Membrane Matrix models

and non-perturbative checks of AdS/CFT

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Based on work with Veselin Filev [arXiv:1506.01366]

I will review the formulation of membranes as matrix models and then present some results based on this formulation.

From Membranes to Matrices

Hoppe in his Ph.D. thesis, with advisor Goldstone, recast the membrane action into a gauge theory of the area-preserving transformations of the membrane surface and then used a matrix regularisation to quantise the model. See [arXiv:hep-th/0002016].

The Membrane action

$$S = -\frac{T}{2}\int d^3\sigma(\sqrt{-h}\left(h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \Lambda\right)$$

Choose $\Lambda = 1$ (rescale X^a and T), and for membranes topology $\mathbb{R}\times\Sigma$ use the gauge $h_{0i} = 0$ and $h_{00} = -\frac{4}{\rho}\det(h_{ij})$.

The action becomes

$$S = \frac{T\rho}{4}\int dt\left(\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - \frac{4}{\rho^2}\det(h_{ij})\right)$$

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In 2-dim $det(h_{ij})$ can be rewritten using $\{f,g\} = \epsilon^{ij}\partial_i f \partial_j g$ as

$$S = \frac{T\rho}{4}\int dt \left(\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - \frac{4}{\rho^2}\{X^\mu, X^\nu\}^2\right)$$

and the constraints become

$$\dot{X}^\mu \partial_i X_\mu = 0 \implies \{\dot{X}^\mu, X_\mu\} = 0$$

$$\text{and} \qquad \dot{X}^\mu \dot{X}_\mu = -\frac{2}{\rho^2}\{X^\mu, X^\nu\}\{X_\mu, X_\nu\}\,.$$

Using lightcone coordinates with $X^\pm = (X^0 \pm X^{D-1})/\sqrt{2}$ with $X^+ = \tau$ we can solve the constraint for $\dot{X}^-$ and Legendre transform to the Hamiltonian to find

$$S = -T\int\sqrt{-G} \longrightarrow H = \int\left(\frac{1}{\rho T}P^a P^a + \frac{T}{2\rho}\{X^a, X^b\}^2\right)$$

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Hoppe then introduced matrix regularisation of the membrane and supermembrane, by treating the membrane as a phase space and quantising it. In this scheme functions on the membrane world-volume at fixed time, $f(\sigma^1, \sigma^2)$ are replaced by $N \times N$ matrices, $f \to F$, with the matrices providing a discrete approximation to the corresponding functions. This is very familiar for those familiar with fuzzy spaces.

The Hamiltonian

$$\mathrm{H} = -\frac{1}{2}\nabla^2 - \frac{1}{4}\sum_{i,j=1}^{p} \mathrm{Tr}[X^i, X^j]^2$$

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The Euclidean finite temperature action for the model is

$$S_b = \frac{1}{g^2} \int_0^\beta dt \, \mathrm{tr} \left\{ \frac{1}{2}(\mathcal{D}_t X^i)^2 - \frac{1}{4}[X^i, X^j]^2 \right\} \ .$$

where $\mathcal{D}_t X^i = \partial_t X^i + [A, X^i]$ and β, the period of the S^1, is the inverse temperature.

It is also the high temperature limit of $1+1$ dimensional $\mathcal{N}=8$ supersymmetric Yang-Mills on $\mathbf{R} \times S^1$ where β, the period of a spatial S^1 and now **not the inverse temperature**. The fermions drop out due to their anti-periodic boundary conditions at finite temperature.

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The Supermembrane

A supermembrane action can only be formulated in $d = 4, 5, 7$ and 11 dimensional space-times was applied

Matrix formulation

"On the Quantum Mechanics of Supermembranes," B. de Wit, J. Hoppe and H. Nicolai, Nucl. Phys. B **305** (1988) 545.

Demise

"The Supermembrane Is Unstable," B. de Wit, M. Luscher and H. Nicolai, Nucl. Phys. B **320** (1989) 135. and then "Supermembranes: A Fond Farewell?," B. de Wit and H. Nicolai, DESY-89-143.

