{RS} = \frac{1}{2\kappa^2}\,\overline{\psi}_a\gamma^{abc}\partial_b\psi_c$$ *Rarita-Schwinger* [κ^{-1}: *mass scale*]

- Majorana condition on ψ_a
- Gauge invariance $\delta\psi_a = \partial_a\lambda$
- Field equation: $\gamma^{abc}\partial_b\psi_c = 0$
- Propagates two massless states with helicities $\pm 3/2$

Gravitino, Rarita-Schwinger lagrangian

Counting states: starting with $4 \times 4 = 16_F$ hermitian fields in $\psi_{\alpha a}$

Rarita-Schwinger:

$$\mathcal{L}_{RS} = \frac{1}{2\kappa^2}\,\overline{\psi}_a \gamma^{abc}\partial_b\psi_c \qquad \delta\psi_a = \partial_a\lambda \qquad \gamma^{abc}\partial_b\psi_c = 0$$

- Use gauge invariance with $\gamma^a\partial_a\lambda = -\gamma^a\psi_a$ to impose $\gamma^a\psi_a = 0$
- Defines λ up to a solution of $\gamma^a\partial_a\widetilde{\lambda} = 0$ (massless Dirac)
- In gauge $\gamma^a\psi_a = 0$: field equation $\gamma^a\partial_b\psi^b = \gamma^b\partial_b\psi^a$
- And then multiply by γ_a to obtain:

$$\gamma^a\psi_a = 0 \quad \text{(gauge choice)} \qquad \partial_b\psi^b = 0 \qquad \gamma^b\partial_b\psi_a = 0 \quad \text{(Dirac)}$$

$$\delta\psi_a = \partial_a\widetilde{\lambda} \qquad \gamma^a\partial_a\widetilde{\lambda} = 0 \quad \text{(residual gauge symmetry)}$$

- Counting on-shell states: $16_F - 4_F - 4_F - 4_F - 2_F = 2_F$
  and these two states have helicities $\pm 3/2$ (use plane waves to check)

The spin connection, Einstein-Hilbert lagrangian

Einstein gravitation formulated in terms of the vierbein e_μ^a and the spin connection $\omega_\mu{}^{ab}$.

- Spin connection curvature (Lorentz gauge field)

$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^{ac}\omega_{\nu\,c}{}^{b} - \omega_\nu{}^{ac}\omega_{\mu\,c}{}^{b}$$

- Lagrangian $\qquad \mathcal{L}_{grav.} = \frac{1}{2\kappa^2}\, e\, R \qquad R = e_a^\mu e_b^\nu R_{\mu\nu}{}^{ab}$

- The spin connection has an algebraic field equation (does not propagate):

$$\begin{aligned} \omega_{\mu\,cd} &= -\tfrac{1}{2}(\partial_\mu e_{\nu c} - \partial_\nu e_{\mu c})e_d^\nu + \tfrac{1}{2}(\partial_\mu e_{\nu d} - \partial_\nu e_{\mu d})e_c^\nu \\ &\quad -\tfrac{1}{2} e_c^\rho e_d^\nu (\partial_\rho e_{\nu a} - \partial_\nu e_{\rho a}) e_\mu^a \;\equiv\; \omega_{\mu\,cd}(e) \end{aligned}$$

- Rewrite then R as a function of the vierbein and its derivative. and $\mathcal{L}_{grav.}$ propagates two states with helicities ± 2 (graviton)

- If the spin connection appears in other lagrangian terms, its algebraic field equation leads to contorsion:

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e) + \kappa_\mu{}^{ab}$$

Pure $\mathcal{N} = 1$ supergravity, construction

Pure $\mathcal{N} = 1$ supergravity is *very simple*:

Einstein-Hilbert + Rarita-Schwinger

$$\mathcal{S}_{ERS}[e_\mu^a, \psi_\mu, \omega_{\mu\,ab}] = \frac{1}{2\kappa_D^2} \int d^D x\, e \left(R + \overline{\psi}_\mu \gamma^{\mu\nu\rho} \tilde{D}_\nu \psi_\rho \right)$$

