SOSpin

a C++ library for Yukawa decomposition in $SO(2N)$ models

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Outline

Introduction

How to break a SO(2N) family group?

$SO(10)$	$\rightarrow$	$SU(5)$	$\rightarrow$	SM
$Y\,16\,16\,10$	$\rightarrow$	$Y_u = Y_d = Y_e^T$	$\rightarrow$	m_u, m_d, m_e
Yukawa	$\rightarrow$	Yukawa	$\rightarrow$	masses

We perform the first task with our SO2pin Library:

- ✎ SO2pin stands for **S**(pecial) **O**(rthogonal) Spin
- ✎ Entirely written in C^{++} language [**c++11**]
- ✎ Its main role is **to decompose** $\mathrm{SO}(2N)$ Yukawa interactions in $\mathrm{SU}(N)$ basis
- ✎ We use the framework of annihilation and creation operators: b_i and $b_i^\dagger$
- ✎ Specific functions are included to address the SO(10)
- ✎ Further simplifications are done invoking FORM

SO$(2N)$ - Special Orthogonal

rotation group in $2N$-dimensional real space

- ✎ $2N \times 2N$ orthogonal matrices, R: $R^T R = R R^T = 1$ and $det(R) = +1$
- ✎ $2N(2N-1)/2$ imaginary antisymmetric generators
- ✎ rank N (i.e. N diagonal, simultaneously diagonalizable generators)
- ✎ Generators: $R(\omega) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}M_{\mu\nu}\right)$
- ✎ Algebra: $[M_{\mu\nu}, M_{\rho\eta}] = i\left(\delta_{\mu\eta}M_{\nu\rho} - \delta_{\mu\rho}M_{\nu\eta} - \delta_{\nu\eta}M_{\mu\rho} + \delta_{\nu\rho}M_{\mu\eta}\right)$

$$\mathrm{SO}(2N) \longrightarrow \begin{cases} \dots \\ \dots \\ \mathrm{SU}(N) \end{cases}$$

How to describe SO$(2N) \to$ SU(N)?

- oscillator technique [Mohapatra, Sakita,1980]
- other methods [He & Meljanac, 1990]

[Fukuyama et al, 2005]

SO($2N$): Repesentations

- ✎ **fundamental:** $\boxed{2N}$
- ✎ **adjoint:** $\boxed{2N(2N-1)/2}$
- ✎ **antisymmetric** of rank r^{th}: $\boxed{\binom{2N}{r}}$
- ✎ **symmetric** of rank r^{th}: $\boxed{\binom{2N+(r-1)}{r} - \binom{2N-(r-3)}{r-2}}$
- ✎ 2 different **spinorial:** $\boxed{2^{N-1}}$ & $\boxed{2^{N-1}}$

  N even: χ and χ' are real (self-conjugated)

  N odd: χ and $\overline{\chi}$ are complex (conjugate of each other)

The **antisymmetric** rep. of rank N is **splitted in two**

SO(2N) in an SU(N) basis

How to write the spinorial SO(2N) representation in terms of SU(N) states?

The Link: annihilation and creation operators: b_i and $b_i^\dagger$!

The description via wave functions is substituted by the use of creation, $b_i^\dagger$, and annihilation, b_i, operators:

$$\Psi \longrightarrow |\Psi\rangle = |0\rangle\, \psi \,+\, b_i^\dagger\, |0\rangle\, \psi^i \,+\, \frac{1}{2}\psi^{ij}\, b_i^\dagger b_j^\dagger\, |0\rangle \,+\, \cdots + \,\frac{1}{N!}\varepsilon^{ij\cdots k}\, \underbrace{b_i^\dagger b_j^\dagger \cdots b_k^\dagger}_{N}\, |0\rangle\, \bar{\psi}$$

Example: $\quad b_1^\dagger\,|0\rangle = |1\rangle \qquad b_1\,|1\rangle = |0\rangle \qquad b_1^\dagger\,|1\rangle = b_1\,|0\rangle = \vec{0}$

Grassmann Algebra:

$$\{b_i, b_j^\dagger\} = \delta_{ij} \qquad \text{and} \qquad \{b_i, b_j\} = 0 = \{b_i^\dagger, b_j^\dagger\}$$

$\{a, b\} \;\equiv\; a\, b \,+\, b\, a$

SO(2N) in an SU(N) basis

One can define,

$$\Gamma_{2j-1} = -i(b_j - b_j^\dagger) \qquad \Gamma_{2j} = (b_j + b_j^\dagger),$$

j=1,...,N

Γ_μ matrices for a Clifford algebra of rank 2N:

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}$$

The $2N(2N-1)/2$ generators of SO($2N$) are defined as

$$\Sigma_{\mu\nu} = \frac{1}{2i}\left[\Gamma_\mu, \Gamma_\nu\right]$$

Projection operator, $P_\pm$

$$P_\pm = \frac{1}{2}(1 \pm \Gamma_0) : \qquad \Psi_\pm = P_\pm \Psi$$

with $\boxed{\Gamma_0 = i^N \Gamma_1 \ldots \Gamma_{2N}}$

Pascal's triangle

$$\sum_{i=1}^{x} \binom{x}{i}$$

<u>Each line</u> of the Pascal's triangle:

1. The $\boxed{SO(x)}$ antisymmetric representations (the position rth gives the antisymmetric representation of rank r)
2. The irreducible representations (irreps) of $\boxed{SU(x)}$