Revival

"M theory as a matrix model: A Conjecture," T. Banks, W. Fischler, S. H. Shenker and L. Susskind, Phys. Rev. D **55** (1997) 5112 [hep-th/9610043].

The BFSS model

Matrix supermembranes propagating in $p+2$ dimensions coincide with $p+1$-dim $SU(N)$ Super Yang-Mills theory dimensionally reduced to 1-dim (only time dependence)

They can be formulated for $p = 2, 3, 5, 9$

The BFSS model is the $p = 9$ case; it also describes a system of N interacting D0 branes.

Hamiltonian Formulation

The 16 supercharges

$$Q_\beta = Tr(\frac{1}{2}\Theta_\alpha \gamma^a_{\alpha\beta} P_a + \frac{i}{4}\Theta_\alpha \gamma^{ab}_{\alpha\beta}[X_a, X_b])$$

$$\{Q_\alpha, Q_\beta\} = \delta_{\alpha\beta}\mathcal{H} + \gamma^a_{\alpha\beta} Tr(X^a J)$$

give the Hamiltonian:

$$\mathrm{H} = Tr(\frac{1}{2}P^a P^a - \frac{1}{4}[X^a, X^b]^2 + \frac{1}{2}\Theta^T \gamma^a [\Theta, X^a])$$

where J is the generator of $SU(N)$ and is zero on physical states

$$J = i[P^b, X^b] + \Theta_\alpha \Theta_\alpha - \delta_{\alpha\alpha}\frac{N^2 - 1}{2N}$$

The 16 fermionic matrices $\Theta_\alpha = \Theta_{\alpha A} t^A$ are quantised as

$$\Theta_{\alpha A}, \Theta_{\beta B} = 2\delta_{\alpha\beta}\delta_{AB}$$

The $\Theta_{\alpha A}$ are $2^{8(N^2-1)}$ and the Fermionic Hilbert space is

$$\mathcal{H}^F = \mathcal{H}_{256} \otimes \cdots \otimes \mathcal{H}_{256}$$

with $\mathcal{H}_{256} = \mathbf{44} \oplus \mathbf{84} \oplus \mathbf{128}$ suggestive of the graviton, anti-symmetric tensor and gravitino of $11-d$ SUGRA.

For an attempt to find the ground state see: J. Hoppe et al arXiv:0809.5270

Lagrangian formulation

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The easiest way to obtain the BFSS matrix model is via dimensional reduction of ten dimensional supersymmetric Yang-Mills theory down to one dimension. The resulting reduced ten dimensional action is given by

$$\begin{aligned} S_M &= \frac{1}{g^2}\int dt\, \mathrm{Tr}\left\{\frac{1}{2}(\mathcal{D}_0 X^i)^2 + \frac{1}{4}[X^i, X^j]^2 \right. \\ &\qquad \left. -\frac{i}{2}\Psi^T C_{10}\,\Gamma^0 D_0 \Psi + \frac{1}{2}\Psi^T C_{10}\,\Gamma^i [X^i, \Psi]\right\} , \end{aligned}$$

where Ψ is a thirty two component Majorana–Weyl spinor, Γ^μ are ten dimensional gamma matrices and C_{10} is the charge conjugation matrix satisfying $C_{10}\Gamma^\mu C_{10}^{-1} = -\Gamma^{\mu\,T}$.

The AdS/CFT dual geometry

Since the model describes the dynamics of $D0$ branes AdS/CFT gives predictions for the strong regime of the theory.