But: local symmetries imply covariant derivatives

$$\tilde{D}_\mu \psi_\nu = \partial_\mu \psi_\nu + \tfrac{1}{2} \omega_\mu{}^{ab} \sigma_{ab} \psi_\nu \qquad \text{(spin connection)}$$

$$\omega_{\mu\,ab} = \omega_{\mu\,ab}(e) + \kappa_{\mu\,ab} \qquad \text{(contorsion tensor)}$$

$$\tilde{D}_\mu \psi_\nu - \tilde{D}_\nu \psi_\mu = D_\mu \psi_\nu - D_\nu \psi_\mu + 2\, S_{\mu\nu}^\lambda \psi_\lambda \qquad \text{(torsion tensor)}$$

with gravitino torsion (for a $D = 4$ Majorana gravitino)

$$S_{\mu\nu}^\lambda = -\tfrac{1}{4} \overline{\psi}_\mu \gamma^\lambda \psi_\nu$$

$$\kappa_{\mu\,ab} = -\tfrac{1}{4} \left[\overline{\psi}_\mu \gamma_a \psi_b - \overline{\psi}_\mu \gamma_b \psi_a + \overline{\psi}_a \gamma_\mu \psi_b \right]$$

Pure $\mathcal{N} = 1$ supergravity, construction

Covariantization $\Longrightarrow$ four-gravitino interaction, and then:

Four-dimensional pure $N = 1$ supergravity is *not so simple*:

$$\begin{aligned}\mathcal{L} \;=\; & \frac{1}{2\kappa_4^2} eR\,(\omega(e)) + \frac{1}{2\kappa_4^2} e\overline{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu(\omega(e))\psi_\rho \\ & + \frac{e}{32\kappa_4^2}\Big[4(\overline{\psi}^\mu\gamma_\mu\psi_\rho)(\overline{\psi}^\nu\gamma_\nu\psi^\rho) - (\overline{\psi}_\mu\gamma_\nu\psi_\rho)(\overline{\psi}^\mu\gamma^\nu\psi^\rho) \\ & -2(\overline{\psi}_\mu\gamma_\nu\psi_\rho)(\overline{\psi}^\mu\gamma^\rho\psi^\nu)\Big]\end{aligned}$$

with now $\tilde{D}_\nu\psi_\rho = \partial_\nu\psi_\rho + \frac{1}{2}\omega_{\nu\,ab}(e)\sigma^{ab}\psi_\rho$, the usual spin connection of pure gravitation theory.

- In four space-time dimensions:
  - Gravitino: $4 \times 4 - 4 = 12_F$ off-shell. 2_F with helicities $\pm 3/2$ on-shell.
  - Graviton: $10 - 4 = 6_B$ on-shell. 2_B with helicities ± 2 on-shell.

Pure $\mathcal{N} = 1$ supergravity, variations, auxiliary fields

More complications: auxiliary fields of $N = 1$ supergravity

The $N = 1$ supergravity action is invariant under local susy variations

$$\begin{array}{lllllll} \delta e_\mu^a & = & -\frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu & \qquad & \delta e_a^\mu & = & \frac{1}{2}\bar{\epsilon}\gamma^\mu\psi_a \\ \delta\psi_\mu & = & D_\mu\epsilon & & \delta\overline{\psi}_\mu & = & D_\mu\bar{\epsilon} \end{array}$$

With standard covariant derivative $D_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{2}\omega_{\mu\,ab}\sigma^{ab}\epsilon$

But it is not an off-shell representation of the supersymmetry algebra:
$[\delta_1, \delta_2]$ is a diffeomorphism only for fields solving the field equations

Another sign is the number of off-shell field components: $12_F \neq 6_B$.

More (auxiliary) fields needed for a linear off-shell representation, with
$n_B^{aux} - n_F^{aux} = 6$

By field equations, vanish for pure supergravity, but produce interactions when coupled to matter or gauge multiplets.