	SO(10)	SU(5)
fundamental	10	5
rank 2	45	10
rank 3	120	$\overline{10}$
rank 4	210	$\overline{5}$
rank 5	$252=126+\overline{126}$	$\overline{1}$
spinorial	$16_{+}=1+\overline{5}+10$	–

N	
	1
	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
8	1 8 28 56 70 56 28 8 1
9	1 9 36 84 126 126 84 36 9 1
10	1 10 45 120 210 252 210 120 45 10 1
11	1 11 55 165 330 462 462 330 165 55 11 1
12	1 12 66 220 495 792 924 792 495 220 66 12 1

SO(2N) in an SU(N) basis

SO(2N)	SU(N)	
$\|0\rangle$	1	ψ
$b_j^\dagger \|0\rangle$	N	ψ^j
$b_j^\dagger b_k^\dagger \|0\rangle$	$\binom{N}{2}$	ψ^{ij}
...	...	...
$b_1^\dagger ... b_N^\dagger \|0\rangle$	$\binom{N}{N}$	$\psi^{i_1 \cdots i_N}$
	dim 2^N	

SO(2N)

For N odd $$|\Psi_+\rangle = |0\rangle\ \psi + \frac{1}{2!} b_i^\dagger b_j^\dagger\ |0\rangle\ \psi^{ij} + ... + \frac{1}{(N-1)!}\ b_{i_1}^\dagger ... b_{N-1}^\dagger\ |0\rangle\ \psi^{i_1 \cdots i_{N-1}}$$

$$|\Psi_-\rangle = b_i^\dagger\ |0\rangle\ \psi + \frac{1}{3!} b_i^\dagger b_j^\dagger b_k^\dagger\ |0\rangle\ \psi^{ijk} + ... + \frac{1}{N!}\ b_{i_1}^\dagger ... b_N^\dagger\ |0\rangle\ \psi^{i_1 \cdots i_N}$$

For N even $$|\Psi_+\rangle = |0\rangle\ \psi + \frac{1}{2!} b_i^\dagger b_j^\dagger\ |0\rangle\ \psi^{ij} + ... + \frac{1}{N!}\ b_{i_1}^\dagger ... b_N^\dagger\ |0\rangle\ \psi^{i_1 \cdots i_N}$$

$$|\Psi_-\rangle = b_i^\dagger\ |0\rangle\ \psi + \frac{1}{3!} b_i^\dagger b_j^\dagger b_k^\dagger\ |0\rangle\ \psi^{ijk} + ... + b_{i_1}^\dagger ... b_{N-1}^\dagger\ |0\rangle\ \frac{\psi^{i_1 \cdots i_{N-1}}}{(N-1)!}$$

Invariants in SO(2N)

✎ $\boxed{\Psi \longrightarrow \Psi^* \quad : \quad |\Psi\rangle \longrightarrow |\Psi^*\rangle}$

$$|\Psi\rangle = |0\rangle\,\psi + b_i^\dagger\,|0\rangle\,\psi^i + \frac{1}{2} b_i^\dagger b_j^\dagger\,|0\rangle\,\psi^{ij} + \frac{1}{12}\varepsilon^{ijklm} b_k^\dagger b_l^\dagger b_m^\dagger\,|0\rangle\,\psi_{ij} + \dots$$

$$|\Psi^*\rangle = |0\rangle\,\psi^* + b_i^\dagger\,|0\rangle\,\psi_i + \frac{1}{2} b_i^\dagger b_j^\dagger\,|0\rangle\,\psi_{ij} + \frac{1}{12}\varepsilon_{ijklm} b_k^\dagger b_l^\dagger b_m^\dagger\,|0\rangle\,\psi^{ij} + \dots$$

$$\left(\psi^{ij}\right)^* = \psi_{ij}$$

✎ $\boxed{\Psi \longrightarrow \Psi^\dagger \quad : \quad |\Psi\rangle \longrightarrow \langle\Psi|}$

$$|\Psi\rangle = |0\rangle\,\psi + b_i^\dagger\,|0\rangle\,\psi^i + \frac{1}{2} b_i^\dagger b_j^\dagger\,|0\rangle\,\psi^{ij} + \frac{1}{12}\varepsilon^{ijklm} b_k^\dagger b_l^\dagger b_m^\dagger\,|0\rangle\,\psi_{ij} + \dots$$

$$\langle\Psi| = \psi^*\,\langle 0| + \psi_i\,\langle 0|\,b_i + \frac{1}{2}\psi_{ij}\,\langle 0|\,b_j b_i + \frac{1}{12}\,\psi^{ij}\varepsilon_{ijklm}\,\langle 0|\,b_m b_l b_k + \dots$$

✎ $\boxed{\Psi \longrightarrow \Psi^T \quad : \quad |\Psi\rangle \longrightarrow \langle\Psi^*|}$

$$|\Psi\rangle = |0\rangle\,\psi + b_i^\dagger\,|0\rangle\,\psi^i + \frac{1}{2} b_i^\dagger b_j^\dagger\,|0\rangle\,\psi^{ij} + \frac{1}{12}\varepsilon^{ijklm} b_k^\dagger b_l^\dagger b_m^\dagger\,|0\rangle\,\psi_{ij} + \dots$$

$$\langle\Psi^*| = \psi\,\langle 0| + \psi^i\,\langle 0|\,b_i + \frac{1}{2}\psi^{ij}\,\langle 0|\,b_j b_i + \frac{1}{12}\psi_{ij}\varepsilon^{ijklm}\,\langle 0|\,b_m\,b_l b_k + \dots$$