The bosonic action for eleven-dimensional supergravity is given by

$$S_{11D} = \frac{1}{2\kappa_{11}^2}\int[\sqrt{-g}R - \frac{1}{2}F_4 \wedge *F_4 - \frac{1}{6}A_3 \wedge F_4 \wedge F_4]$$

where $2\kappa_{11}^2 = 16\pi G_N^{11} = \frac{(2\pi l_p)^9}{2\pi}$.
The equations of motion of this system are

$$R_{MN} - \frac{1}{2}g_{MN}R = \frac{1}{2}F_{MN}^2 - \frac{1}{4}g_{MN}|F_4|^2 \qquad (1)$$

$$d * F_4 + \frac{1}{2}F_4 \wedge F_4 = 0, \qquad dF_4 = 0. \qquad (2)$$

Then dimensionally reducing an S^1 gives us IIA supergravity. The reduction involves

$$g^{11}_{MN} dx^M dx^N = \mathrm{e}^{-\frac{2}{3}\Phi} g^{10}_{mn} dx^m dx^n + \mathrm{e}^{\frac{4}{3}\Phi} (dx_{10} + C_m dx^m)^2 \qquad (3)$$

$$A_{10mn} dx^m \wedge dx^n = \frac{B_2}{2\pi R} \qquad A_{lmn} dx^l \wedge dx^m \wedge dx^n = C_3 \qquad (4)$$

and where the constant giving the string coupling has been removed from the dilaton. Then with $2\kappa_0^2 g_s^2 = \frac{2\kappa_{11}^2}{2\pi R}$ where R is the radius of the X_{10} circle on which the compactification is done one obtains the IIA supergravity action.

The leading $\alpha' = l_s^2$ "low" energy effective field theory on the dual gravity side is given by IIA supergravity the bosonic part of whose action is given in the string frame by

$$S_{IIA} = \frac{1}{2\kappa_0^2 g_s^2} \int d^{10}x\sqrt{-g}\{\mathrm{e}^{-2\Phi}[R+4|d\phi|^2 - \frac{1}{12}|H_3|^2 - \frac{1}{4}|G_2|^2 - \frac{1}{48}|G_4|^2]\}$$

where

$$H_3 = dB_2 \ , \qquad G_2 = dC_1 \ , \qquad G_4 = dC_3 + H_3 \wedge C_1$$

Eleven dimensional supergravity is the natural strong coupling limit of the IIA superstring. The fields (ϕ, g_{mn}, B_{mn}) are from the $NS \otimes NS$ sector of the IIA string while the fields (C_1, C_3) are from the $R \otimes R$ sector.

The relevant solution to eleven dimensional supergravity for the dual geometry to the BFSS model corresponds to N coincident $D0$ branes in the IIA theory. It is given by

$$ds^2 = -H^{-1}dt^2 + dr^2 + r^2 d\Omega_8^2 + H(dx_{10} - Cdt)^2$$

with $A_3 = 0$
The one-form is given by $C = H^{-1} - 1$ and $H = 1 + \frac{\alpha_0 N}{r^7}$ where $\alpha_0 = (2\pi)^2 14\pi g_s l_s^7$.

$$ds^2 = \alpha'(-\frac{F}{\sqrt{H}}dt^2 + \frac{\sqrt{H}}{F}dU^2 + \sqrt{H}U^2 d\Omega_8)$$

$H(U) = \frac{240\pi^5\lambda}{U^7}$ and the black hole time dilation factor $F(U) = 1 - \frac{U_0^7}{U^7}$ with $U_0 = 240\pi^5\alpha'^5\lambda$. The temperature

$$\frac{T}{\lambda^{1/3}} = \frac{1}{4\pi\lambda^{1/3}}H^{-1/2}F'(U_0) = \frac{7}{2^4 15^{1/2}\pi^{7/2}}(\frac{U_0}{\lambda^{1/3}})^{5/2}.$$

From black hole entropy to AdS prediction for the Energy

$$S = \frac{A}{4G_N} \sim \left(\frac{T}{\lambda^{1/3}}\right)^{9/2} \implies \frac{E}{\lambda N^2} \sim \left(\frac{T}{\lambda^{1/3}}\right)^{14/5}$$