Pure $\mathcal{N} = 1$ supergravity, auxiliary fields

Several choices of auxiliary fields:

- Minimal schemes with $12_B + 12_F$ $\quad (n_B^{aux} = 6, \quad n_F^{aux} = 0)$
  - Old minimal: A^μ, not a gauge field (4_B), couples to a non-conserved (in general) current and f_0, complex scalar (2_B)
  - New minimal: A^μ with gauge symmetry (3_B), couples to a conserved current, antisymmetric tensor $B_{\mu\nu}$ with gauge symmetry (3_B).
- Non minimal schemes have $16_B + 16_F$ (somewhat, loosely relevant to superstring theories), $20_B + 20_F, \ldots$
- Each scheme generates a particular class of interactions. For instance: R–symmetric with new minimal.

- The most general is the simplest, old minimal (describes all classes).
- Each scheme can be constructed from superconformal theories with various compensating fields to gauge-fix dilatation, special conformal and R symmetries

Supergravities, summary

Four-dimensional supersymmetry (linear) representations

SUSY	Supergravity	$\lvert$Hel.$\rvert \leq 1$	$\lvert$Hel.$\rvert \leq 1/2$	Chirality	
$\mathcal{N}=1$	2_B+2_F	$\checkmark$	$\checkmark$ *	$\checkmark$	
$\mathcal{N}=2$	4_B+4_F	$\checkmark$ *	$\checkmark$ *	-	$D=6$
$\mathcal{N}=3$	8_B+8_F	$\checkmark$ *	-	-	
$\mathcal{N}=4$	16_B+16_F *	$\checkmark$ *	-	-	$D=10$
$\mathcal{N}=5$	32_B+32_F *	-	-	-	
$\mathcal{N}=6$	64_B+64_F *	-	-	-	
$\mathcal{N}=8$	128_B+128_F *	-	-	-	$D=11$

- *: scalar fields in supermultiplet
- Number of supercharges is $4\mathcal{N}$
- 16 supercharges ($\mathcal{N}=4$): type I, heterotic strings
- 32 supercharges ($\mathcal{N}=8$): type IIA, IIB strings, M–“theory"
- $\mathcal{N}=7$ does not exist (it is the $\mathcal{N}=8$ theory)
- $\mathcal{N}=0,1$ only for realistic models, or nonlinear, (or truncated. . .)

$\mathcal{N} = 1$ supergravity and matter couplings

$\mathcal{N} = 1$ supergravity couples to:

all gauge groups (gauge superfield A_μ, λ, helicities $\pm 1, \pm 1/2$)

all representations for chiral multiplets (ψ and z, helicities $\pm 1/2, 0, 0$)

and allows chirality of fermion representations.

The idea is then:

- Couple the SSM to supergravity, add a “hidden" sector to break supersymmetry.
- Generate a susy breaking scale $m_{3/2}$ and scalar vev's in the hidden sector $\langle \phi \rangle$
- Decouple gravity: expand, take $M_P \longrightarrow \infty$, keep $m_{3/2}$ fixed, ...
- The result is a global $\mathcal{N} = 1$ theory with soft breaking terms.
- However: $\mathcal{N} = 1$ Poincaré only if the cosmological constant at the breaking point is zero. In general, AdS global $\mathcal{N} = 1$ ...
- A severe constraint on the hidden sector ...

$\mathcal{N} = 1$ supergravity and matter couplings

The complete Lagrangian has been obtained by Cremmer, Ferrara, Girardello and Van Proeyen (1982). It needs 1.5 pages in Nucl. Phys. B 212.

In the superconformal formulation, it is symbolically

$$\mathcal{L} = -\frac{3}{2}\Big[S_0\overline{S}_0\, e^{-\mathcal{K}/3}\Big]_D + \Big[S_0^3 W(\Phi) + \frac{1}{4}f(\Phi)\,\mathrm{Tr}\,\mathcal{W}\mathcal{W}\Big]_F$$

where S_0 is the chiral compensating multiplet of the old minimal formalism.