SO(2N) in an SU(N) basis

$$Y_{ab}\ \langle \Psi^*_{k\,a} |\, B\, C^{-1}\, \Gamma_{[\mu_1} \Gamma_{\mu_2} \ldots \Gamma_{\mu_m]}\ | \Psi_{l\,b} \rangle\ \phi_{\mu_1 \mu_2 \cdots \mu_m}$$

- a, b flavour indices
- C: Dirac charge conjugation matrix
- $\Gamma_{[\mu} \Gamma_\nu \Gamma_\rho \cdots \Gamma_{\lambda]} = \frac{1}{m!} \sum_P (-1)^{\delta P} \Gamma_{\mu_{P(1)}} \Gamma_{\nu_{P(2)}} \Gamma_{\rho_{P(3)}} \cdots \Gamma_{\lambda_{P(m)}}$
  $\delta P = 0$ for even permutations and $\delta P = 1$ for odd permutations
- Operator B (charge conjugation matrix):

$$B = \prod_{\mu = odd} \Gamma_\mu = -i \prod_{k=1}^{N} (b_k - b_k^\dagger)$$

where for N odd $\{B, \Gamma_0\} = 0$ and for N even $[B, \Gamma_0] = 0$
Thus, in the case $N + m$ is an odd number

$$\langle \Psi^*_a |\, B\, C^{-1}\, \Gamma_{\mu_1} \Gamma_{\mu_2} \cdots \Gamma_{\mu_m}\ | \Psi_a \rangle = 0$$

while in the case $N + m$ is an even number

$$\langle \Psi^*_a |\, B\, C^{-1}\, \Gamma_{\mu_1} \Gamma_{\mu_2} \cdots \Gamma_{\mu_m}\ \left| \Psi_{b \neq a} \right\rangle = 0$$

SO(2N) in an SU(N) basis

"The Basic Theorem"

[Nath, Syed, 2001]

[Nath, Syed, Nucl.Phys.B618, 2001]

$$\begin{aligned}
\Gamma_\mu \Gamma_\nu \Gamma_\lambda ... \Gamma_\sigma \phi_{\mu\nu\lambda...\sigma} &= b_i^\dagger b_j^\dagger b_k^\dagger ... b_n^\dagger \phi_{c_i c_j c_k ... c_n} \\
&+ (b_i b_j^\dagger b_k^\dagger ... b_n^\dagger \phi_{\bar{c}_i c_j c_k ... c_n} + \text{perms.}) \\
&+ (b_i b_j b_k^\dagger ... b_n^\dagger \phi_{\bar{c}_i \bar{c}_j c_k ... c_n} + \text{perms.}) \\
&+ ... \\
&+ (b_i b_j b_k ... b_{n-1} b_n^\dagger \phi_{\bar{c}_i \bar{c}_j \bar{c}_k ... \bar{c}_{n-1} c_n} + \text{perms.}) \\
&+ (b_i b_j b_k ... b_n \phi_{\bar{c}_i \bar{c}_j \bar{c}_k ... \bar{c}_n}),
\end{aligned}$$

$$\phi_{c_i} = \phi_{2i} + i\phi_{2i-1} \qquad \phi_{\bar{c}_i} = \phi_{2i} - i\phi_{2i-1}$$

SO(10) in SU(5) basis

e.g. of gauge-invariant Yukawa coupling: $\langle \Psi^* | \; B\, C^{-1}\, \Gamma_\mu \; |\Psi\rangle \; \phi_\mu$

$$\Psi \to |\Psi\rangle = |0\rangle\, \psi \; + \; b_i^\dagger\, |0\rangle\, \psi^i \; + \; \frac{1}{2}\psi^{ij}\, b_i^\dagger b_j^\dagger\, |0\rangle \; + \; \frac{1}{6}\psi^{ijk}\, b_i^\dagger b_j^\dagger b_k^\dagger\, |0\rangle$$

$$+ \; \frac{1}{24}\psi^{ijkl}\, b_i^\dagger b_j^\dagger b_k^\dagger b_l^\dagger\, |0\rangle + \; \frac{1}{120}\psi^{ijklm}\, b_i^\dagger b_j^\dagger b_k^\dagger b_l^\dagger b_m^\dagger\, |0\rangle$$

$$\psi^{ijkl} = \frac{1}{1!}\varepsilon^{ijklm}\psi_m\,, \qquad \psi^{ijk} = \frac{1}{2!}\varepsilon^{ijklm}\psi_{lm}\,, \qquad \psi^{ijklm} = \frac{1}{0!}\varepsilon^{ijklm}\psi'$$

and

$$\varepsilon^{ijklm} b_i^\dagger b_j^\dagger b_k^\dagger b_l^\dagger b_m^\dagger \psi' = 5!\, b_1^\dagger b_2^\dagger b_3^\dagger b_4^\dagger b_5^\dagger \psi'$$