The observables that we focus on

$$
\begin{aligned}
E/N^2 &= \left\langle -\frac{3}{4N\beta}\int_0^\beta dt\,\mathrm{Tr}\left([X^i,X^j]^2\right)\right\rangle \,, \\
\langle R^2\rangle &= \left\langle \frac{1}{N\beta}\int_0^\beta dt\,\mathrm{Tr}\left(X^i\right)^2\right\rangle \,, \\
<|P|> &= \langle|\frac{1}{N}\mathrm{Tr}U\,|\rangle, \\
U &\equiv \mathcal{P}\exp\left(i\int_0^\beta dt\,A_0(t)\right) \,.
\end{aligned}
$$

Non-perturbative study via lattice simulations

Discretise time to $t_n = an$, $(n = 0, \ldots, \Lambda - 1)$, $a = \beta/\Lambda$, and periodic boundary conditions $t_\Lambda = \Lambda a = \beta \equiv 0$.

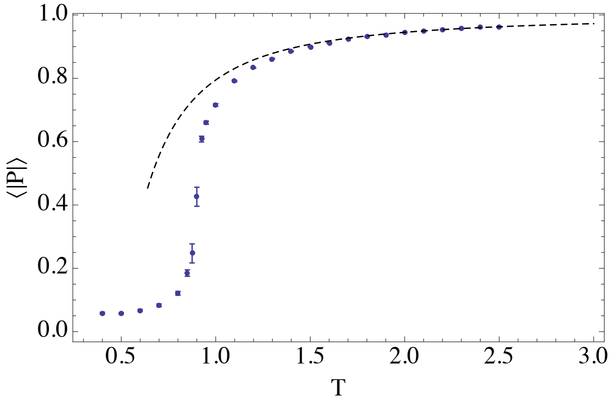
$$\mathcal{D}_t \to \frac{U_{n,n+1} X^i_{n+1} U_{n+1,n} - X^i_n}{a} \,, \qquad U_{n+1,n} = U^\dagger_{n,n+1}$$

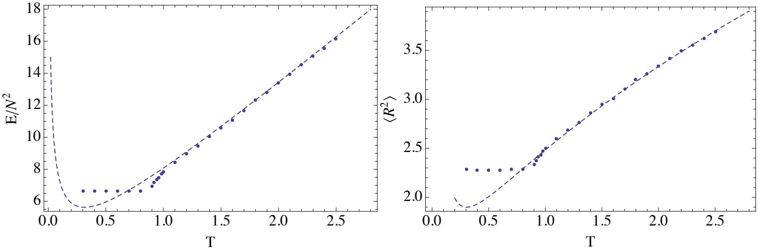
where $\quad U_{n,n+1} = \mathcal{P} \exp \left[i \int_{na}^{(n+1)a} dt\, A(t) \right] \quad$ parallel transport.

Discretised Goldstone Hoppe regulated bosonic membrane action:

$$S_b = N \sum_{n=0}^{\Lambda-1} \mathrm{tr} \left\{ -\frac{1}{a} X^i_n U_{n,n+1} X^i_{n+1} U^\dagger_{n,n+1} + \frac{1}{a} (X^i_n)^2 - \frac{a}{4} [X^i_n, X^j_n]^2 \right\} ,$$