- Similar to the global superspace Lagrangian

$$\mathcal{L} = \int d^2\theta d^2\overline{\theta}\,\mathcal{K}(\overline{\Phi}^{\mathcal{A}},\Phi) + \int d^2\theta\,\Big[W(\Phi) + \frac{1}{4}f(\Phi)\,\mathrm{Tr}\,\mathcal{W}\mathcal{W}\Big] + \mathrm{h.c.}$$

Superconformal and superspace calculus turn these symbolic expressions into Lagrangians ....

Curiously, most applications use only the scalar potential and few fermion mass terms.

Local versus global, Kähler symmetry

- Global supersymmetry: three independent functions

$$\mathcal{L} = \int d^2\theta d^2\overline{\theta}\, \mathcal{K}(\overline{\Phi}^{\mathcal{A}}, \Phi) + \int d^2\theta\, [W(\Phi) + \frac{1}{4} f(\Phi)\, \mathrm{Tr}\, \mathcal{W}\mathcal{W}] + \mathrm{h.c.}$$

Invariant under Kähler transformations $\mathcal{K} \longrightarrow \mathcal{K} + \Lambda(\Phi) + \overline{\Lambda}(\overline{\Phi})$.

- $N = 1$ supergravity, in the superconformal formulation:

$$\mathcal{L} = -\frac{3}{2}\Big[S_0 \overline{S}_0\, e^{-\mathcal{K}/3}\Big]_D + \Big[S_0^3 W + \frac{1}{4} f(\Phi)\, \mathrm{Tr}\, \mathcal{W}\mathcal{W}\Big]$$

Kähler transformation is a gauge invariance:

$$\mathcal{K}' = \mathcal{K} + \Lambda(\Phi) + \overline{\Lambda}(\overline{\Phi}) \qquad S_0{}' = S_0 e^{\Lambda(\Phi)/3} \qquad W' = W e^{-\Lambda(\Phi)}$$

Supergravity depends then on two functions only:

$$\mathcal{L} = -\frac{3}{2}\Big[S_0 \overline{S}_0\, e^{-\mathcal{G}/3}\Big]_D + \Big[S_0^3 + \frac{1}{4}\, \mathrm{Tr}\, \mathcal{W}\mathcal{W}\Big] \qquad \mathcal{G} = \mathcal{K} + \ln |W|^2$$

Also true for global susy in an Anti-de Sitter geometry,
an effect of the large radius limit to Poincaré

The $\mathcal{N} = 1$ supergravity scalar potential

$$V = \frac{1}{\kappa^4}\Big[e^{\mathcal{K}}\, {\mathcal{K}^{-1}}^i{}_j (W_i + \mathcal{K}_i W)(W^j + \mathcal{K}^j W) - 3\, e^{\mathcal{K}}\, \overline{W} W\Big] + \frac{1}{2} f(\Phi)^{-1} \mathcal{K}_i (T^A z)^i \mathcal{K}_j (T^A z)^j$$

- Blue terms are positive or zero: they are generated by chiral (f) or gauge (D) auxiliary fields. Susy breaks if they are not zero.
- The red term is negative or zero, it is generated by a supergravity auxiliary field. Unbroken susy: Anti-de Sitter or Minkowski ($W = 0$).
- Supergravity and supersymmetry are actually extensions of Anti-de Sitter symmetry: $SO(2,3) \sim Sp(4,\mathbb{R}) \longrightarrow OSp(n|4)$
  Poincaré is obtained in the large AdS radius limit only.
- De Sitter ground state: with broken supersymmetry only.

Supersymmetry breaking

In Nature, supersymmetry is at best a broken symmetry . . .