$\boxed{\Psi = \Psi_+ + \Psi_-}$ $\quad \Psi_+ \sim 16$ and $\Psi_- \sim \overline{16}$

$$\Psi_+ \to |\Psi_+\rangle \; = \; |0\rangle\; \psi \; + \; \frac{1}{2} b_i^\dagger b_j^\dagger\, |0\rangle\; \psi^{ij} \; + \; \frac{1}{24}\varepsilon^{ijklm}\, b_i^\dagger b_j^\dagger b_k^\dagger b_l^\dagger\, |0\rangle\; \overline{\psi}_m$$

$$\Psi_- \to |\Psi_-\rangle \; = \; b_i^\dagger\, |0\rangle\; \psi^i \; + \; \frac{1}{12}\varepsilon^{ijklm}\, b_k^\dagger b_l^\dagger b_m^\dagger\, |0\rangle\; \overline{\psi}_{ij} \; + \; b_1^\dagger b_2^\dagger b_3^\dagger b_4^\dagger b_5^\dagger\, |0\rangle\; \overline{\psi}$$

SO(10) in an SU(5) basis

✎ Action of B in ⟨bra| and |ket⟩:

$$B\,|\Psi_+\rangle = i\,b_1^\dagger b_2^\dagger b_3^\dagger b_4^\dagger b_5^\dagger\,|0\rangle\,\psi - i\,\frac{1}{12}\varepsilon^{ijklm} b_k^\dagger b_l^\dagger b_m^\dagger\,|0\rangle\,\psi^{ij} + i b_i^\dagger\,|0\rangle\,\overline{\psi}_i$$

$$B\,|\Psi_-\rangle = -\frac{i}{24}\varepsilon^{ijklm} b_j^\dagger b_k^\dagger b_l^\dagger b_m^\dagger\,|0\rangle\,\psi^i + \frac{i}{2} b_i^\dagger b_j^\dagger |0\,|0\rangle\,\overline{\psi}_{ij} - i\,|0\rangle\,\overline{\psi}$$

and

$$\left\langle \Psi_+^* \right| B \;=\; -i\,\psi\;\langle 0|\;b_1 b_2 b_3 b_4 b_5 \;-\; \frac{i}{12}\varepsilon^{ijklm}\psi_{ij}\,\langle 0|\;b_k b_l b_m - i\,\overline{\psi}_i\,\langle 0|\;b_i$$

$$\left\langle \Psi_-^* \right|\,B \;=\; \frac{i}{24}\varepsilon^{ijklm}\psi^i\,\langle 0|\;b_j b_k b_l b_m \;+\; \frac{i}{2}\overline{\psi}_{ij}\,\langle 0|\;b_i b_j \;+\; i\,\overline{\psi}\,\langle 0|$$

Antisymmetric representations:

$$\begin{aligned}
&dim(\phi_\mu) = 10\,,\\
&dim(\phi_{\mu\nu}) = 45\,,\\
&dim(\phi_{\mu\nu\lambda}) = 120\,,\\
&dim(\phi_{\mu\nu\lambda\sigma}) = 210\,,\\
&dim(\phi_{\mu\nu\lambda\sigma\gamma}) = 252 = 126\oplus\overline{126}\,,
\end{aligned}$$

SO(10) in SU(5) basis

The 120 field representation $\phi_{\mu\nu\lambda}$:

$$\Gamma_\mu \Gamma_\nu \Gamma_\lambda \phi_{\mu\nu\lambda} = b_i b_j b_k \phi_{\bar{c}_i \bar{c}_j \bar{c}_k} + b_i^\dagger b_j^\dagger b_k^\dagger \phi_{c_i c_j c_k} + 3 \left(b_i^\dagger b_j b_k \phi_{c_i \bar{c}_j \bar{c}_k} + b_i^\dagger b_j^\dagger b_k \phi_{c_i c_j \bar{c}_k} \right) + \left(3 b_i \phi_{\bar{c}_n c_n \bar{c}_i} + 3 b_i^\dagger \phi_{\bar{c}_n c_n c_i} \right),$$

where

$$\phi_{c_i c_j \bar{c}_k} = f_k^{ij} + \tfrac{1}{4} \left(\delta_k^i f^j - \delta_k^j f^i \right), \qquad \phi_{c_i \bar{c}_j \bar{c}_k} = f_{jk}^i - \tfrac{1}{4} \left(\delta_j^i f_k - \delta_k^i f_j \right),$$
$$\phi_{c_i c_j c_k} = \varepsilon^{ijklm} f_{lm}, \qquad \phi_{\bar{c}_i \bar{c}_j \bar{c}_k} = \varepsilon_{ijklm} f^{lm},$$
$$\phi_{\bar{c}_n c_n c_i} = f^i, \qquad \phi_{\bar{c}_n c_n \bar{c}_i} = -f_i,$$

SU(5) normalization:

$$f^i = \frac{4}{\sqrt{3}} h^i, \qquad f_i = \frac{4}{\sqrt{3}} h_i,$$
$$f^{ij} = \frac{1}{\sqrt{3}} h^{ij}, \qquad f_{ij} = \frac{1}{\sqrt{3}} h_{ij},$$
$$f_k^{ij} = \frac{2}{\sqrt{3}} h_k^{ij}, \qquad f_{jk}^i = \frac{2}{\sqrt{3}} h_{jk}^i,$$