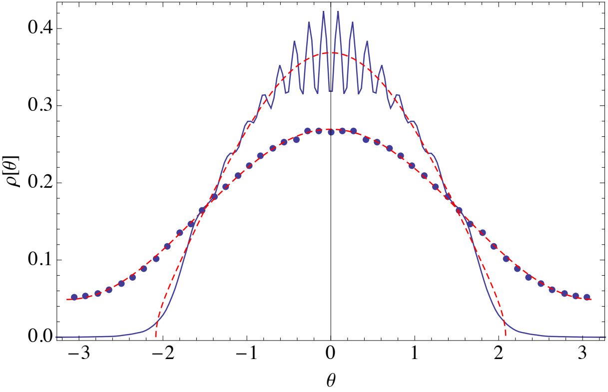
Polyakov loop





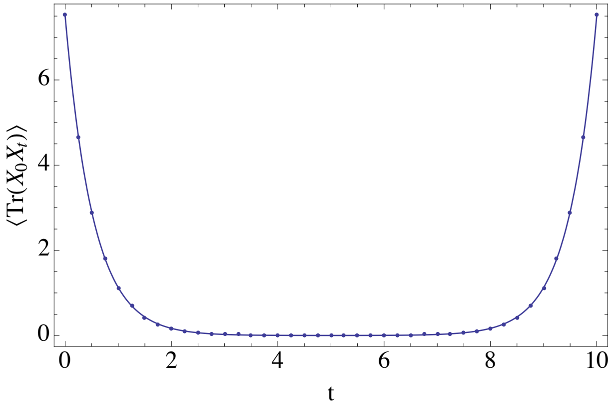
Plots of the scaled energy E/N^2 and the "extent of space" $\langle R^2 \rangle$ as functions of the temperature. The dashed curves correspond to the high temperature expansion. One can see that near $T \approx 0.9$ the plots suggest the existence of a second order phase transition. The energy and temperature in the plots are in units of $\lambda^{1/3}$.

The eigenvalue distribution of the holonomy



Plots of the distribution of the holonomy P for temperatures $T = 0.900$ (the gapped phase) and $T = .9006$ (the ungapped phase). The plots are for size $N = 16$ and lattice spacing $a \approx 0.05$. The dashed curves correspond to fits to the Gross-Witten gapped and untapped distributions.

Correlation function



The correlator $\left\langle \mathrm{Tr}\left(X^1(0)X^1(t)\right)\right\rangle$ for $N = 30$, $\beta = 10$ and lattice spacing $a = 0.25$. Fitting to $A\left(e^{-m\,t} + e^{-m(\beta - t)}\right)$
$\implies \; m = E_1 - E_0 \approx (1.90 \pm .01)\,\lambda^{1/3}$

The effective dynamics of the Bosonic membrane is given by the action

$$S_{\text{eff}} \approx N \int_{-\infty}^{\infty} dt \, \mathrm{Tr} \left(\frac{1}{2} \dot{X}^2 - \frac{1}{2} m^2 X^2 \right)$$

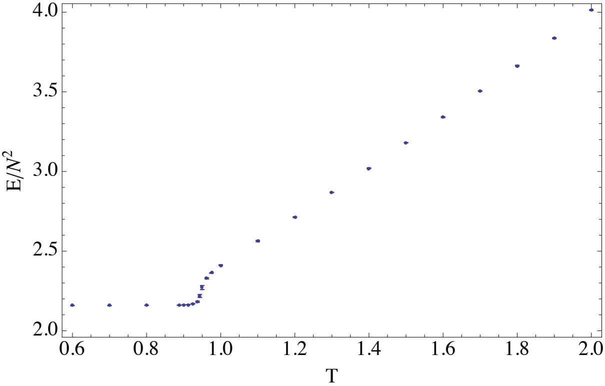
One can derive this using a large $1/p$ expansion which to leading order in large p gives the Euclidean finite temperature action

$$S_b = N \int_0^{\beta} dt \, \mathrm{Tr} \left(\frac{1}{2} (\mathcal{D}_t X)^2 + \frac{p^{2/3}}{2} X^2 \right)$$

This model can of course be solved analytically.

The gauged Gaussian model has a phase transition:

The energy for $N = 32$ and $a = 0.05$.



The gauge field that is responsible for the phase transition. At low temperatures the eigenvalues of A_0 are uniformly distributed but at high temperatures it becomes another matrix whose eigenvalues have a Wigner distribution.

A detailed $1/D$ analysis of the membrane model is given in G. Mandal, M. Mahato and T. Morita, JHEP **1002** (2010) 034 [arXiv:0910.4526]. Numerical studies were performed in N. Kawahara, J. Nishimura and S. Takeuchi, JHEP **0710** (2007) 097 [arXiv:0706.3517]
and refined in
T. Azuma, T. Morita and S. Takeuchi, Phys. Rev. Lett. **113** (2014) 091603 [arXiv:1403.7764 [hep-th]]

Comments

The bosonic relativistic membrane has a mass gap and at low temperatures is very well described by a system of oscillators.