- Global $\mathcal{N} = 1$ supersymmetry

Spontaneous susy breaking induced by auxiliary field vev's: $\langle f \rangle$, $\langle D^a \rangle$

Scalar potential: $V = \overline{f} f + \frac{1}{2} D^a D^a \geq 0$

An algebraic problem: $f = \frac{\partial W}{\partial z} = 0$ and $D^a = 0$ should have no solution.

Disastrous mass relations:

$\langle f \rangle \quad \Rightarrow \quad \mathrm{STr}\, \mathcal{M}^2 = 0 \qquad (\mathrm{STr}\, \mathcal{M}^2 = \mathrm{Tr}_{\mathrm{bosons}}\, \mathcal{M}^2 - \mathrm{Tr}_{\mathrm{fermions}}\, \mathcal{M}^2)$

$\langle D^a \rangle$: needs Fayet-Iliopoulos terms for gauged $U(1)$
in conflict with data, phenomenology, anomalies, esthetics . . .

$$\mathrm{STr}\, \mathcal{M}^2 \sim -\langle D^a \rangle \, \mathrm{Tr}\, T^a$$

And anyway a massless Goldstino spinor.

Consider then spontaneous breaking of local susy in supergravity models

Spontaneous breaking of local supersymmetry

- Analogy: $U(1)$ gauge theory with a charged complex $\phi(x)$:

$$\phi(x) = e^{i\sigma(x)}[v + h(x)]$$

v: $U(1)$-breaking order parameter (gauge-invariant)

gauge symmetry: $\delta\sigma = \Lambda \qquad \delta A_\mu = -\partial_\mu \Lambda$

$$(D_\mu\phi)^\dagger(D^\mu\phi) \quad \longrightarrow \quad v^2(A_\mu + \partial_\mu\sigma)(A^\mu + \partial^\mu\sigma) + \ldots$$

The gauge-invariant massive field is $\widetilde{A}_\mu = A_\mu + \partial_\mu\sigma$, σ is the Goldstone boson, and the helicity zero component of the spin one $\widetilde{A}_\mu$

- Super-higgs mechanism: (neither Higgs nor “et al." found this)

Induced by auxiliary field vev's $\langle f^i\rangle$, $\langle D^a\rangle$

Massless Goldstino would be $\eta_G = \langle \mathcal{K}_i^j \overline{f}_j\rangle\psi^i - \frac{i}{2}\langle \mathrm{Re}\, f(z) D^a\rangle\lambda^a$

Use local susy parameter ϵ to gauge-away η_G.

Since $\delta\psi_\mu = \partial_\mu\epsilon + \ldots$ η_G absorbed in ψ_μ: a massive spin 3/2 state.

Susy-breaking order parameter. $m^2_{3/2} = \frac{1}{\kappa^2}\langle |W|^2 e^{\mathcal{K}}\rangle$

A “no-scale" example

Two chiral supermultiplets with scalar fields S and T, and

$$\mathcal{K} = -n\ln(T+\overline{T}) + \widehat{\mathcal{K}}(S,\overline{S})$$ Kähler potential,

$$W = W(S)$$ Superpotential.

Scalar potential (using $\kappa = 1$ in Einstein frame)

$$V = (T+\overline{T})^{-n}\, e^{\widehat{\mathcal{K}}} \left[\widehat{\mathcal{K}}^{-1}_{S\overline{S}}\, |W_S + \widehat{\mathcal{K}}_S W|^2 + (n-3)W\overline{W}\right]$$

Positive if $n \geq 3$. Choose $n = 3$ and solve $\langle W_S + \widehat{\mathcal{K}}_S W\rangle = 0$

$\Longrightarrow$ A stable ground state with $\langle V\rangle = \Lambda = 0$

$\langle W_S + \widehat{\mathcal{K}}_S W\rangle = 0$: fixes $\langle S\rangle$ (in general), $\langle f_S\rangle = 0$, leaves $\langle T+\overline{T}\rangle$ arbitrary, as well as

$$\langle f_T\rangle = \langle (T+\overline{T})^{-1/2} e^{\widehat{\mathcal{K}}/2}\, \overline{W}\rangle$$

$\langle W\rangle \neq 0$: susy broken by T with arbitrary susy-breaking order parameter

$$m_{3/2} = \langle e^{\mathcal{K}/2} W\rangle = \langle (T+\overline{T})^{-3/2} e^{\widehat{\mathcal{K}}/2} W\rangle$$

$m_{3/2}$ is the gravitino mass since $\Lambda = 0$.