Generic example:

$$\psi_{ij}^{k}\ \langle 0|\ b_i\, b_j\, b_k^\dagger\, b_l\, b_m^\dagger\, b_n^\dagger\, |0\rangle\ \phi_l\, \psi^{mn} \longrightarrow \psi_{ij}^{k}\, \phi_l\, \psi^{mn}\ \boxed{\langle 0|\ b_i\, b_j\, b_k^\dagger\, b_l\, b_m^\dagger\, b_n^\dagger\, |0\rangle}$$

Normal ordering

$$\langle 0|\ b_i\, b_j\, b_k^\dagger\, b_l\, b_m^\dagger\, b_n^\dagger\, |0\rangle =$$

$$= \langle 0|\ b_i\, b_j\, \left(\delta_{kl} - b_l\, b_k^\dagger\right) b_m^\dagger\, b_n^\dagger\, |0\rangle$$

$$= \delta_{kl}\, \langle 0|\ b_i\, b_j\, b_m^\dagger\, b_n^\dagger\, |0\rangle - \langle 0|\ b_i\, b_j\, b_l\, b_k^\dagger\, b_m^\dagger\, b_n^\dagger\, |0\rangle$$

$\downarrow$ in SU(5) $\downarrow$

$$\frac{1}{3!}\delta_{kl}\, \varepsilon_{ij\alpha\beta\gamma}\, \varepsilon_{nm\alpha\beta\gamma} \qquad \frac{1}{2!}\varepsilon_{ijl\alpha\beta}\, \varepsilon_{nmk\alpha\beta}$$

Reverse ordering

$$\langle 0|\ b_i\, b_j\, b_k^\dagger\ b_l\, b_m^\dagger\, b_n^\dagger\, |0\rangle =$$

$$= \langle 0|\ b_i\, b_j\, b_k^\dagger\, \left(\delta_{lm} - b_m^\dagger b_l\right)\, b_n^\dagger\, |0\rangle$$

$$= \delta_{lm}\, \langle 0|\ b_i\, b_j\, b_k^\dagger\, b_n^\dagger\, |0\rangle - \langle 0|\ b_i\, b_j\, b_k^\dagger\, b_m^\dagger b_l b_n^\dagger\, |0\rangle$$

$$= \delta_{lm}\, \langle 0|\ b_i\, b_j\, b_k^\dagger\, b_n^\dagger\, |0\rangle$$

$$- \langle 0|\ b_i\, b_j\, b_k^\dagger\, b_m^\dagger (\delta_{ln} - \underbrace{b_n^\dagger\, b_l}_{0})\, |0\rangle$$

$$= \delta_{lm}\, \langle 0|\ b_i\, b_j\, b_k^\dagger\, b_n^\dagger\, |0\rangle - \delta_{ln}\, \langle 0|\ b_i\, b_j\, b_k^\dagger\, b_m^\dagger\, |0\rangle$$

$\downarrow$ in SU(5) $\downarrow$

$$\frac{1}{3!}\delta_{lm}\, \varepsilon_{ij\alpha\beta\gamma}\, \varepsilon_{nk\alpha\beta\gamma} \qquad \frac{1}{3!}\delta_{ln}\, \varepsilon_{ij\alpha\beta\gamma}\, \varepsilon_{mk\alpha\beta\gamma}$$

Summary

In SU(5)

$$\langle 0|\, b_i\, b_j^\dagger\, |0\rangle = \frac{1}{24}\varepsilon_{i\alpha\beta\gamma\delta}\,\varepsilon_{j\alpha\beta\gamma\delta}$$

$$\langle 0|\, b_i\, b_j\, b_k^\dagger\, b_l^\dagger\, |0\rangle = \frac{1}{6}\varepsilon_{ij\alpha\beta\gamma}\,\varepsilon_{lk\alpha\beta\gamma}$$

$$\langle 0|\, b_i\, b_j\, b_k\, b_l^\dagger\, b_m^\dagger\, b_n^\dagger\, |0\rangle = \frac{1}{2}\varepsilon_{ijk\alpha\beta}\,\varepsilon_{nml\alpha\beta}$$

$$\langle 0|\, b_i\, b_j\, b_k\, b_l\, b_m^\dagger\, b_n^\dagger\, b_o^\dagger\, b_p^\dagger\, |0\rangle = \varepsilon_{ijkl\alpha}\,\varepsilon_{ponm\alpha}$$

$$\langle 0|\, b_i\, b_j\, b_k\, b_l\, b_m\, b_n^\dagger\, b_o^\dagger\, b_p^\dagger\, b_q^\dagger\, b_r^\dagger\, |0\rangle = \varepsilon_{ijklm}\,\varepsilon_{rqpon}$$

$$\langle 0|\, b_1\, b_2\, b_3\, b_4\, b_5\, b_n^\dagger\, b_o^\dagger\, b_p^\dagger\, b_q^\dagger\, b_r^\dagger\, |0\rangle = \varepsilon_{rqpon}$$

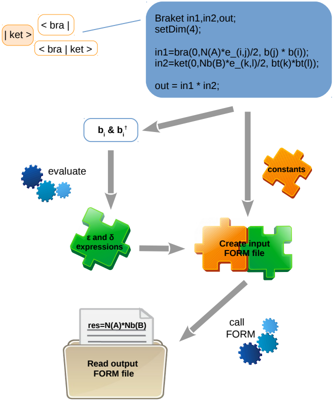
Rules

A complete expression of $\langle 0| \ldots |0\rangle$ must have :