The result is that the relativistic bosonic membrane has only Planck mass excitations!

Adding Fermions – Lattice implementation

the fermionic part of the action:

$$S_f = \frac{1}{2g^2} \int d\tau \, \mathrm{tr} \left\{ \psi^\alpha C_{9\,\alpha\beta} \, \mathcal{D}_\tau \psi^\beta - \psi^\alpha (C_9 \gamma^i)_{\alpha\beta} \, [X^i, \psi^\beta] \right\} \; . \quad (5)$$

We begin by splitting the fermions into two eight component fermions: $\psi = (\psi_1, \psi_2)$ and defining the forward and backward derivatives $D_\pm$:

$$\begin{aligned} (\mathcal{D}_- W)_n &= (W_n - U_{n,n-1} W_{n-1} U_{n-1,n})/a \; , \\ (\mathcal{D}_+ W)_n &= (U_{n,n+1} W_{n+1} U_{n+1,n} - W_n)/a \; . \end{aligned} \quad (6)$$

The discretised kinetic term takes the form:

$$
\begin{aligned}
S_f^{\rm kin} &= \frac{1}{2g^2}\int d\tau\, \mathrm{tr}\left(\psi^\alpha\, C_{9\,\alpha\beta}\, \mathcal{D}_\tau \psi^\beta\right) \\
&= \frac{a}{2g^2}\sum_{n=0}^{\Lambda-1} \mathrm{tr}\left\{\psi^T_{1,n}(\mathcal{D}_-\psi_2)_n + \psi^T_{2,n}(\mathcal{D}_+\psi_1)_n\right\} \\
&= \frac{1}{g^2}\mathrm{tr}\left\{-\sum_{n=0}^{\Lambda-1}\psi^T_{2,n}\psi_{1,n} + \sum_{n=0}^{\Lambda-2}\psi^T_{2,n}\, U_{n,n+1}\psi_{1,n+1}\, U_{n+1,n}\right. \\
&\qquad\qquad \left. \pm\psi^T_{2,\Lambda-1}\, U_{\Lambda-1,0}\psi_{1,0}\, U_{0,\Lambda-1}\right\} \quad .
\end{aligned}
$$

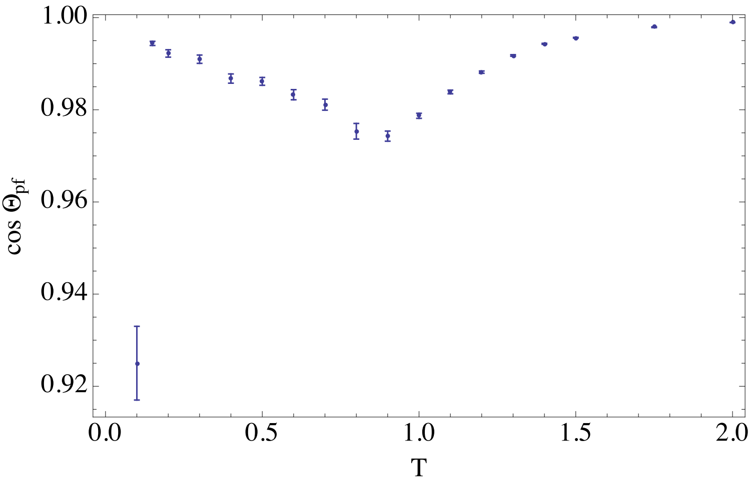
The $\pm$ gives periodic/anti-periodic boundary conditions.
In a static gauge the holonomy is concentrated on a singe link:

$$
S_f^{\rm kin} = \frac{1}{g^2}\mathrm{tr}\left\{-\sum_{n=0}^{\Lambda-1}\psi^T_{2,n}\psi_{1,n} + \sum_{n=0}^{\Lambda-2}\psi^T_{2,n}\psi_{1,n+1} \pm \psi^T_{2,\Lambda-1} D\,\psi_{1,0}\, D^\dagger\right\} .
$$