Supergravity scalar potentials again

For all $\mathcal{N}$, the typical supergravity potential is the sum of three terms:

$$V = V_m + V_g + V_0$$

- $V_m \geq 0$: generated when matter fermions are present. Exists for $\mathcal{N} = 1, 2$ (chiral and hyper multiplets).

$$\delta\psi = \mathcal{A}\epsilon + \ldots \quad \longrightarrow \quad V_m \sim +|\mathcal{A}|^2$$

- $V_g \geq 0$: generated when gauginos are present. Exists for all $\mathcal{N} = 1, \ldots, 8$.

$$\delta\lambda = \mathcal{B}\epsilon + \ldots \quad \longrightarrow \quad V_g \sim +|\mathcal{B}|^2$$

- $V_0 \leq 0$: generated by the gravitinos (helicities $\pm 2, \pm 1/2$): Exists for all $\mathcal{N} = 1, \ldots, 8$.

$$\delta\psi_\mu = \mathcal{C}\gamma_\mu\epsilon + \ldots \quad \longrightarrow \quad V_0 \sim -|\mathcal{C}|^2$$

- Some or all supersymmetries break if V_m and/or V_g are positive on the vacuum.

Supergravity scalar potentials again

- The gauge potential V_g plays a fundamental role in the vacuum structure of $\mathcal{N}$-extended supergravities.
- Produced by gauging a symmetry of the theory: abelian (for instance, R–symmetry in $\mathcal{N}=1$) or non-abelian. Compact or non-compact.
- Flat directions in the potential, with no-scale behaviour and Minkowski space, algebraically characterized: a condition on the gauged algebra.

$(\ldots)$: number of abelian, ungauged, gauge fields in the supermultiplet:

SUSY	Supergravity	|Hel.|≤ 1	|Hel.|$\leq 1/2$	
$\mathcal{N}=1$	2_B+2_F (0)	✓ (n)	✓	
$\mathcal{N}=2$	4_B+4_F (1)	✓ (n)	✓	$D=6$
$\mathcal{N}=3$	8_B+8_F (3)	✓ (n)	-	
$\mathcal{N}=4$	16_B+16_F (6)	✓ (n)	-	$D=10$
$\mathcal{N}=5$	32_B+32_F (10)	-	-	
$\mathcal{N}=6$	64_B+64_F (15)	-	-	
$\mathcal{N}=8$	128_B+128_F (28)	-	-	$D=11$

Some dates and names, ~ 40 years ago

- Field theories with linear supersymmetry, 1974 (Wess and Zumino).
- Soon found to have softer divergences than ordinary gauge theories (logarithmic renormalization only) and powerful all-order non-renormalization theorems (Iliopoulos, Zumino, Wess, Ferrara).
- Superspace techniques (Salam, Strathdee; Wess, Zumino, Ferrara).
- Spontaneous supersymmetry breaking (Fayet, Iliopoulos, 1974; O'Raifeartaigh, 1975).
- Currents and supercurrents, approaches to gravity coupling (Ferrara, Zumino, 1974).
- Supergravity was created in 1976 (Ferrara, Freedman and Van Nieuwenhuizen; Deser and Zumino).
- Matter and gauge couplings to $N = 1$ supergravity, 1982, Cremmer, Ferrara, Girardello, Van Proeyen

  (also Arnowitt, Chamseddine, Nath; Bagger, Witten)