- ✎ equal number of upper and lower indices
- ✎ the difference between the number of upper and lower indices must be zero or N for $SO(2N)$
- ✎ contracted indices (equal upper and lower indices) do not account
- ✎ the number of creation and annihilation operator in each term must be equal
- ✎ the number of contiguous creation (or annihilation) operators must be equal or less than N of $SO(2N)$
- ✎ The b_i operator is written on the left and $b_i^\dagger$ on the right (normal ordering) otherwise the result is zero

SO2pin: the scheme

- Provided under the terms of the GNU Lesser General Public License (Free Software Foundation)
- Hosted by the Hepforge: http://sospin.hepforge.org



SOSpin: Data structures

Data structures used to store and evaluate expressions should:

- Optimised memory usage - since expressions are extremely long
- Optimised flexibility of permutations - adjacency in memory is not relevant
- Standard description of all elements - to ease interpretation and evaluation and to reduce memory waste in contraction and expansion operations
- ...

Solution:

- Linked lists

Implemented operations:

- in multiplications:
  - $\langle \text{bra}| \cdot |\text{ket}\rangle$
  - $\langle \text{bra}| \cdot \text{free}$
  - $\text{free} \cdot |\text{ket}\rangle$
  - $\text{free} \cdot \text{free}$
- in sums:
  - $\langle \text{bra}| + \langle \text{bra}|$
  - $|\text{ket}\rangle + |\text{ket}\rangle$
  - $\text{free} + \text{free}$
  - $\langle \text{bra} \,|\, \text{ket}\rangle + \langle \text{bra} \,|\, \text{ket}\rangle$

SOSpin: Elements of the Linked List

Each element can be of one of the following types:

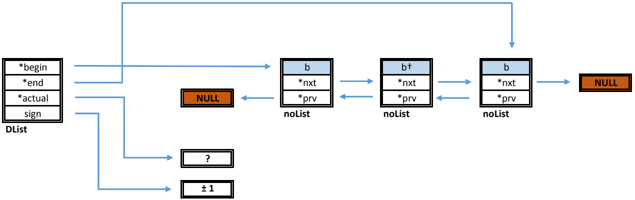
- type 0 - b
- type 1 - $b^\dagger$
- type 2 - δ
- type 3 - constant

# bits	3	1	10	10	8
elemtype	type	sig.	id.x	id.y	free bits

The internal organisation of each elements (32 bits) is as follows:

- Type - 3 bits (it allows for future expansion with 4 additional element types)
- Signal - 1 bit
- id.x - 10 bits (allow values between 0 and 1023)
- id.y - 10 bits (allow values between 0 and 1023)
- Free bits - 8 bits (for future expansion)

SO2pin: class DList - the linked list



Each node (`struct noList`) is composed of:

- `elemtype data` - element itself
- `noList *prv` - pointer to the previous node in list
- `noList *nxt` - pointer to the next node in list

The linked list is then composed of none, one or several nodes connected by the following pointers structure:

- `noList *beg` - pointer to the first node in list
- `noList *end` - pointer to the last node in list
- `noList *act` - pointer to any node in list - used for list manipulation and scanning

This linked list is implemented by the `class DList`.

SOSpin: How to declare ...

a field

FieldNames#$^{\text{UpperIndices}}$#$_{\text{LowerIndices}}$(FlavorIndex,UpperIndices,LowerIndices)

- ✎ M^{ijk}_{lm} with flavor index A: M32(A,i,j,k,l,m)

  ```
  Field(M,3,2,ASYM_WITH_FLAVOR);
  ```

- ✎ N with flavor index A: N(A)

  ```
  Field(N,0,0,ASYM_WITH_FLAVOR);
  ```

- ✎ H^{ij} symmetric and no flavor: Hs20(i,j)

  ```
  Field(H,2,0,SYM);
  ```

- ✎ M_{ij} symmetric and flavor index B: Ms02(B,i,j)

  ```
  Field(M,0,2,SYM_WITH_FLAVOR);
  ```

How to declare ...

Bra $\langle 0|\, \frac{1}{24}\varepsilon^{opqrs}\overline{M}_{oA}\; b_p b_q b_r b_s$

```
Braket exp = bra(4, 1/24 * e_(o,p,q,r,s) * Mb01(A,o), b(p) * b(q) * b(r) * b(s));
Field(Mb,0,1,SYM_WITH_FLAVOR);
```

Ket $\frac{1}{120}\varepsilon_{jklmn}\overline{M}_B\; b_j^\dagger b_k^\dagger b_l^\dagger b_m^\dagger b_n^\dagger\,|0\rangle$

```
Braket exp = ket(5,1/120 * e_(j,k,l,m,n) * Mb(B), bt(j) * bt(k) * bt(l) * bt(m) * bt(n) );
Field(Mb,0,0,SYM_WITH_FLAVOR);
```

free operators: $i\; b_j^\dagger b_k^\dagger b_l^\dagger b_m^\dagger b_n^\dagger$

```
Braket exp = free(0,i, bt(j) * bt(k) * bt(l) * bt(m) * bt(n) );
```

How to declare ...