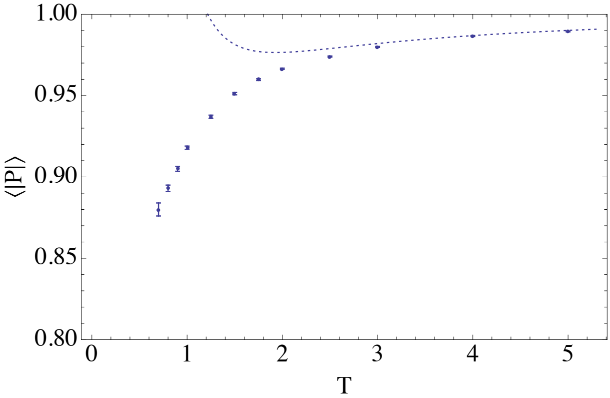
Discretised BFSS model:

$$
\begin{aligned}
S_{BFFS}^{Lattice} \quad = \quad N\,\mathrm{tr}\Bigg\{ & -\frac{1}{a}\sum_{n=0}^{\Lambda-2} X_n^i X_{n+1}^i - \frac{1}{a} X_{\Lambda-1}^i D\, X_0^i D^\dagger \\
& + \sum_{n=0}^{\Lambda-1}\left(\frac{1}{a}(X_n^i)^2 - \frac{a}{4}[X_n^i, X_n^j]^2\right) \\
& - \sum_{n=0}^{\Lambda-1}\psi_{2,n}^T\psi_{1,n} + \sum_{n=0}^{\Lambda-2}\psi_{2,n}^T\psi_{1,n+1} \pm \psi_{2,\Lambda-1}^T D\,\psi_{1,0}\, D^\dagger \\
& - a\sum_{n=0}^{\Lambda-1}\psi_{\alpha,n}(C_9\gamma^i)_{\alpha\beta}\,[X_n^i, \psi_{\beta,n}]\Bigg\}
\end{aligned}
$$

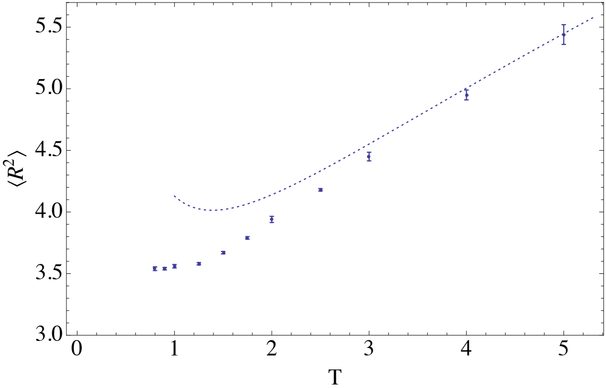
X_n^i traceless $N \times N$ Hermitian matrices,
$\psi_{\alpha,n}$ traceless $N \times N$ Hermitian matrices with Grassmann entries.
$D = \mathrm{diag}\{e^{i\theta_1}, \ldots, e^{i\theta_N}\}$,



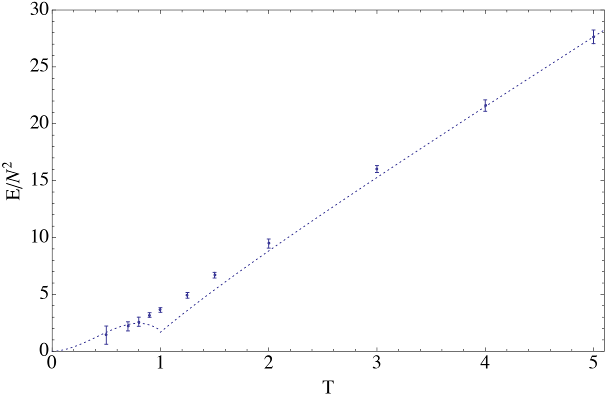
The pfaffian phase for $N = 3$ and $\Lambda = 4$. The phase remains small for all temperature, but drops at very low temperatures. We believe that this is a lattice effect and is not present in the continuum limit.