Dilaton supergravity, no-scale models

For a single chiral superfield S and a constant superpotential W,

$$V = \frac{1}{\kappa^4} e^{\mathcal{K}} \Big[K^{-1}_{S\overline{S}} \mathcal{K}_S \mathcal{K}_{\overline{S}} - 3 \Big] \overline{W} W$$

is identically zero if

$$\mathcal{K} = -3 \ln(S + \overline{S}) \qquad\qquad \forall\, W$$

but the auxiliary field f_S and the gravitino mass are

$$f_S = \overline{W} (S + \overline{S})^{-1/2} \neq 0 \qquad\qquad m_{3/2} = W (S + \overline{S})^{-3/2}$$

Hence, W induces supersymmetry breaking in Minkowski space, to obtain:

Broken supersymmetry in Minkowski space with a free scale $\langle S + \overline{S} \rangle$

The prototype of no-scale models:

Tree-level susy breaking scale arbitrary, radiative corrections may define it with some logarithmic factor and then with an induced scale hierarchy.

(Cremmer, Ferrara, Kounnas, Nanopoulos, 1983)

Dilaton supergravity, no-scale models

Consider now a string compactification:

In general, it produces a real dilaton scalar and an antisymmetric tensor $B_{\mu\nu}$ with gauge invariance $\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$ in the universal gravitation sector (in type II, in NS–NS sector).

The antisymmetric tensor is equivalent to a real scalar with shift symmetry:

$$\partial_{[\mu} B_{\nu\rho]} \quad \leftrightarrow \quad \partial_\mu \, \mathrm{Im}\, s \qquad\qquad C \quad \leftrightarrow \quad \mathrm{Re}\, s$$

and there should be a description in terms of a chiral multiplet S, with however an auxiliary field f_S which could be a source of supersymmetry breaking.

The relation is a Legendre transformation between supermultiplets.

The behaviour of the dilaton scalar in the effective supergravity Lagrangian is important: its value is the string coupling. Does it stabilize, does it slide to zero (run away), are further moduli fields needed ?

Dilaton supergravity, no-scale models

Within supergravity, two descriptions and a duality generated by a Legendre transformation:

- Description with $B_{\mu\nu}$: (The superpotential is constant)

$$\mathcal{L} = -\frac{3}{2}\Big[S_0\overline{S}_0\mathcal{H}(X)\Big]_D + \Big[S_0^3\, W\Big]_F \qquad X = \frac{L}{S_0\overline{S}_0}$$

- Description with chiral multiplet S:

$$\widetilde{\mathcal{L}} = -\frac{3}{2}\Big[S_0\overline{S}_0 e^{-\frac{1}{3}\mathcal{K}(S+\overline{S})/3}\Big]_D + \Big[S_0^3 W\Big]_F$$

- Legendre transformation: $e^{-\frac{1}{3}\mathcal{K}(S+\overline{S})} = \mathcal{H}(X) - X(S+\overline{S})$

- Dilaton supergravities:

Heteretic: $\mathcal{H} \sim X^{-1/2}$ $\qquad \mathcal{K} = -ln(S+\overline{S})$

Type II: $\mathcal{H} \sim X^{4}$ $\qquad \mathcal{K} = -4\,ln(S+\overline{S})$

Dilaton supergravity, no-scale models

- The Legendre transformation implies:

$$f_S = -C\mathcal{H}_{CC}\overline{z}_0 f_0$$

  and f_S is not an independent auxiliary field. Generalization to many fields:
  The auxiliary field f_S of a chiral multiplet dual to a linear superfield with an antisymmetric rensor is a linear combination of other auxiliary fields.

$$f_S \sim \frac{\partial}{\partial z^i}\mathcal{H}_C f^i$$

- The dilaton is not stabilized. More fields and interactions required.
- The single field no-scale model with $\mathcal{K} = -3\ln(S+\overline{S})$ does not describe a $B_{\mu\nu}$ + dilaton sector.
- Hence, low-energy scenarios in which supersymmetry breaking is induced by the dilaton superfield S only are forbidden by supergravity arguments.