Braket $\overline{M}_{iA} \langle 0| \, b_i \, b_n^\dagger \, |0\rangle \, M_B \, H^n$

```
Braket exp = braket(0, Mb01(A,i) * M(B) * H10(n), b(i) * bt(n) );
Field(Mb,0,1,SYM_WITH_FLAVOR);
Field(M,0,0,SYM_WITH_FLAVOR);
Field(H,1,0,ASYM);
```

Example

$$exp0 = \langle 0| \, \frac{1}{24} \varepsilon^{opqrs} \overline{M}_o \, b_p b_q b_r b_s \qquad exp1 = \frac{1}{120} \varepsilon_{jklmn} \overline{M} \, b_j^\dagger b_k^\dagger b_l^\dagger b_m^\dagger b_n^\dagger \, |0\rangle$$

```
Braket exp0 = bra(4, 1/24 * e_(o,p,q,r,s) * Mb01(A,o), b(p) * b(q) * b(r) * b(s));
Braket exp1 = ket(5,1/120 * e_(j,k,l,m,n) * Mb(B), bt(j) * bt(k) * bt(l) * bt(m) * bt(n) );
Field(Mb,0,1,ASYM_WITH_FLAVOR);
Field(Mb,0,0,ASYM_WITH_FLAVOR);

newId("A");newId("B");newId("o");
Braket product = exp0 * exp1;
```

An example in SO(4)

$$|\Psi\rangle \;=\; |0\rangle\, M \;+\; b_i^\dagger\, |0\rangle\, N^i \;+\; \frac{1}{2}\varepsilon^{ij} b_i^\dagger b_j^\dagger\, |0\rangle\, \overline{M}$$

In $SU(2)$: $\quad M, \overline{M} \rightsquigarrow 1 \quad$ and $\quad N^i \rightsquigarrow 2$

$$|\psi\rangle \;=\; |\psi_1\rangle + |\psi_2\rangle$$

where

$$|\psi_1\rangle \;=\; M\,|0\rangle \;+\; b_1^\dagger b_2^\dagger\, |0\rangle\, \overline{M} \quad \text{and} \quad |\psi_2\rangle \;=\; b_i^\dagger N^i\, |0\rangle$$

The Clifford Algebra:

$$\Gamma_1 \;=\; -i(b_1 - b_1^\dagger), \quad \Gamma_2 \;=\; b_1 + b_1^\dagger$$
$$\Gamma_3 \;=\; -i(b_2 - b_2^\dagger), \quad \Gamma_4 \;=\; b_2 + b_2^\dagger$$

Charge conjugation operator:

$$B \;=\; -i(b_1 - b_1^\dagger)(b_2 - b_2^\dagger) \quad \text{and} \quad [B, \Gamma_0] \;=\; 0$$

$$iB\,|\psi_1\rangle \;=\; \frac{1}{2}\varepsilon^{ij} b_i^\dagger b_j^\dagger\, |0\rangle\, M \;+\; |0\rangle\, \overline{M} \quad \text{and} \quad iB\,|\psi_2\rangle \;=\; \varepsilon^{ij} b_i^\dagger\, |0\rangle\, N^j$$

An example in SO(4)

Computing $\langle \psi^*_{1a} \,|\, B\Gamma_\mu \,|\, \psi_{2b} \rangle \langle \psi^*_{1c} \,|\, B\Gamma_\mu \,|\, \psi_{2d} \rangle$ in SO(4)

```cpp
#include <sospin/son.h>
using namespace sospin;

int main(int argc, char *argv[]){

  setDim(4);

  Braket L1, R1, L2, R2;
  Braket in1E, in1O, in2E, in2O, res;

  L1 = bra(0,M(a),identity);
  L1+= bra(0,Mb(a)*e_(i,j)/2,b(j)*b(i));
  R1 = ket(0,N10(b,k),bt(k));

  L2 = bra(0,M(c),identity);
  L2+= bra(0,Mb(c)*e_(l,m)/2,b(m)*b(l));
  R2 = ket(0,N10(d,o),bt(o));

  Field(M,  0, 0, ASYM_WITH_FLAVOR);
  Field(Mb, 0, 0, ASYM_WITH_FLAVOR);
  Field(N,  1, 0, ASYM_WITH_FLAVOR);

  newId("a");  newId("b");
  newId("c");  newId("d");

  in1E = L1 * Bop("i") * G(true, "j") * R1;
  in2E = L2 * Bop("k") * G(true, "j") * R2;
  in1O = L1 * Bop("i") * G(false,"j") * R1;
  in2O = L2 * Bop("k") * G(false,"j") * R2;

  in1E.evaluate();
  in2E.evaluate();
  in1O.evaluate();
  in2O.evaluate();

  res = in1E * in2E + in1O * in2O ;

  unsetFormIndexSum();
  CallForm(res,false, true, "i");

  res.setON();
  std::cout << "Output result:\n" << res << std::endl;

  CleanGlobalDecl();

  exit(0);
}
```

```
Output result:

    Local R1 = +2*M(a)*N10(b,i)*Mb(c)*N10(d,j)*e_(i,j);
    Local R2 = -2*Mb(a)*N10(b,i)*M(c)*N10(d,j)*e_(i,j);
```

The results are:

$$\langle \psi^*_{1a} \,|\, B \,|\, \psi_{2b} \rangle = \langle \psi^*_{2} \,|\, B \,|\, \psi_{1} \rangle = 0\,,$$

$$\langle \psi^*_{2a} \,|\, B \,|\, \psi_{2b} \rangle = \varepsilon^{ij} N^i_a N'^j_b\,.$$