The average Polyakov loop $\langle |P| \rangle$.
All dashed curves represent the predictions of the high temperature expansion of N. Kawahara, J. Nishimura and S. Takeuchi, [arXiv:0710.2188]



$\langle R^2 \rangle = \left\langle \frac{1}{N\beta} \int_0^\beta dt \, \mathrm{Tr} \left(X^i \right)^2 \right\rangle$ which measures the extent of the eigenvalues of X^i.



The internal energy from simulations of $8 \leq N \leq 14$ and $8 \leq \Lambda \leq 16$.

The data seems to converge on the low temperature prediction of the AdS/CFT correspondence—especially when $1/\alpha'$ corrections are included.

Our results agrees with other groups:

S. Catterall and T. Wiseman, Phys. Rev. D **78** (2008) 041502 [arXiv:0803.4273 [hep-th]]

K. N. Anagnostopoulos, M. Hanada, J. Nishimura and S. Takeuchi, Phys. Rev. Lett. **100** (2008) 021601 [arXiv:0707.4454 [hep-th]].

D. Kadoh and S. Kamata, arXiv:1503.08499 [hep-lat].

V. G. Filev and D. O'Connor, arXiv:1506.01366 [hep-th].

The Berkooz Douglas model

M. Berkooz and M. R. Douglas, [hep-th/9610236].
M. Van Raamsdonk, [hep-th/0112081].
Describes $D0 - D4$ systems. The IKKT version describes a $D(-1) - D3$ system see M. Van Raamsdonk, [hep-th/0305145].
The more general framework involves $Dp - D(p+4)$ systems.

Add new bosonic degrees of freedom Φ_α as two complex $N \times N_f$ matrices

$$\mathcal{S}^{\mathrm{E}}_{\Phi} = \frac{1}{g^2}\int_0^\beta d\tau\,\mathrm{tr}\Big(D_\tau\bar{\Phi}^\rho D_\tau\Phi_\rho - \bar{\Phi}^\alpha(\sigma^A)_\alpha^{\ \beta}\,J^A_{ab}\,[X^a, X^b]\Phi_\beta$$

$$-\frac{1}{2}\bar{\Phi}^\alpha\Phi_\beta\bar{\Phi}^\beta\Phi_\alpha + \bar{\Phi}^\alpha\Phi_\alpha\bar{\Phi}^\beta\Phi_\beta\Big)\ .$$

J^A are $SU(2)$ generators

$$SO(4)\text{ generators}\quad (L_{ab})_{cd} = i(\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd})$$

$$J^A_{ab} = \frac{1}{2}(L_{A4})_{ab} + \frac{1}{4}\varepsilon^{ABC}(L_{BC})_{ab}\ ,\ K^A_{ab} = -\frac{1}{2}(L_{A4})_{ab} + \frac{1}{4}\varepsilon^{ABC}(L_{BC})_{ab}$$

$$S_\chi = \frac{1}{g^2}\int \mathrm{tr}\Big(i\chi^\dagger D_0\chi + \bar\chi\gamma^a X^a\chi$$
$$+\sqrt{2}\, i\,\varepsilon_{\alpha\beta}\,\bar\chi\lambda_\alpha\Phi_\beta - \sqrt{2}\, i\,\varepsilon_{\alpha\beta}\,\bar\Phi^\alpha\bar\lambda_\beta\chi\Big)\ .$$

where $\lambda_\alpha = P^\iota_\alpha\psi_\iota$.

The full model is

$$S_{BD} = S_{BFSS} + S_\Phi + S_\chi\,.$$

The lattice discretisation is again delicate but works and simulations are in progress.

Conclusions

- Bosonic membranes when quantised are massive $m \simeq p^{1/3} l_p$.
- Supersymmetric membranes are highly non-trivial with infra-red divergences. Appear to be consistent with AdS/CFT and the interpretation as a non-perturbative formulation of M-theory is promising.
- Keep tuned!

Thank you for your attention!