Gauged supergravities

- All *ungauged* supergravities have been constructed long ago. They depend on the abelian field strengths $F_{\mu\nu}$ only and have then (in four dimensions) electric-magnetic duality.
- A symmetry of an ungauged theory can be gauged using the abelian gauge fields of the theory. One selects an algebra and associates a (electric or magnetic) gauge field A_μ^M of the theory with each generator

$$[T_A, T_B] = f_{AB}{}^C \qquad X_M = \Theta_M{}^A T_A \qquad \Theta_M{}^A\text{: embedding tensor}$$

- The consistency conditions for the procedure have been established for a generic field theory in a fundamental paper by de Wit, Samtleben and Trigiante (hep-th/0507289).
- Large classes of gauged supergravities have been constructed, large classes are missing.
- Particularly interesting for 16 ($\mathcal{N} = 4$) and 32 ($\mathcal{N} = 8$) supercharges related to superstrings and M theories.

An example, maximal supergravity with $SO(8)$

- Can be obtained by S_7 sphere compactification of 11-dimensional supergravity. (de Wit, Nicolai)
- $\mathcal{N} = 8$ supergravity has 28 abelian gauge fields $F^I_{\mu\nu}$ and then 28 duals $\widetilde{F}^I_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{I\rho\sigma}$ and 70 scalars..

  Obvious gauging: the 28 gauge fields in the adjoint of $SO(8)$: electric gauging.
- Is the gauging unique ?
- Starting point:
  - The electric-magnetic duality group is $Sp(56, \mathbb{R})$ (Gaillard, Zumino).
  - The 70 scalar fields are in $E_{7,7}/SU(8)$ with $E_{7,7} \subset Sp(56, \mathbb{R})$.
  - Fermions reduce the symmetry to $SU(8)$
  - Gauge group $SO(8) \subset SU(8)$, $\mathbf{28} = \mathbf{28}$.

An example, maximal supergravity with $SO(8)$

Group theory:

First embedding chain, relevant to gauge fields:

$$Sp(56,\mathbb{R}) \supset SU(28)\times U(1) \supset SU(8)\times U(1)$$

$$\begin{aligned} \mathbf{56} &= \mathbf{28_1} + \overline{\mathbf{28}}_{-1} = \mathbf{28_1} + \overline{\mathbf{28}}_{-1} \\ \mathbf{1596} &= \mathbf{783_0} + \mathbf{1_0} + \mathbf{406_2} + \overline{\mathbf{406}}_{-2} \\ &= \mathbf{63_0} + \mathbf{1_0} + \mathbf{720_0} + \mathbf{336_2} + \overline{\mathbf{336}}_{-2} + \mathbf{70_2} + \mathbf{70_{-2}} \end{aligned}$$

Second embedding chain, relevant to scalar fields:

$$Sp(56,\mathbb{R}) \supset E_{7,7} \supset SU(8)$$

$$\begin{aligned} \mathbf{56} &= \mathbf{56} = \mathbf{28} + \overline{\mathbf{28}} \\ \mathbf{1596} &= \mathbf{133} + \ldots = \mathbf{63} + \mathbf{70} + \ldots \end{aligned}$$

$E_{7,7}$ is not unique in $Sp(56,\mathbb{R})$: for a given $SU(8)$, the $\mathbf{70}$ component is complex with a $U(1)$ charge: a phase choice to adapt the $E_{7,7}$ of the scalars inside the electric-magnetic duality group.

An example, maximal supergravity with $SO(8)$

- Leads to a one-parameter family of $SO(8)$, $\mathcal{N} = 8$ gauged supergravity. (Dall'Agata, Inverso, Trigiante; Borghese, Guarino, Roest)
- Invisible at the $SO(8)$ level: there is only one $\mathcal{N} = 8$, $SO(8)$ theory, a different definition of electric/magnetic.
- But visible when a second parameter is introduced in the embedding tensor, reducing the gauged algebra.

A very simple (but surprising) example of the gauging procedure in extended supergravities, with the largest compact gauging $SO(8)$.