Example in SO(10))

[Nath, Syed, Nucl.Phys.B618, 2001]

Fermion masses coming from $16_a\, 16_b\, 120_H$:
For the 120 field representation one has,

$$\begin{aligned}\Gamma_\mu \Gamma_\nu \Gamma_\lambda \phi_{\mu\nu\lambda} &= b_i b_j b_k \phi_{\bar{c}_i \bar{c}_j \bar{c}_k} + b_i^\dagger b_j^\dagger b_k^\dagger \phi_{c_i c_j c_k} \\ &+ 3(b_i^\dagger b_j b_k \phi_{c_i \bar{c}_j \bar{c}_k} + b_i^\dagger b_j^\dagger b_k \phi_{c_i c_j \bar{c}_k}) \\ &+ (3 b_i \phi_{\bar{c}_n c_n \bar{c}_i} + 3 b_i^\dagger \phi_{\bar{c}_n c_n \bar{c}_i}) ,\end{aligned} \tag{1}$$

where

$$\phi_{c_i c_j \bar{c}_k} = f^{ij}_k + \frac{1}{4}\left(\delta^i_k f^j - \delta^j_k f^i\right), \qquad \phi_{c_i \bar{c}_j \bar{c}_k} = f^i_{jk} - \frac{1}{4}\left(\delta^i_j f_k - \delta^i_k f_j\right), \tag{2}$$

$$\phi_{c_i c_j c_k} = \varepsilon^{ijklm} f_{lm}, \qquad \phi_{\bar{c}_i \bar{c}_j \bar{c}_k} = \varepsilon_{ijklm} f^{lm}, \tag{3}$$

$$\phi_{\bar{c}_n c_n \bar{c}_i} = f^i, \quad \phi_{\bar{c}_n c_n \bar{c}_i} = -f_i, \tag{4}$$

with the following normalization

$$f^i = \frac{4}{\sqrt{3}} h^i, \qquad f_i = \frac{4}{\sqrt{3}} h_i, \tag{5}$$

$$f^{ij} = \frac{1}{\sqrt{3}} h^{ij}, \qquad f_{ij} = \frac{1}{\sqrt{3}} h_{ij}, \tag{6}$$

$$f^{ij}_k = \frac{2}{\sqrt{3}} h^{ij}_k, \qquad f^i_{jk} = \frac{2}{\sqrt{3}} h^i_{jk}, \tag{7}$$

Example in SO(10)

```
#include <sospin/son.h>
#include <sospin/tools/so10.h>

using namespace sospin;

int main(int argc, char *argv[]){

setDim(10);

Braket res = psi_16p(bra) * Bop() * GammaH(3) * psi_16p(ket);

res.evaluate();
CallForm(res, true, true);

CleanGlobalDecl();
exit(0);
}
```

Results:

```
R= - M(a)*H10(j1)*Mb01(b,j1)*sqrt(2)*i_
   + M20(a,j1,j2)*H01(j2)*Mb01(b,j1)*sqrt(2)*i_
   + 1/4*M20(a,j1,j2)*H10(j3)*M20(b,j4,j5)*sqrt(2)*e_(j1,j2,j3,j4,j5)*i_
   + Mb01(a,j1)*H01(j2)*M20(b,j1,j2)*sqrt(2)*i_
   - Mb01(a,j1)*H10(j1)*M(b)*sqrt(2)*i_ ;
```

$$i \, \frac{2}{\sqrt{3}} \, Y^{-}_{ab} \, \left(2 \, M_a \overline{M}_b \, H^i \; + \; M_a^{ij} \, M_b \, H_{ij} \; + \; \overline{M}_{i\,a} \, \overline{M}_{j\,b} \, H^{ij} \; - \; M_a^{ij} \, M_b \, H_j \; + \; \overline{M}_{i\,a} \, M_b^{jk} \, H^i_{jk} \; - \; \frac{1}{4} \, \varepsilon_{ijklm} \, M_a^{ij} \, M_b^{mn} \, H_n^{kl} \right)$$

where $Y^{-}_{ab} \equiv \dfrac{Y_{ab} - Y_{ba}}{2}$

Conclusions and Prospects

- we have presented the SO10pin library, a C++ tool whose main goal is to decompose Yukawa interactions, invariant under SO($2N$), in terms of SU(N) fields
- This library makes use of the oscillator expansion formalism
- The SO10pin code simulates the non-commutativity of the operators and their products via the implementation of **doubly-linked-list data structures**
- Data storage in the memory does not need to be adjacent, this is one of the reason why the doubly-linked-lists led to high performances in our tests;
  Example: [Intel(R) Core(TM) i5-3317U CPU @ 1.70GHz] $9! = 362880$ terms need only 454.15 MB and 2.80 s
- We added support for SO(10) dedicated-functions including 144 and $\overline{144}$
- **Future Plans**
  - Enhance the use of the memory
  - Make the simplifications of the final expressions independent of external programs in order to reach more performance
  - More automatisations to deal with SO($2N$)
  - The library should be easily adapted to SO($2N+1$) groups or other systems using creation and annihilation operators


ΒΙΒΛΙΟΧΑΡΤΟΠΩΛΕΙΟ
ΣΥΜΜΕΤΡΙΑ
Ευχαριστώ!